Approximate QCAs and a converse to the Lieb-Robinson bounds

Michael Walter

joint work with Daniel Ranard (MIT) and Freek Witteveen (Amsterdam)

Workshop on Topology and Entanglement in Many-Body Systems
Banff, October 2021

UNIVERSITEIT VAN AMSTERDAM

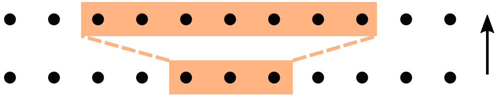
QuSoft Research Center for Quantum Software

NWO

Motivation

Quantum cellular automata model strictly local dynamics. However:

Lieb-Robinson: Local Hamiltonian evolution obeys *approximate* light cone.



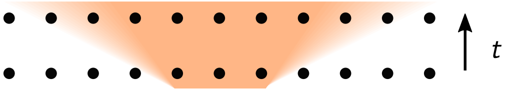
For short-range interactions, there is Lieb-Robinson velocity v such that support of local operators grows as vt, up to exponential tails.

Can the theory of QCAs be generalized to this setting?

Motivation

Quantum cellular automata model strictly local dynamics. However:

Lieb-Robinson: Local Hamiltonian evolution obeys *approximate* light cone.



For short-range interactions, there is Lieb-Robinson velocity v such that support of local operators grows as vt, up to exponential tails.

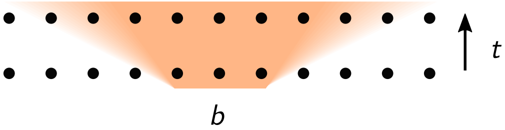
Can the theory of QCAs be generalized to this setting?

Motivation

Quantum cellular automata model strictly local dynamics. However:

Lieb-Robinson: Local Hamiltonian evolution obeys *approximate* light cone.

$$b(t) = e^{iHt} b e^{-iHt}$$



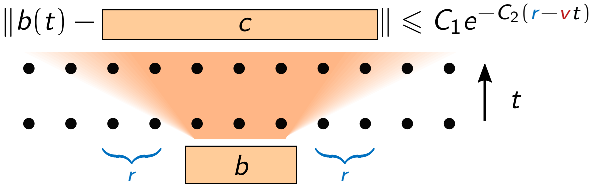
For short-range interactions, there is Lieb-Robinson velocity v such that support of local operators grows as vt, up to exponential tails.

Can the theory of QCAs be generalized to this setting?

Motivation

Quantum cellular automata model strictly local dynamics. However:

Lieb-Robinson: Local Hamiltonian evolution obeys *approximate* light cone.



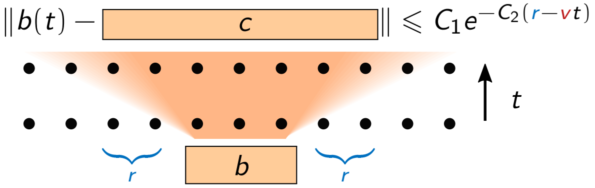
For short-range interactions, there is Lieb-Robinson velocity v such that support of local operators grows as vt, up to exponential tails.

Can the theory of QCAs be generalized to this setting?

Motivation

Quantum cellular automata model strictly local dynamics. However:

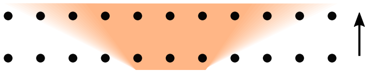
Lieb-Robinson: Local Hamiltonian evolution obeys *approximate* light cone.



For short-range interactions, there is Lieb-Robinson velocity v such that support of local operators grows as vt, up to exponential tails.

Can the theory of QCAs be generalized to this setting?

Motivation and summary



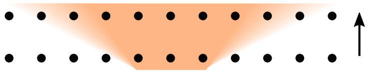
Physics question: Are local dynamics generated by local Hamiltonians?

- That is, can we find converse to Lieb-Robinson bounds?
- How about lattice translations?
- Boundary dynamics generated by bulk local Hamiltonian?

Mathematics question: Classify approximately local dynamics.

Our results: Approximately local dynamics in 1D have structure & index theory similar to QCAs. In particular, obtain a converse to LR bounds.

Motivation and summary



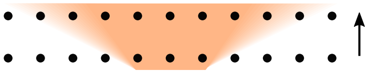
Physics question: Are local dynamics generated by local Hamiltonians?

- That is, can we find converse to Lieb-Robinson bounds?
- How about lattice translations?
- Boundary dynamics generated by bulk local Hamiltonian?

Mathematics question: Classify approximately local dynamics.

Our results: Approximately local dynamics in 1D have structure & index theory similar to QCAs. In particular, obtain a converse to LR bounds.

Motivation and summary



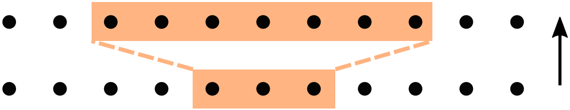
Physics question: Are local dynamics generated by local Hamiltonians?

- That is, can we find converse to Lieb-Robinson bounds?
- How about lattice translations?
- Boundary dynamics generated by bulk local Hamiltonian?

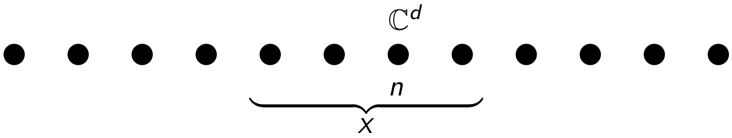
Mathematics question: Classify approximately local dynamics.

Our results: Approximately local dynamics in 1D have structure & index theory similar to QCAs. In particular, obtain a converse to LR bounds.

