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Topology and Entanglement in Many-Body Systems

Spectral gaps, stability and $O(n)$ spin chains

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joint work with

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Outline

- Locality, quasi-locality, etc.
- Ground states of infinite systems
- Gapped ground states
- What do you mean by 'stability'?
- Stability theorem for the bulk gap
- Stability of dimerized $O(n)$ spin chains
- Phase diagram of $O(n)$ spin chains

Quantum lattice systems

Here, 'lattice' is some nice discrete metric space (Γ, d), such as $\mathbb{Z}^\nu$ with the usual ℓ^1 distance, or a Delone subset of $\mathbb{R}^\nu$.
For each $x \in \Gamma$, observables are finite-dimensional matrix algebra $\mathcal{A}_{\{x\}}$; for finite $\Lambda \subset \Gamma$,

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_{\{x\}}, \quad \mathcal{A}_{\rm loc} = \bigcup_{\text{finite } \Lambda\subset\Gamma} \mathcal{A}_\Lambda, \quad \mathcal{A}_\Gamma = \overline{\mathcal{A}_{\rm loc}}^{\|\cdot\|}.$$

$A \in \mathcal{A}_\Lambda$ is said to be supported in Λ, any $A \in \mathcal{A}_{\rm loc}$ is a local observable, and $A \in \mathcal{A}_\Gamma$ are the quasi-local observables.
A system is defined by its Heisenberg dynamics $\tau_t^\Phi, t \in \mathbb{R}$, in terms of an interaction Φ: $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$, for all finite $X \subset \Gamma$, through the derivation $\delta : \mathcal{A}_{\rm loc} \to \mathcal{A}_\Gamma$. Formally:

$$\delta(A) = \sum_X [\Phi(X), A]; \quad \frac{d}{dt}\tau_t^\Phi(A) = i\delta(\tau_t^\Phi(A)),\ \tau_0^\Phi = \mathbb{1}; \quad \tau_t^\Phi = e^{it\delta}.$$

This requires sufficient decay of $\|\Phi(X)\|$ for large $X \subset \Gamma$ to ensure δ is well-defined and that its closure generates a group of automorphisms τ_t^Φ.
An invariant domain for δ will be strictly larger than $\mathcal{A}_{\rm loc}$ but smaller than $\mathcal{A}_\Gamma$.

Locality, Quasi-Locality, Almost-Locality

By construction, for all $A \in \mathcal{A}_\Gamma$ and any sequence $\Lambda_n \uparrow \Gamma$, there exist $\mathcal{A}_{\Lambda_n} \ni A_n \to A$. A concrete sequence of local approximations of any $A \in \mathcal{A}_\Gamma$ can be obtained by using the conditional expectations Π_Λ:

$$\Pi_\Lambda = \mathrm{id}_{\mathcal{A}_\Lambda} \otimes \rho \restriction_{\mathcal{A}_{\Gamma \setminus \Lambda}}, \text{ where } \rho \text{ is the tracial state.}$$

Given (Λ_n) (for example $\Lambda = b_x(n)$, $n \geq 0$, balls centered at $x \in \Gamma$), $A \in \mathcal{A}_\Gamma$, one has f, decreasing to 0, for which

$$\|A - \Pi_{\Lambda_n}(A)\| \leq \|A\| f(n), n \geq 1.$$

For a fixed sequence (Λ_n) and f, positive and decreasing to 0, we can define

$$\mathcal{A}_f = \{A \in \mathcal{A}_\Gamma \mid \exists C > 0, \|A - \Pi_{\Lambda_n}(A)\| \leq C\|A\| f(n), \text{all } n \geq 1\}.$$

Useful relation connection between locality and Lieb-Robinson bounds:

$$\|A - \Pi_\Lambda(A)\| \leq \sup_{B \in \mathcal{A}_{\Gamma \setminus \Lambda}, \|B\|=1} \|[A, B]\| \leq 2\|A - \Pi_\Lambda(A)\|.$$

If (Γ, d) is a Delone subset of $\mathbb{R}^\nu$, one can assume that Φ is supported on balls $b_x(n) \subset \Gamma$ and express decay by a conditions of the form

$$\|\Phi(b_x(n))\| \leq \|\Phi\|_f f(n), \text{ for all } x \in \Gamma, n \geq 0.$$

For suitable f and g, there is h for which $\delta(A) \in \mathcal{A}_h$, for all $A \in \mathcal{A}_g$.

Examples: if f and g are characteristic functions, h can be taken to be a characteristic function; if $f(n) = g(n) = e^{-an^\theta}$, one can take $h(n) = e^{-a'n^\theta}$, with $a' < a$.

