Stability of ground state degeneracy to long range interactions

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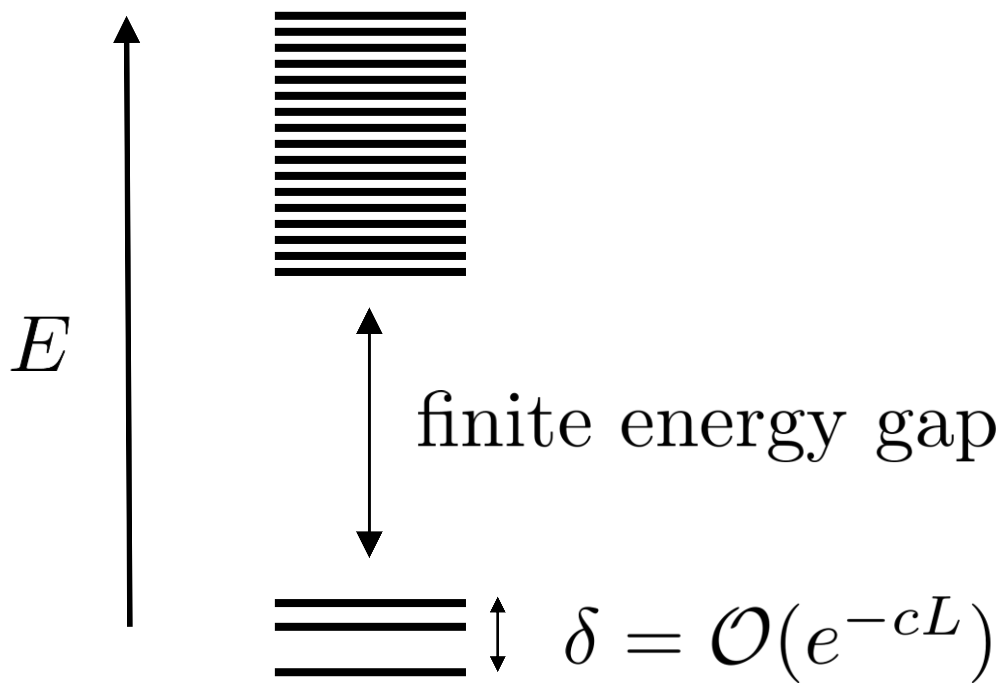
Some gapped many-body systems have exponentially small ground state splitting:



Robust phenomenon: no fine tuning

Examples:

- 2D topological phases (in torus geometry)
- 2D topological phases with non-Abelian anyons
- 1D topological superconductors (with open b.c.)
- (Discrete) symmetry breaking phases

Main question

Arguments for robust, exponentially small splitting apply to systems with *short-range* interactions

What about long-range (power law) interactions?

Why worry about long-range interactions?

- Relevant to many experimental systems

- Conceptual question: how much locality is necessary for topological phenomena?

Stability formulation

Consider:

$$H = H_0 + \lambda V$$

H_0: exactly solvable; short-range; exact GSD

V: generic interaction with short-range and long-range parts

What happens when we turn on $\lambda \neq 0$?

Two questions:

1. Does the gap stay open?
2. If so, is ground state splitting δ exponentially small in system size?

Short-range V: Yes, in many cases

(Kirkwood, Thomas, 1983) (Kennedy, Tasaki, 1992) (Klich, 2010)
(Bravyi, Hastings, Michalakis, 2010) (Bravyi, Hastings, 2011)
(Michalakis, Zwolak, 2013) (Nachtergaele, Sims, Young, 2019)

Long-range V: ??