Quantum Cellular Automata



Setup: Infinite spin chains



It is convenient to work in the Heisenberg picture:

$$\mathcal{A}_n = \mathrm{Mat}(d) \quad \rightsquigarrow \quad \mathcal{A}_X = \bigotimes_{n \in X} \mathcal{A}_n \quad \rightsquigarrow \quad \mathcal{A}_{loc} = \bigcup_{X \in \Lambda} \mathcal{A}_X$$

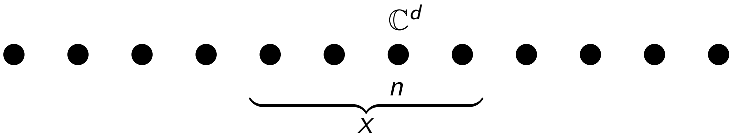
Quasi-local C^*-algebra:

$$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\|\cdot\|} = \text{“}\bigotimes_{n \in \mathbb{Z}} \mathcal{A}_n\text{”}$$

We can also define $\mathcal{A}_{\geqslant n} = \mathcal{A}_{\{n,n+1,\dots\}} \subseteq \mathcal{A}$, etc.

Local dynamics are naturally modeled by automorphisms $\alpha \colon \mathcal{A} \to \mathcal{A}$.

Setup: Infinite spin chains



It is convenient to work in the Heisenberg picture:

$$\mathcal{A}_n = \mathrm{Mat}(d) \quad \rightsquigarrow \quad \mathcal{A}_X = \bigotimes_{n \in X} \mathcal{A}_n \quad \rightsquigarrow \quad \mathcal{A}_{loc} = \bigcup_{X \in \Lambda} \mathcal{A}_X$$

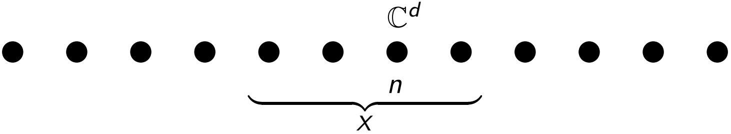
Quasi-local C^*-algebra:

$$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\|\cdot\|} = \text{“} \bigotimes_{n \in \mathbb{Z}} \mathcal{A}_n \text{”}$$

We can also define $\mathcal{A}_{\geqslant n} = \mathcal{A}_{\{n, n+1, \dots\}} \subseteq \mathcal{A}$, etc.

Local dynamics are naturally modeled by automorphisms $\alpha \colon \mathcal{A} \to \mathcal{A}$.

Setup: Infinite spin chains



It is convenient to work in the Heisenberg picture:

$$\mathcal{A}_n = \mathrm{Mat}(d) \quad \rightsquigarrow \quad \mathcal{A}_X = \bigotimes_{n \in X} \mathcal{A}_n \quad \rightsquigarrow \quad \mathcal{A}_{loc} = \bigcup_{X \in \Lambda} \mathcal{A}_X$$

Quasi-local C^***-algebra**:

$$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\|\cdot\|} = \text{“}\bigotimes_{n \in \mathbb{Z}} \mathcal{A}_n\text{”}$$

We can also define $\mathcal{A}_{\geqslant n} = \mathcal{A}_{\{n,n+1,\dots\}} \subseteq \mathcal{A}$, etc.

Local dynamics are naturally modeled by automorphisms $\alpha \colon \mathcal{A} \to \mathcal{A}$.

Quantum cellular automata (QCAs)

[Margolus, Schumacher-Werner, ...]

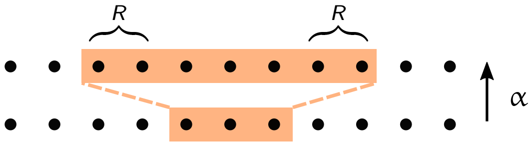
Quasi-local algebra on infinite 1D lattice:

$$\mathcal{A} = \bigotimes_{n\in\mathbb{Z}} \mathcal{A}_n, \quad \mathcal{A}_n = \mathrm{Mat}(d)$$

An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is a **quantum cellular automaton (QCA)** or locality preserving unitary (LPU) with radius $R > 0$ if:

$$\alpha(\mathcal{A}_n) \subseteq \mathcal{A}_{\{n-R,\dots,n+R\}}$$

That is, the support of *any* local operator grows by at most R:



Quantum cellular automata (QCAs)

[Margolus, Schumacher-Werner, ...]

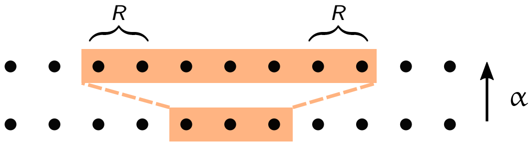
Quasi-local algebra on infinite 1D lattice:

$$\mathcal{A} = \bigotimes_{n\in\mathbb{Z}} \mathcal{A}_n, \quad \mathcal{A}_n = \mathrm{Mat}(d)$$

An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is a **quantum cellular automaton (QCA)** or locality preserving unitary (LPU) with radius $R > 0$ if for all $X \subseteq \mathbb{Z}$:

$$\alpha(\mathcal{A}_X) \subseteq \mathcal{A}_{R\text{-Neighborhood}(X)}$$

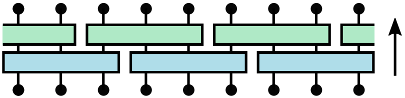
That is, the support of *any* local operator grows by at most R:



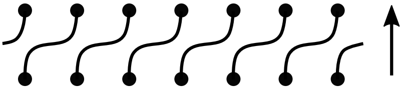
Classification of QCAs in 1D

Examples:

Quantum circuit



Shift

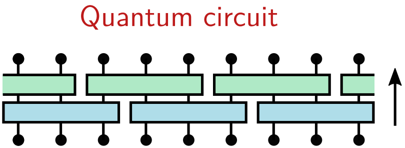


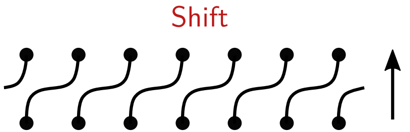
Theorem (Gross-Nesme-Vogts-Werner, GNVW):

- Any QCA is a composition of circuit and shift.
- Shift cannot be implemented by circuit.
- QCAs modulo circuits are classified by quantized index.