Upshot: for infinite systems with sufficiently short-range interactions, we can define

$$h_x = \sum_n \Phi(b_x(n))$$

and

$$\delta(A) = \sum_x [h_x, A], \quad A \in \mathcal{A}_g \subset \operatorname{dom} \delta.$$

(N-Sims-Young, JMP 2019, Moon-Ogata JFA 2019, Kapustin-Sopenko JMP 2020, Bachmann-Lange arXiv:2105.14168, Henheik-Teufel arXiv:2012.15238/9.)

Ground states

A state ω on $\mathcal{A}$ is a ground state for the dynamics τ_t with generator δ if

$$\omega(A^*\delta(A)) \geq 0, \text{ for all } A \in \mathrm{dom}\,\delta.$$

It is sufficient to check this condition for A in a core for δ, such as $\mathcal{A}_{\mathrm{loc}}$.

The GNS representation

The GNS representation of a state on $\mathcal{A}_\Gamma$ is given by a Hilbert space $\mathcal{H}$, a representation π of $\mathcal{A}$ on $\mathcal{H}$, and a cyclic vector $\Omega \in \mathcal{H}$ such that, for all $A \in \mathcal{A}$

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle, \quad A \in \mathcal{A}_\Gamma.$$

For ground states one finds that τ_t is implemented by a strongly continuous group of unitaries on $\mathcal{H}$:

$$\pi(\tau_t(A)) = U_t^* \pi(A) U_t = e^{itH_\omega} \pi(A) e^{-itH_\omega}$$
$$H_\omega \geq 0, \quad H_\omega \Omega = 0$$

If there is only one ground state for τ_t, we necessarily have that it is a pure state (hence, π is irreducible) and that $\ker H_\omega = \mathbb{C}\Omega$.

Gapped ground states

Consider the case of a pure ground state with $\ker H_\omega = \mathbb{C}\Omega$. Then, for any $\gamma > 0$

$$\operatorname{spec} H_\omega \cap (0,\gamma) = \emptyset \text{ iff } \omega(A^*\delta(A)) \geq \gamma\omega(A^*A), A \in \mathcal{A}_{\rm loc} \text{ with } \omega(A) = 0$$

If this condition holds for some $\gamma > 0$, the ground state is gapped. Then

$$\mathrm{gap}(H_\omega) = \sup\{\gamma > 0 \mid \operatorname{spec} H_\omega \cap (0,\gamma) = \emptyset\}.$$

For infinite systems with Γ without boundary, e.g., $\Gamma = \mathbb{Z}^\nu$: $\mathrm{gap}(H_\omega)$ is the bulk gap. If Γ is a half-space of $\mathbb{Z}^\nu$, it may be referred to as the edge gap etc.

Stability of Spectral Gaps

Stability of the bulk gap

Suppose $\{h_x\}_{x\in\Gamma}$ defines generator δ with (for simplicity) a unique ground state ω and a gap $\gamma_0 > 0$:

$$\omega(A^*\delta(A)) \geq \gamma_0(\omega(A^*A) - |\omega(A)|^2), A \in \mathrm{dom}\,\delta \Leftrightarrow \mathrm{gap}(H_\omega) \geq \gamma_0.$$

Define perturbations of the form

$$h_x(s) = h_x + s\Phi_x, s \in \mathbb{R}, \Phi_x = \sum_n \Phi(b_x(n)), \text{ with } \|\Phi(b_x(n))\| \leq g(n).$$

The gap of the model is stable under such perturbations if for all $\gamma \in (0, \gamma_0)$, there exists $s_0(\gamma) > 0$ such that the gap for the perturbed model, γ_s, satisfies

$$\gamma_s \geq \gamma, \text{ for all } |s| < s_0(\gamma).$$

Stability theorem for frustration free finite range interactions

We consider perturbations of finite-range (R) frustration-free models with Hamiltonians of the form

$$H_\Lambda(s) = \sum_{x\in\Lambda} h_x + s \sum_{x\in\Lambda, n\geq 0} \Phi(b_x(n))$$