Review of arguments in short-range case

Focus on simple example:

$$H_0 = -\sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z, \qquad V = \sum_{j=1}^{L} \sigma_j^x$$

(Transverse field Ising model)

Ising symmetry:

$$\mathcal{S} = \prod_{j=1}^{L} \sigma_j^x$$

H_0 has 2 exactly degenerate ground states:

$$|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle - |\downarrow\downarrow\cdots\downarrow\rangle)$$

$|+\rangle \in \mathcal{H}_+$

$|-\rangle \in \mathcal{H}_-$

subspaces with $\mathcal{S} = \pm 1$

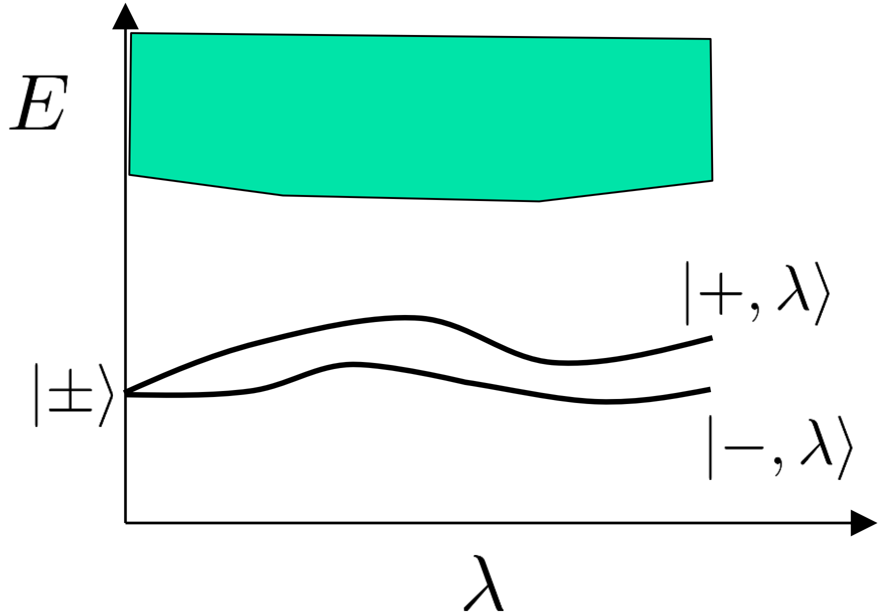
How do these states split when we turn on $\lambda \neq 0$?

(Hastings, Wen, 2005)

Approach #1: Quasi-adiabatic continuation

Assume: energy gap stays open for $\lambda \neq 0$.

Energy spectrum:



QAC $\implies$ there exists unitary U_λ such that

1. $|\pm, \lambda\rangle = U_\lambda |\pm\rangle$

2. U_λ is locality preserving:

$$U_\lambda : \quad O \quad \to \quad U_\lambda^\dagger O U_\lambda$$

local "superpolynomially local"

(derived from Lieb-Robinson bounds)

Energy splitting:

$$\delta = \langle + | U_\lambda^\dagger H U_\lambda | + \rangle \; - \; \langle - | U_\lambda^\dagger H U_\lambda | - \rangle$$

$$= 2 \ \mathrm{Re} \left(\langle \uparrow\uparrow \cdots \uparrow | U_\lambda^\dagger H U_\lambda | \downarrow\downarrow \cdots \downarrow \rangle \right)$$

$$= \mathcal{O}(L^{-\infty})$$

sum of "superpolynomially local" operators

(Michalakis, Zwolak, 2013)

Now consider the case where V is long-range (power law):

$$U_\lambda : \quad O \quad \rightarrow \quad U_\lambda^\dagger O U_\lambda$$

local "polynomially local"

$\Longrightarrow$ $U_\lambda^\dagger H U_\lambda$ has power-law tails

$\Longrightarrow$ can only get *power-law* bound: $\delta = \mathcal{O}(L^{-\alpha})$

Approach #2: Perturbation theory

Key observation – no splitting until Lth order in perturbation theory:

$$\langle \Uparrow | V^p | \Downarrow \rangle = 0 \text{ for } p < L$$

$\Longrightarrow \; \delta \sim \lambda^L$

$\Longrightarrow$ exponential bound

To make rigorous, need finite radius of convergence. Can be established using "polymer expansion."

(Kennedy, Tasaki, 1992) (Borgs, Kotecki, Ueltschi, 1996)
(Datta, Fernandez, Frohlich, 1996) (Klich, 2010)

Can we extend to long-range V?

Yes!