Classification of QCAs in 1D

Examples:





Theorem (Gross-Nesme-Vogts-Werner, GNVW):

- Any QCA is a composition of circuit and shift.
- Shift cannot be implemented by circuit.
- QCAs modulo circuits are classified by quantized index.

Index of QCAs

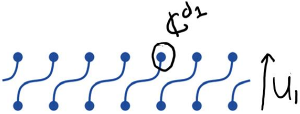
GNVW gave axiomatic, algebraic, and analytic definitions. Intuively:

$$\text{index} = \text{amount of quantum information flowing right} \\ - \text{amount of quantum information flowing left}$$

Index of QCAs

GNVW gave axiomatic, algebraic, and analytic definitions. Intuively:

$$\text{index} = \text{amount of quantum information flowing right}$$
$$- \text{amount of quantum information flowing left}$$

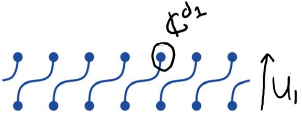


index $= \log d_1$

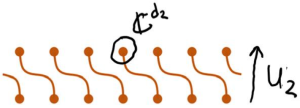
Index of QCAs

GNVW gave axiomatic, algebraic, and analytic definitions. Intuively:

$$\text{index} = \text{amount of quantum information flowing right} - \text{amount of quantum information flowing left}$$



index = $\log d_1$

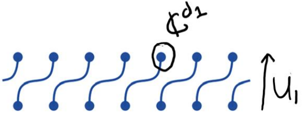


index = $-\log d_2$

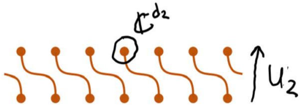
Index of QCAs

GNVW gave axiomatic, algebraic, and analytic definitions. Intuively:

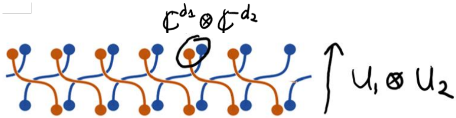
index = amount of quantum information flowing right
− amount of quantum information flowing left



index = $\log d_1$



index = $-\log d_2$

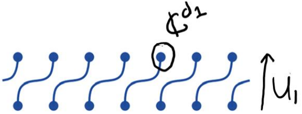


index = $\log d_1 - \log d_2 = \log \frac{d_1}{d_2}$

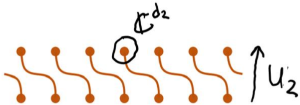
Index of QCAs

GNVW gave axiomatic, algebraic, and analytic definitions. Intuively:

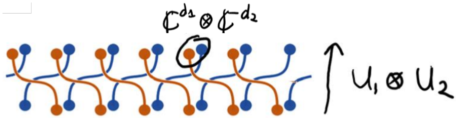
$$\text{index} = \text{amount of quantum information flowing right} - \text{amount of quantum information flowing left}$$



index $= \log d_1$



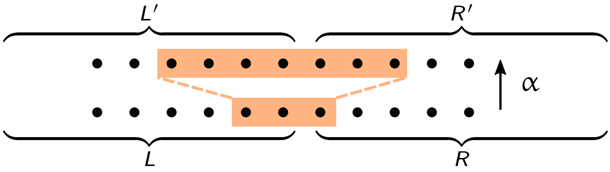
index $= -\log d_2$



index $= \log d_1 - \log d_2 = \log \frac{d_1}{d_2}$

This intuition can be made precise. . .

(Re)defining the index



Cut chain in halves and consider corresponding Choi state $\rho_{LRL'R'}$. Then:

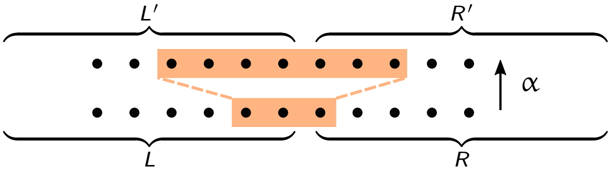
$$\text{index}\,\alpha = \frac{1}{2}\left(I(L:R') - I(L':R)\right)$$

where $I(A:B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$ is the quantum mutual information.

Properties:

- quantized: $\text{index}\,\alpha \in \mathbb{Z}[\{\log p_i\}]$, p_i = prime factors of local dimension
- additive: $\text{index}\,\alpha \otimes \beta = \text{index}\,\alpha + \text{index}\,\beta$
- robust: if $\alpha \approx \beta$ then $\text{index}\,\alpha = \text{index}\,\beta$

(Re)defining the index



Cut chain in halves and consider corresponding Choi state $\rho_{LRL'R'}$. Then:

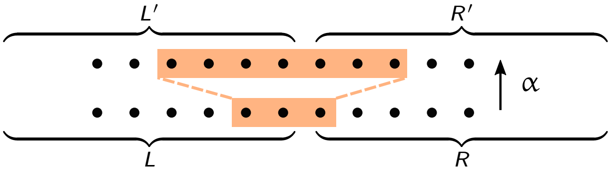
$$\text{index}\,\alpha = \frac{1}{2}\big(I(L:R') - I(L':R)\big)$$

where $I(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B)$ is the quantum mutual information.

Properties:

- quantized: index $\alpha \in \mathbb{Z}[\{\log p_i\}]$, p_i = prime factors of local dimension
- additive: index $\alpha \otimes \beta$ = index α + index β
- robust: if $\alpha \approx \beta$ then index α = index β

(Re)defining the index



Cut chain in halves and consider corresponding Choi state $\rho_{LRL'R'}$. Then:

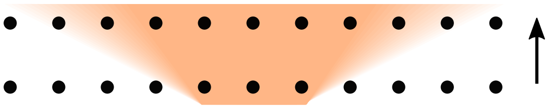
$$\text{index}\,\alpha = \frac{1}{2}\big(I(L:R') - I(L':R)\big)$$

where $I(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B)$ is the quantum mutual information.