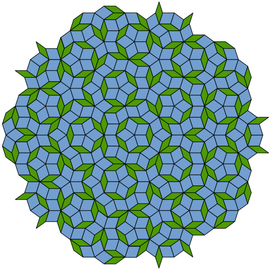
with uniformly bounded $h_x \in \mathcal{A}_{b_x(R)}$, $\sup_x \|h_x\| < \infty$. $\Gamma \subset \mathbb{R}^\nu$, Delone.
C1: There are $C > 0, q \geq 0$ such that $\mathrm{gap}(H_{b_x(n)}(0)) \geq Cn^{-q}$ (non-zero edge modes do not vanish faster than a power law).
C2: $\mathrm{gap}(H^{GNS}(0)) = \gamma_0 > 0$.
C3: $\|\Phi(b_x(n))\| \leq \|\Phi\| e^{-an^\theta}$, for some $a > 0, \theta > 0$.
C4: LTQO. Denote by P_Λ the projection onto $\ker H_\Lambda(0)$. There exists a positive decreasing function G_0 for which, for all $A \in \mathcal{A}_{b_x(k)}$,

$$\|P_{b_x(m)} A P_{b_x(m)} - \omega_0(A) P_{b_x(m)}\| \leq \|A\|(k+1)^\nu G_0(m-k).$$

and

$$\sum_{n\geq 1} n^{q+3\nu/2} \sqrt{G_0(n)} < \infty.$$

Not assuming a uniform gap in finite volume!



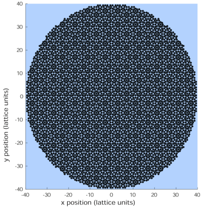


FIG. 2. Round section of the quasilattice, of radius 40 lattice units.

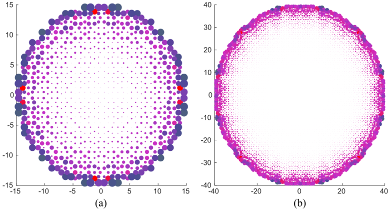


FIG. 7. (a) The radius is 15, with eigenvalue 0.018. (b) The radius is 40 with eigenvalue 0.007.

Figure: Penrose tiling. Ammann-Beenker tiling. Edges state or not? (T. Loring, J. Math. Phys. **60**, 081903 (2019))

Theorem

(Stability of the bulk gap, N-Sims-Young, arXiv:arXiv:2102.07209)

If conditions C1-C4 are satisfied, then, for all $\gamma \in (0, \gamma_0)$, there is a constant $\beta > 0$, such that the ground state of $H(s)$ with

$$|s| \leq \frac{\gamma_0 - \gamma}{\beta \gamma_0}$$

is unique, and $\mathrm{gap} H^{GNS}(s) > \gamma$.

Proved using the strategy of Bravyi-Hastings-Michalakis 2010, applied to the GNS Hamiltonian. β is explicit.

$O(n)$ spin chains

$$H = -\sum_x Q_{x,x+1}$$

with Q the rank-1 projection determined by

$$\psi = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle.$$

- translation-invariant nn 1D model of n-dimensional spins
- manifestly $O(n)$-invariant and not frustration-free
- case $n = 2$ is Heisenberg anti-ferromagnet, Bethe-ansatz solvable, unique gapless ground state
- for $n \geq 3$, H has (at least) two 2-periodic gapped ground states:

$$\omega_\pm(Q_{x,x+1}) = -e_n \pm (-1)^x \delta_n, \quad \delta_n > 0, n \geq 3$$

recently proved by Aizenman, Duminil-Copin, and Warzel (AHP 2020).

A little bit of stability

Perturbing H with nn swap operator $T|\alpha,\beta\rangle = |\beta,\alpha\rangle$, we obtain the family of Hamiltonians:

$$H(u,v) = \sum_x uT_{x,x+1} + vQ_{x,x+1}, \quad u,v \in \mathbb{R}$$

This is the $O(n)$ extension of the spin-1 bilinear-biquadratic chain, with the most general $O(n)$-symmetric nn interaction.
We now proved that these dimerized ground states and the spectral gap above them persists for $|u|$ small and n large.

Theorem (Björnberg-Mühlbacher-N-Ueltschi, CMP 2021)
There exist constants $n_0, u_0 > 0$ such that for all $n > n_0$, for the $O(n)$ chain with $v = -1$ and $|u| < u_0$, there are two pure 2-periodic gapped ground states.

Previous stability results do not apply since these models are not frustration-free.