Our main result: exponential splitting bound for a large class of V of the form

$$V = V_{\text{short}} + V_{\text{long}}$$

where H_0 is symmetry breaking Hamiltonian

Our result for a prototypical example

$$H = H_0 + \lambda V$$

$$H_0 = -\sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z$$

$$V = h \sum_{j=1}^{L} \sigma_j^x + \sum_{jk} f(j-k) \sigma_j^x \sigma_k^x$$

We require that f obeys:

$$\sum_{k=1}^{L} |f(j-k)| \leq C_0$$

"summability condition"

e.g. $f(r) \sim \frac{1}{|r|^\alpha}$ with $\alpha > 1$

Condition guarantees that

$$\| \sum_{jk} f(j-k) \sigma_j^x \sigma_k^x \| \leq C_0 L$$

$\implies \|V\|$ is *extensive*

(Lapa, Levin, arXiv:2107.11396)

Theorem:

There exists $\lambda_0 > 0$ such that, if $|\lambda| < \lambda_0$, then:

1. H has a unique ground state and a finite energy gap in $\mathcal{H}_{\pm}$.

2. The ground state splitting between sectors obeys the bound

$$|E_+(\lambda) - E_-(\lambda)| = \mathcal{O}(e^{-cL})$$

Idea of proof

Define:

$$Z_\pm = \mathrm{Tr}_\pm(e^{-\beta H}) \qquad Z_0 = \mathrm{Tr}_\pm(e^{-\beta H_0})$$

trace over $\mathcal{H}_\pm$

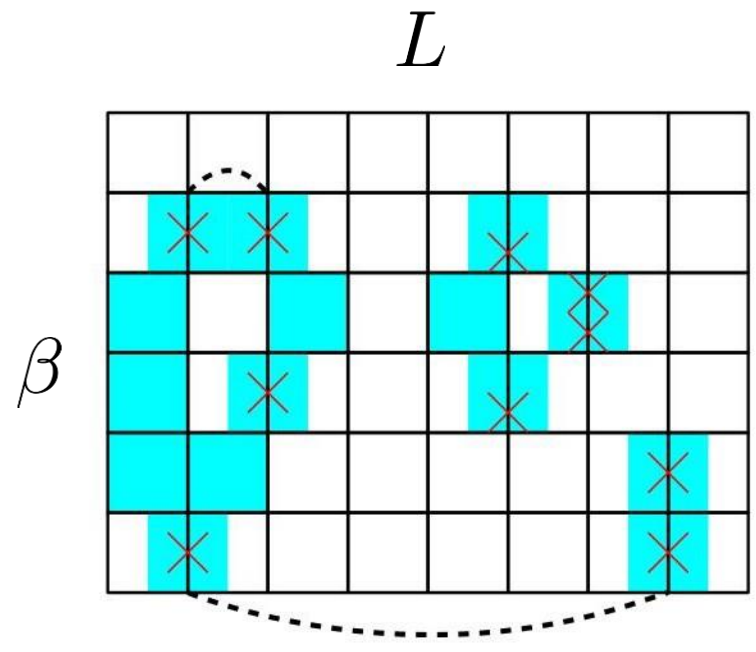
Can write:

$$Z_\pm = Z_0 \cdot \sum_X W_\pm(X)$$

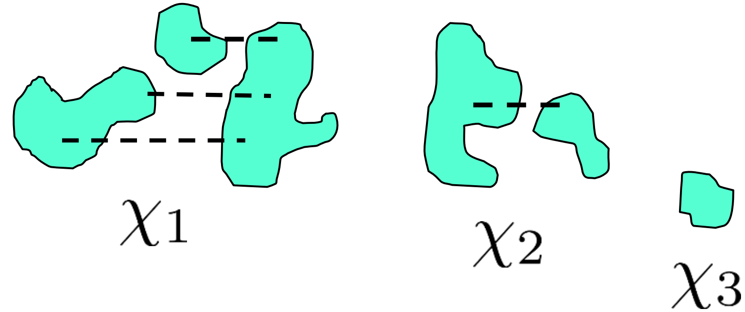
"support sets"

"weight of X"

A typical support set X:



X = collection of "plaquettes", "boxes", "dashed lines"



Weights factorize:

$$X = \chi_1 \cup \chi_2 \cup \dots$$

connected components of X

$$W_\pm(X) = W_\pm(\chi_1) W_\pm(\chi_2) \cdots$$

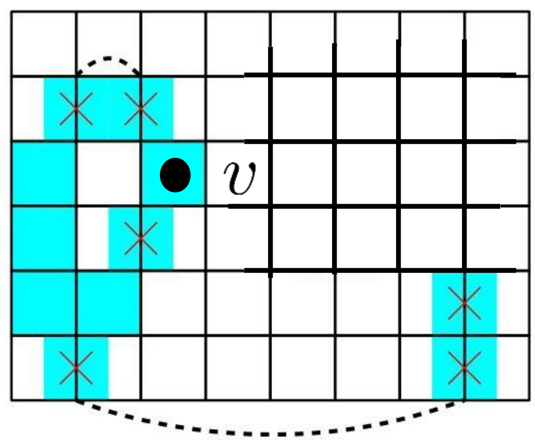
Factorization $\implies$ "polymer expansion" for $\log(Z_\pm / Z_0)$

Polymers = *connected* support sets χ

Need to show polymer expansion converges for small $|\lambda|$

Key bound: There exists a constant $\lambda_0 > 0$ such that for all $|\lambda| < \lambda_0$,

$$\sum_{\chi \ni v} |W_{\pm}(\chi)| < 1$$



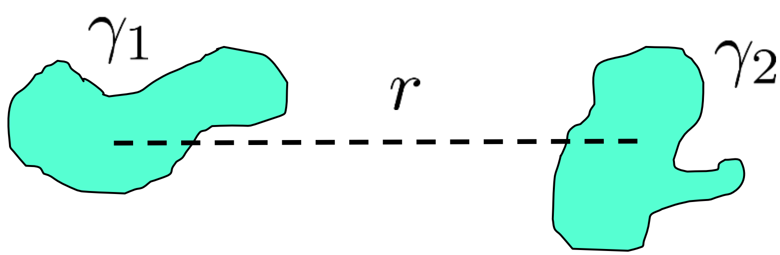
Similar to energy-entropy condition of Peierls

Can show:

$$|W_{\pm}(\chi)| \leq |\lambda|^{|\chi|} \cdot \prod_r |\lambda f(r)|^{d_r(\chi)}$$

$|\chi|$ = total number of plaquettes and boxes in χ

$d_r(\chi)$ = number of dashed lines of length r in χ



$$|\lambda|^{|\gamma_1|+|\gamma_2|} \cdot \underbrace{\sum_r |\lambda f(r)|}_{\leq C_0 |\lambda|}$$

Generalization

one dimension $\Longrightarrow$ D dimensions

2-body interactions $\Longrightarrow$ K-body interactions

$$H_0 = -\sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \sigma^z_{\mathbf{r}} \sigma^z_{\mathbf{r}'} \qquad V = \sum_{\mathbf{r}} h_{\mathbf{r}} V_{\mathbf{r}} \qquad V_{\mathbf{r}} = T_{\mathbf{r}} V_0 T_{\mathbf{r}}^{-1}$$

$$V_0 = \sum_Y V_Y$$

subsets of at most K sites containing 0

We require that:

$$\sum_Y \|V_Y\| \leq C_0$$

generalized summability condition

Also, we impose the symmetry requirement:

$$[V_Y, \mathcal{S}_{Y_i}] = 0 \qquad \mathcal{S}_{Y_i} = \prod_{\mathbf{r} \in Y_i} \sigma_{\mathbf{r}}^x$$

$$Y_i = \text{connected components of } Y$$

Note our result does *not* apply to long-range $\sigma_{\mathbf{r}}^z \sigma_{\mathbf{r}'}^z$ interactions.

Open questions

1. Can we generalize to topologically ordered models?
    - 1D topological superconductors ✓
    - 2D toric code ?

2. What happens in cases where summability condition is violated?
    - If gap stays open, is the splitting always exponentially small?