Properties:

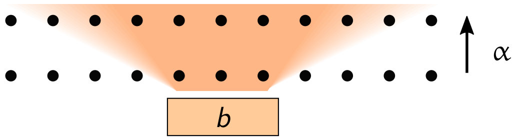
- quantized: $\text{index}\,\alpha \in \mathbb{Z}[\{\log p_i\}]$, p_i = prime factors of local dimension
- additive: $\text{index}\,\alpha \otimes \beta = \text{index}\,\alpha + \text{index}\,\beta$
- robust: if $\alpha \approx \beta$ then $\text{index}\,\alpha = \text{index}\,\beta$

Approximately Locality-Preserving Unitaries



Approximately locality-preserving unitaries (ALPUs)

Idea: Replace strict locality $\to$ Lieb-Robinson type bounds.

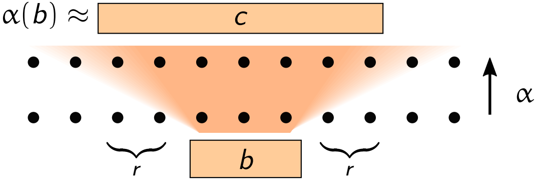


An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is an **approximately locality preserving unitary (ALPU)** with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\forall b \in \mathcal{A}_X \colon \exists c \in \mathcal{A}_{r\text{-Neighborhood}(X)} \colon \|\alpha(b) - c\| \leqslant f(r)\|b\|$$

Approximately locality-preserving unitaries (ALPUs)

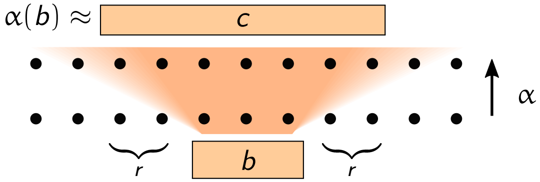
Idea: Replace strict locality $\to$ Lieb-Robinson type bounds.



An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is an **approximately locality preserving unitary (ALPU)** with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\forall b \in \mathcal{A}_X\colon\ \exists c \in \mathcal{A}_{r\text{-Neighborhood}(X)}\colon\ \|\alpha(b) - c\| \leqslant f(r)\|b\|$$

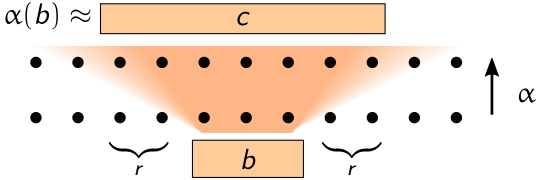
Approximately locality-preserving unitaries (ALPUs)



An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is an **approximately locality preserving unitary (ALPU)** with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\forall b \in \mathcal{A}_X\colon\ \exists c \in \mathcal{A}_{r\text{-Neighborhood}(X)}\colon\ \|\alpha(b) - c\| \leqslant f(r)\|b\|$$

Approximately locality-preserving unitaries (ALPUs)



An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is an **approximately locality preserving unitary (ALPU)** with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\alpha(\mathcal{A}_X) \subseteq_{f(r)} \mathcal{A}_{r\text{-Neighborhood}(X)}$$

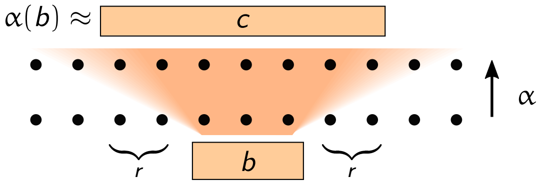
$\uparrow$

Useful notation: $\mathcal{B} \subseteq_\varepsilon \mathcal{C}$ means

$$\forall b \in \mathcal{B}\colon \exists c \in \mathcal{C}\colon \|b - c\| \leqslant \varepsilon \|b\|.$$

Examples: QCAs, local Hamiltonian dynamics (Lieb-Robinson!), …?

Approximately locality-preserving unitaries (ALPUs)



An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ is an **approximately locality preserving unitary (ALPU)** with $f(r)$-tails if for all $X \subseteq \mathbb{Z}$ and all $r > 0$:

$$\alpha(\mathcal{A}_X) \subseteq_{f(r)} \mathcal{A}_{r\text{-Neighborhood}(X)}$$

↑

Useful notation: $\mathcal{B} \subseteq_\varepsilon \mathcal{C}$ means

$$\forall b \in \mathcal{B}\colon \exists c \in \mathcal{C}\colon \|b - c\| \leqslant \varepsilon \|b\|.$$

Examples: QCAs, local Hamiltonian dynamics (Lieb-Robinson!), . . . ?

Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics. . .
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

Classification of ALPUs?

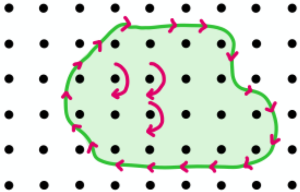
Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics. . .
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics...
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]



Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics...
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

Why not obvious?

- Previous work only treats exact QCAs.
- Previous techniques sensitive to perturbations.
- Local Hamiltonian dynamics are not quantum circuit.

- Previous definitions of index do not apply to ALPUs.

Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics...
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

Why not obvious?

- Previous work only treats exact QCAs.
- Previous techniques sensitive to perturbations.
- Local Hamiltonian dynamics are not quantum circuit.
  But: Can always approximate by circuits. How about ALPUs?
- Previous definitions of index do not apply to ALPUs.

Classification of ALPUs?

Why do we care?

- A theory of local dynamics should allow local Hamiltonian dynamics...
- Converse to Lieb-Robinson bounds?
- Is there a local Hamiltonian that generates lattice translation (shift)?
- Stability of chiral many-body localized 2D Floquet systems? [Po et al]

Why not obvious?

- Previous work only treats exact QCAs.
- Previous techniques sensitive to perturbations.
- Local Hamiltonian dynamics are not quantum circuit.
  But: Can always approximate by circuits. How about ALPUs?
- Previous definitions of index do not apply to ALPUs.
  But: Mutual information defn. applies! Does index remain quantized?