Graphical respresentation

On finite chains of even length, $[-\ell+1,\ell]\subset\mathbb{Z}$, The ground state of $H(u,-1)$, with $|u|$ not too large, is unique. Call it Ω. Then

$$|\Omega\rangle\langle\Omega| = \lim_{\beta\to\infty}\frac{e^{-2\beta H}}{\mathrm{Tr}e^{-2\beta H}},$$

and

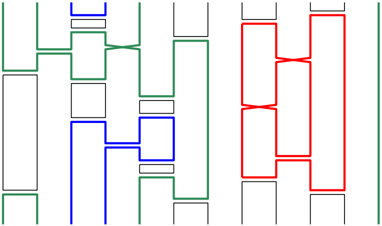
$$\langle\Omega|A|\Omega\rangle = \lim_{\beta\to\infty}\frac{\mathrm{Tr}e^{-\beta H}Ae^{-\beta H}}{\mathrm{Tr}e^{-2\beta H}}.$$

By writing (for integer β)

$$\begin{aligned} e^{-\beta H} &= \lim_{N\to\infty}(\mathbb{1}-\frac{1}{N}H)^{\beta N} \\ &= \lim_{N\to\infty}\left(\mathbb{1}-\frac{u}{N}\sum_{x=-\ell+1}^{\ell-1}T_{x,x+1}-\frac{v}{N}\sum_{x=-\ell+1}^{\ell-1}Q_{x,x+1}\right)^{\beta N}. \end{aligned}$$

and expanding the product we get a weighted sum of terms that are a products of βN factors $\mathbb{1}$, $T_{x,x+1}$ and $Q_{x,x+1}$.

$$
\begin{aligned}
\langle \alpha', \beta' | Q | \alpha, \beta \rangle &= \frac{1}{n} \delta_{\alpha\beta} \delta_{\alpha'\beta'} \\
\langle \alpha', \beta' | T | \alpha, \beta \rangle &= \delta_{\alpha\beta'} \delta_{\alpha'\beta} \\
\langle \alpha', \beta' | \mathbb{1} | \alpha, \beta \rangle &= \delta_{\alpha\alpha'} \delta_{\beta\beta'}
\end{aligned}
$$



Graphically we represent the operators by crosses and double bars:

$$T = \bowtie, \quad Q = |\!\!=\!\!|, \quad \mathbb{1} = | \quad |$$

Basis labels are constant along lines. After $\lim_{N\to\infty}$, one gets a space-time picture of loops:

$$\mathrm{Tr} e^{-2\beta H(u,-1)} = \int_{\Omega_{\ell,\beta}} d\rho_u(\omega) n^{\mathcal{L}(\omega) - |\omega_{|\!=\!|}|}$$

with

$$d\rho_u(\omega) = e^{(1+u)2\beta(2\ell-1)} u^{|\omega_{\bowtie}|} dx^{\otimes|\omega|}.$$

Correlations

The basic correlation functions are integrals of indicator functions of 'events' for loop configurations.

$x \overset{+}{\longleftrightarrow} y$: the set of configurations ω where the top of $(x, 0)$ is connected to the bottom of $(y, 0)$;

$x \overset{-}{\longleftrightarrow} y$: the set of configurations ω where the top of $(x, 0)$ is connected to the top of $(y, 0)$

Define $L^{\alpha,\alpha'} = |\alpha\rangle\langle\alpha'| - |\alpha'\rangle\langle\alpha|$.

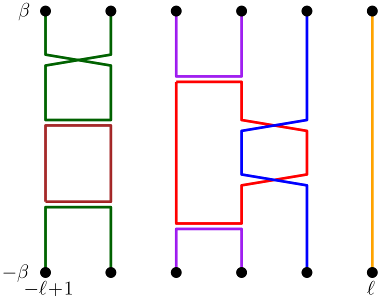
Proposition

For the spin chain of length 2ℓ with interaction $h_{x,x+1} = -uT_{x,x+1} - Q_{x,x+1}$, we have for all $1 \leq \alpha < \alpha' \leq n$,

$$\frac{Tr \;\; L_x^{\alpha,\alpha'} L_y^{\alpha,\alpha'} e^{-2\beta H}}{Tre^{-2\beta H}}$$

$$= \frac{2}{n} \frac{\int_{\Omega_{\ell,\beta}} d\rho_u(\omega)\, n^{\mathcal{L}(\omega)-|\omega_{\boxminus}|} \big(1\!\!1[x \overset{-}{\longleftrightarrow} y] - 1\!\!1[x \overset{+}{\longleftrightarrow} y] \big)}{\int_{\Omega_{\ell,\beta}} d\rho_u(\omega) n^{\mathcal{L}(\omega)-|\omega_{\boxminus}|}}$$

short loops, long loops, winding loops



- the winding loop are those that are not contractible (blue and orange)

- the long loops are those that are winding or visit 3 or more sites (red, blue, orange)

- short loops are those that are not long (green, brown, purple)

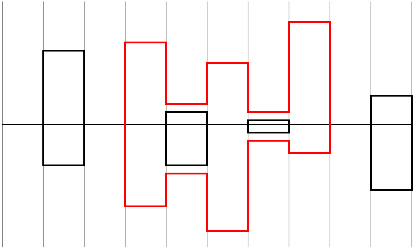
For large β, winding loops become negligible.