Our results: Classification of ALPUs

ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
$\cong$ QCAs *modulo* circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
- Any ALPU can be approximated by a sequence of QCAs.

- **Converse to Lieb-Robinson bound:** ALPU generated by quasi-local Hamiltonian iff index $= 0$. Always the case for *finite* open chain!
- Shift cannot be approximated by quasi-local Hamiltonian dynamics.

Our results: Classification of ALPUs

ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
$\cong$ QCAs *modulo* circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
  index α = index β iff α = *quasi-local Hamiltonian dynamics* $\circ\, \beta$
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
- Any ALPU can be approximated by a sequence of QCAs.

- **Converse to Lieb-Robinson bound:** ALPU generated by quasi-local Hamiltonian iff index $= 0$. Always the case for *finite* open chain!
- Shift cannot be approximated by quasi-local Hamiltonian dynamics.

Our results: Classification of ALPUs

ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
$\cong$ QCAs *modulo* circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
  index α = index β iff α = *quasi-local Hamiltonian dynamics* $\circ\, \beta$
  iff α, β can be 'blended'
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
- Any ALPU can be approximated by a sequence of QCAs.

- **Converse to Lieb-Robinson bound:** ALPU generated by quasi-local Hamiltonian iff index = 0. Always the case for *finite* open chain!
- Shift cannot be approximated by quasi-local Hamiltonian dynamics.

Our results: Classification of ALPUs

ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
$\cong$ QCAs *modulo* circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
  index α = index β iff α = *quasi-local Hamiltonian dynamics* $\circ\, \beta$
  iff α, β can be 'blended'
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
- Any ALPU can be approximated by a sequence of QCAs.

- **Converse to Lieb-Robinson bound:** ALPU generated by quasi-local Hamiltonian iff index $= 0$. Always the case for *finite* open chain!
- Shift cannot be approximated by quasi-local Hamiltonian dynamics.

Our results: Classification of ALPUs

ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
$\cong$ QCAs *modulo* circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
  index α = index β iff α = *quasi-local Hamiltonian dynamics* $\circ\, \beta$
  iff α, β can be 'blended'
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
- Any ALPU can be approximated by a sequence of QCAs.

- **Converse to Lieb-Robinson bound:** ALPU generated by quasi-local Hamiltonian iff index $= 0$. Always the case for *finite* open chain!
- Shift cannot be approximated by quasi-local Hamiltonian dynamics.

Our results: Classification of ALPUs

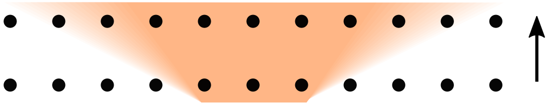
ALPUs *modulo* (time-dependent) quasi-local Hamiltonian dynamics
$\cong$ QCAs *modulo* circuits

Theorem:

- ALPUs are classified by index that is quantized, additive, robust:
  index α = index β iff $\alpha = \textit{quasi-local Hamiltonian dynamics} \circ \beta$
  iff α, β can be 'blended'
- Any ALPU is composition of quasi-local Hamilt. dynamics and shift.
- Any ALPU can be approximated by a sequence of QCAs.

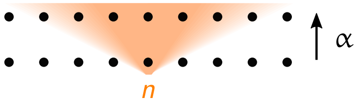
- **Converse to Lieb-Robinson bound:** ALPU generated by quasi-local Hamiltonian iff index = 0. Always the case for *finite* open chain!
- Shift cannot be approximated by quasi-local Hamiltonian dynamics.

Proof Ideas



A first attempt

Suppose we have an ALPU:



For any *fixed* n, can truncate tails to obtain *approximate* morphism

$$\mathcal{A}_n \to \mathcal{A}_{\{n-r,\ldots,n+r\}}.$$

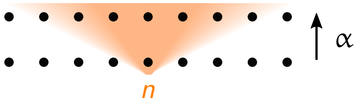
By a version of Ulam stability, can even find exact such morphism nearby.

However, for different sites n, the images of these morphisms need *not* commute → unclear how to **patch together**!

Need a more clever strategy. . .

A first attempt

Suppose we have an ALPU:



For any *fixed* n, can truncate tails to obtain *approximate* morphism

$$\mathcal{A}_n \to \mathcal{A}_{\{n-r,\ldots,n+r\}}.$$

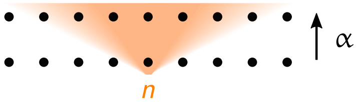
By a version of Ulam stability, can even find exact such morphism nearby.

However, for different sites n, the images of these morphisms need *not* commute → unclear how to **patch together**!

Need a more clever strategy…

A first attempt

Suppose we have an ALPU:



For any *fixed* n, can truncate tails to obtain *approximate* morphism

$$\mathcal{A}_n \to \mathcal{A}_{\{n-r,\ldots,n+r\}}.$$

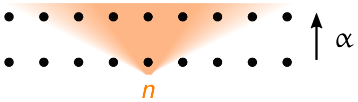
By a version of Ulam stability, can even find exact such morphism nearby.

However, for different sites n, the images of these morphisms need *not* commute → unclear how to **patch together**!

Need a more clever strategy...

A first attempt

Suppose we have an ALPU:



For any *fixed* n, can truncate tails to obtain *approximate* morphism

$$\mathcal{A}_n \to \mathcal{A}_{\{n-r,\ldots,n+r\}}.$$

By a version of Ulam stability, can even find exact such morphism nearby.

However, for different sites n, the images of these morphisms need *not* commute $\to$ unclear how to **patch together**!

Need a more clever strategy. . .

Main tool: Stability of inclusion

Theorem (Christensen, 80s): If $\mathcal{B} \subseteq_\varepsilon \mathcal{C}$ for hyperfinite von Neumann algebras and $\varepsilon < \frac{1}{8}$, then there is a unitary $u \in (\mathcal{B} \cup \mathcal{C})''$ such that

$$u\mathcal{B}u^* \subseteq \mathcal{C} \quad \text{and} \quad \|u - I\| \leqslant 12\varepsilon.$$

We extend this to show that, moreover:

- If $x \in_\delta \mathcal{B}$ and $x \in_\delta \mathcal{C}$, then $\|x - uxu^*\| = O(\delta\|x\|)$.
- If $x \in_\delta \mathcal{B}'$ and $x \in_\delta \mathcal{C}'$, then $\|x - uxu^*\| = O(\delta\|x\|)$.

Applied to ALPU, can localize image of any region, while preserving tails.

Main tool: Stability of inclusion

Theorem (Christensen, 80s): If $\mathcal{B} \subseteq_\varepsilon \mathcal{C}$ for hyperfinite von Neumann algebras and $\varepsilon < \frac{1}{8}$, then there is a unitary $u \in (\mathcal{B} \cup \mathcal{C})''$ such that

$$u\mathcal{B}u^* \subseteq \mathcal{C} \quad \text{and} \quad \|u - I\| \leqslant 12\varepsilon.$$

We extend this to show that, moreover:

- If $x \in_\delta \mathcal{B}$ and $x \in_\delta \mathcal{C}$, then $\|x - uxu^*\| = O(\delta\|x\|)$.
- If $x \in_\delta \mathcal{B}'$ and $x \in_\delta \mathcal{C}'$, then $\|x - uxu^*\| = O(\delta\|x\|)$.

Applied to ALPU, can localize image of any region, while preserving tails.

Main tool: Stability of inclusion

Theorem (Christensen, 80s): If $\mathcal{B} \subseteq_\varepsilon \mathcal{C}$ for hyperfinite von Neumann algebras and $\varepsilon < \frac{1}{8}$, then there is a unitary $u \in (\mathcal{B} \cup \mathcal{C})''$ such that

$$u\mathcal{B}u^* \subseteq \mathcal{C} \quad \text{and} \quad \|u - I\| \leqslant 12\varepsilon.$$

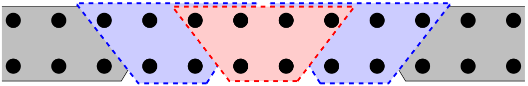
We extend this to show that, moreover:

- If $x \in_\delta \mathcal{B}$ and $x \in_\delta \mathcal{C}$, then $\|x - uxu^*\| = O(\delta\|x\|)$.
- If $x \in_\delta \mathcal{B}'$ and $x \in_\delta \mathcal{C}'$, then $\|x - uxu^*\| = O(\delta\|x\|)$.

Applied to ALPU, can localize image of any region, while preserving tails.

How to use this?

Key idea: For any *fixed* cut, can apply unitaries near identity to construct automorphism that looks like QCA *near this cut*:



Left and right are decoupled – stronger than what we had before!

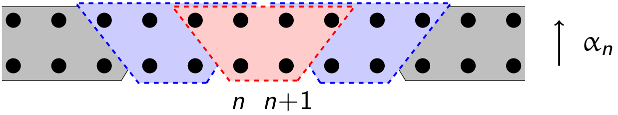
This allows us to glue different α_n, α_{n+2}, ... together.

Approximation Theorem: For any 1D ALPU α, there are QCAs β_r of radius $2r$ such that $\beta_r \to \alpha$ strongly. In fact, if $f(r)$ are the tails of α,

$$\|(\alpha - \beta_r)_{\mathcal{A}_X}\| \leqslant C_f\, f(r)\, \frac{\operatorname{diam}(X)}{r}.$$

How to use this?

Key idea: For any *fixed* cut, can apply unitaries near identity to construct automorphism that looks like QCA *near this cut*:



Left and right are decoupled – stronger than what we had before!

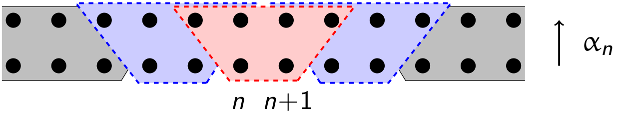
This allows us to glue different α_n, α_{n+2}, … together.

Approximation Theorem: For any 1D ALPU α, there are QCAs β_r of radius $2r$ such that $\beta_r \to \alpha$ strongly. In fact, if $f(r)$ are the tails of α,

$$\|(\alpha - \beta_r)_{\mathcal{A}_X}\| \leqslant C_f\, f(r)\, \frac{\text{diam}(X)}{r}.$$

How to use this?

Key idea: For any *fixed* cut, can apply unitaries near identity to construct automorphism that looks like QCA *near this cut*:



Left and right are decoupled – stronger than what we had before!

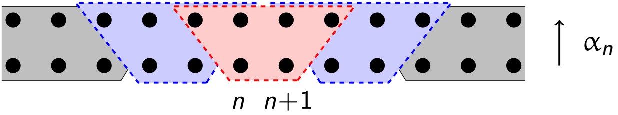
This allows us to glue different α_n, α_{n+2}, … together.

Approximation Theorem: For any 1D ALPU α, there are QCAs β_r of radius $2r$ such that $\beta_r \to \alpha$ strongly. In fact, if $f(r)$ are the tails of α,

$$\|(\alpha - \beta_r)_{\mathcal{A}_X}\| \leqslant C_f\, f(r)\, \frac{\operatorname{diam}(X)}{r}.$$

How to use this?

Key idea: For any *fixed* cut, can apply unitaries near identity to construct automorphism that looks like QCA *near this cut*:



Left and right are decoupled – stronger than what we had before!

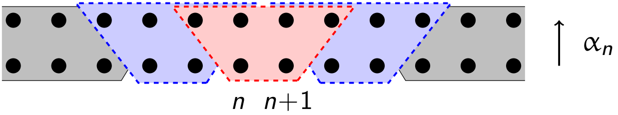
This allows us to glue different α_n, α_{n+2}, ... together.