If there were only short loops, the measure would clearly be dominated by a perfectly dimerized state.

The challenge is to show that dimerization survives in spite of the non-vanishing contributions of long loops.

Contours

In the case $u = 0$, long loops can serve as contours separating one dimerized phase from the other:



The short loops outside and inside the contour are out of phase.
A Peierls argument using such contours was used to prove dimerization for $n \geq 17$ (N-Ueltschi, 2017).
Later, special properties of the random loop measure were used to prove dimerization for all $n \geq 3$ (Aizenman, Duminil-Copin, Warzel, 2020).

Clusters

For $u \neq 0$, configurations contain crosses ($\bowtie$), which may be crossings of different loops or self-crossings. Similarly, the top and bottom part of a double bar ($\asymp$), may belong to the same loop or to different loops. Since these distinction are non-local, we define clusters of long loops that share a $\bowtie$ or a $\asymp$.

As in the case $u = 0$, the short loops describe the reference dimerized states. A convergent cluster expansion of the partition function is the tool that allows us to prove that short loops dominate (for large n and small $|u|$).

Theorem (Björnberg-Mühlbacher-N-Ueltschi, CMP 2021)

There exist constants $n_0, u_0, c_1, c_2, C > 0$ (independent of ℓ) such that for $n > n_0$ and $|u| < u_0$, we have

$$\langle \Omega | L_x^{\alpha,\alpha'} e^{-tH} L_y^{\alpha,\alpha'} e^{tH} | \Omega \rangle \leq C e^{-c_1|x-y|-c_2|t|}$$

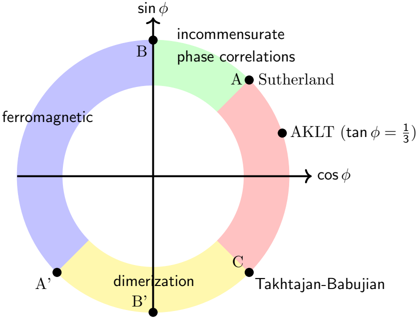


Figure: Ground state phase diagram for the $S = 1$ chain ($n = 3$) with nearest-neighbor interactions $\cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi(\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

- $\phi = 0$ Heisenberg AF chain, Haldane phase (Haldane, 1983)
- $\tan\phi = 1/3$, AKLT point (Affleck-Kennedy-Lieb-Tasaki, 1987,1988), FF, MPS, gapped
- $\tan\phi = 1$, solvable, gapless, SU(3) invariant, (Sutherland, 1975)
- $\phi \in [\pi/2, 3\pi/2]$, ferromagnetic, FF, gapless
- $\phi = -\pi/2$, solvable, SU(3) invariant, Temperley-Lieb algebra, dimerized, gapped (Klümper; Affleck,1990)
- $\phi- = -\pi/4$ gapless, Bethe-ansatz, (Takhtajan; Babujian, 1982)

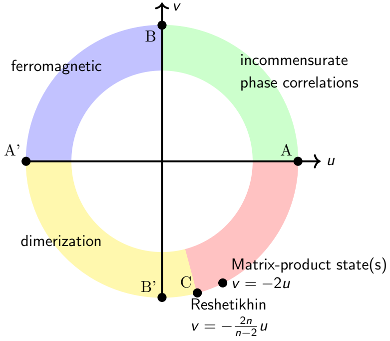


Figure: Ground state phase diagram for the chain with nearest-neighbor interactions $uT + vQ$ for $n \geq 3$, studied by Tu & Zhang, 2008.

- $v = -2nu/(n-2), n \geq 3$, Bethe ansatz point (Reshetikhin, 1983)
- $v = -2u$: frustration free point, equivalent to $\perp$ projection onto symmetric vectors $\ominus$ one. Unique g.s. if n odd; two 2-periodic g.s. for even n; spectral gap in all cases and stable phase (N-Sims-Young, 2021).
- $u = 0, v = -1$. Equivalent to the $SU(n) - P^{(0)}$ models aka Temperley-Lieb chain; Affleck, 1990, Nepomechie-Pimenta 2016). Dimerized for all $n \geq 3$ (Aizenman, Duminil-Copin, Warzel, 2020). New result: a proof of stability for large n (Björnberg-Mühlbacher-N-Ueltschi, 2021).

Concluding Comments

- gap for $O(n)$ chains is more stable than that little bit we proved
- need a good general method for non-frustration-free models, for proving gaps and their stability
- more general formulation of LTQO?