Approximation Theorem: For any 1D ALPU α, there are QCAs β_r of radius $2r$ such that $\beta_r \to \alpha$ strongly. In fact, if $f(r)$ are the tails of α,

$$\|(\alpha - \beta_r)_{\mathcal{A}_X}\| \leqslant C_f \, f(r) \, \frac{\mathrm{diam}(X)}{r}.$$

How to use this?

Key idea: For any *fixed* cut, can apply unitaries near identity to construct automorphism that looks like QCA *near this cut*:



Left and right are decoupled – stronger than what we had before!

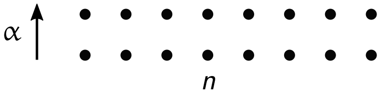
This allows us to glue different α_n, α_{n+2}, ... together.

Approximation Theorem: For any 1D ALPU α, there are QCAs β_r of radius $2r$ such that $\beta_r \to \alpha$ strongly. In fact, if $f(r)$ are the tails of α,

$$\|(\alpha - \beta_r)_{\mathcal{A}_X}\| \leqslant C_f \, f(r) \, \frac{\mathrm{diam}(X)}{r}.$$

1. How to create QCA near cut?

1. How to create QCA near cut?



Consider an ALPU that is ε-nearest neighbor. In particular:

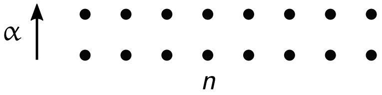
$$\alpha(\mathcal{A}_{\geqslant n}) \subseteq_\varepsilon \mathcal{A}_{\geqslant n-1}$$

By Christensen's theorem, we can find unitary $u \approx I$ s.th.

$$u\alpha(\mathcal{A}_{\geqslant n})u^* \subseteq \mathcal{A}_{\geqslant n-1}.$$

We can visualize this as above...

1. How to create QCA near cut?



Consider an ALPU that is ε-nearest neighbor. In particular:

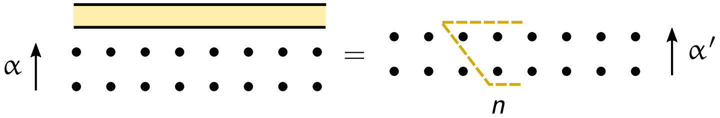
$$\alpha(\mathcal{A}_{\geqslant n}) \subseteq_\varepsilon \mathcal{A}_{\geqslant n-1}$$

By Christensen's theorem, we can find unitary $u \approx I$ s.th.

$$u\alpha(\mathcal{A}_{\geqslant n})u^* \subseteq \mathcal{A}_{\geqslant n-1}.$$

We can visualize this as above...

1. How to create QCA near cut?



Consider an ALPU that is ε-nearest neighbor. In particular:

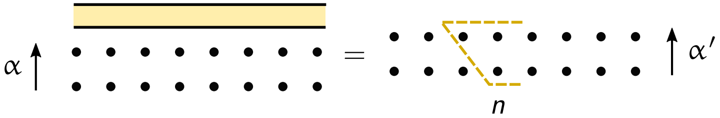
$$\alpha(\mathcal{A}_{\geqslant n}) \subseteq_\varepsilon \mathcal{A}_{\geqslant n-1}$$

By Christensen's theorem, we can find unitary $u \approx I$ s.th.

$$u\alpha(\mathcal{A}_{\geqslant n})u^* \subseteq \mathcal{A}_{\geqslant n-1}.$$

We can visualize this as above...

1. How to create QCA near cut?



The new ALPU α' is still ε'-nearest neighbor. In particular:

$$\alpha'(\mathcal{A}_{\geqslant n}) \supseteq_{\varepsilon'} \mathcal{A}_{\geqslant n+1}$$

and both algebras are in $\mathcal{A}_{\geqslant n-1}$. Thus, the latter contains unitary v s.th.

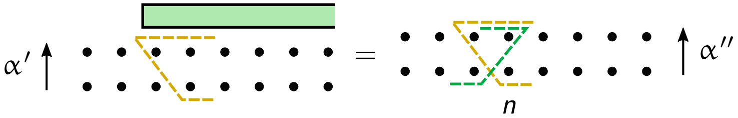
$$v\alpha'(\mathcal{A}_{\geqslant n})v^* \supseteq \mathcal{A}_{\geqslant n+1}$$

and hence

$$v\alpha'(\mathcal{A}_{\leqslant n-1})v^* \subseteq \mathcal{A}_{\leqslant n}$$

Key fact: Second unitary does not destroy locality achieved in first step!

1. How to create QCA near cut?



The new ALPU α' is still ε'-nearest neighbor. In particular:

$$\alpha'(\mathcal{A}_{\geqslant n}) \supseteq_{\varepsilon'} \mathcal{A}_{\geqslant n+1}$$

and both algebras are in $\mathcal{A}_{\geqslant n-1}$. Thus, the latter contains unitary v s.th.

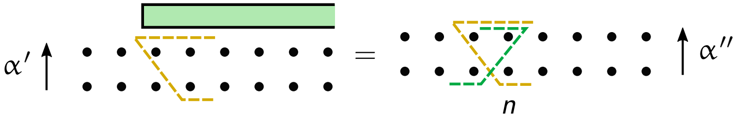
$$v\alpha'(\mathcal{A}_{\geqslant n})v^* \supseteq \mathcal{A}_{\geqslant n+1}$$

and hence

$$v\alpha'(\mathcal{A}_{\leqslant n-1})v^* \subseteq \mathcal{A}_{\leqslant n}$$

Key fact: Second unitary does not destroy locality achieved in first step!

1. How to create QCA near cut?



The new ALPU α' is still ε'-nearest neighbor. In particular:

$$\alpha'(\mathcal{A}_{\geqslant n}) \supseteq_{\varepsilon'} \mathcal{A}_{\geqslant n+1}$$

and both algebras are in $\mathcal{A}_{\geqslant n-1}$. Thus, the latter contains unitary v s.th.

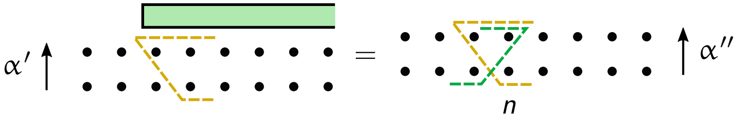
$$v\alpha'(\mathcal{A}_{\geqslant n})v^* \supseteq \mathcal{A}_{\geqslant n+1}$$

and hence

$$v\alpha'(\mathcal{A}_{\leqslant n-1})v^* \subseteq \mathcal{A}_{\leqslant n}$$

Key fact: Second unitary does not destroy locality achieved in first step!

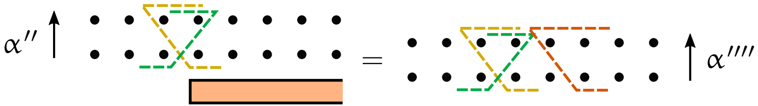
1. How to create QCA near cut?



We continue in this way, successively rotating images and preimages...

...until we obtain automorphism that looks like a QCA near fixed cut:

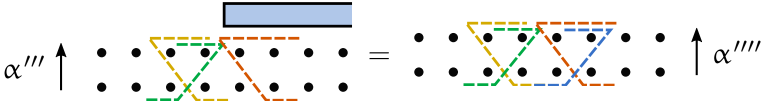
1. How to create QCA near cut?



We continue in this way, successively rotating images and preimages. . .

. . . until we obtain automorphism that looks like a QCA near fixed cut:

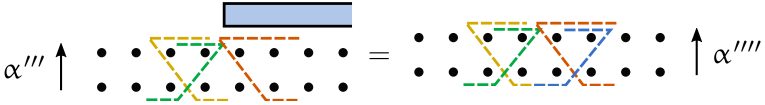
1. How to create QCA near cut?



We continue in this way, successively rotating images and preimages. . .

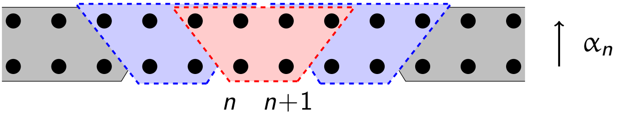
. . . until we obtain automorphism that looks like a QCA near fixed cut:

1. How to create QCA near cut?



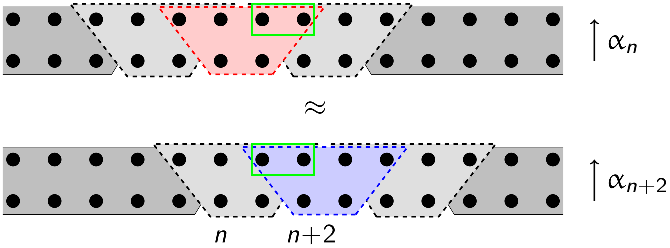
We continue in this way, successively rotating images and preimages...

...until we obtain automorphism that looks like a QCA near fixed cut:



2. Why can we glue?

Compare two such local QCAs:



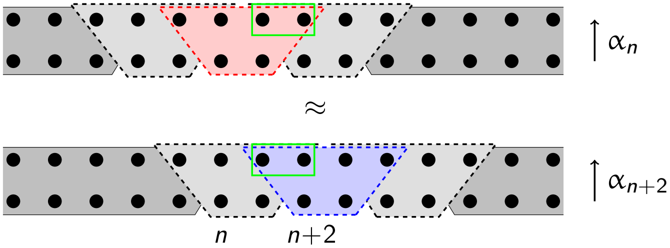
We can glue the red and the blue morphism by applying a unitary

$$u \in \mathcal{A}_{n+1,n+2}.$$

Inductively we obtain a QCA.

2. Why can we glue?

Compare two such local QCAs:



We can glue the red and the blue morphism by applying a unitary

$$u \in \mathcal{A}_{n+1,n+2}.$$

Inductively we obtain a QCA.

Index of an ALPU

Thus we proved that any ALPU α in 1D can be approximated by sequence of QCAs β_r (sufficiently fast). This allows us to define the **index**:

$$\operatorname{index} \alpha := \lim_{r \to \infty} \operatorname{index} \beta_r$$

- well-defined, independent of choice of $\{\beta_r\}$
- inherits properties of GNVW index: quantized, additive, continuous, ...

If $O(\frac{1}{r^{1+\delta}})$-tails, can also compute as mutual information difference:

$$\operatorname{index} \alpha = \frac{1}{2}\big(I(L:R') - I(L':R)\big)$$

Index of an ALPU

Thus we proved that any ALPU α in 1D can be approximated by sequence of QCAs β_r (sufficiently fast). This allows us to define the **index**:

$$\operatorname{index} \alpha := \lim_{r \to \infty} \operatorname{index} \beta_r$$

- well-defined, independent of choice of $\{\beta_r\}$
- inherits properties of GNVW index: quantized, additive, continuous, ...

If $O(\frac{1}{r^{1+\delta}})$-tails, can also compute as mutual information difference:

$$\operatorname{index} \alpha = \frac{1}{2}\left(I(L:R') - I(L':R)\right)$$

Index of an ALPU

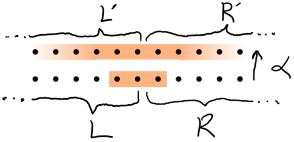
Thus we proved that any ALPU α in 1D can be approximated by sequence of QCAs β_r (sufficiently fast). This allows us to define the **index**:

$$\operatorname{index}\alpha := \lim_{r\to\infty} \operatorname{index}\beta_r$$

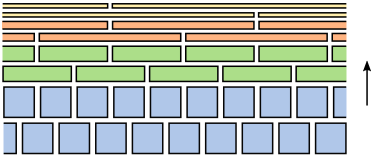
- well-defined, independent of choice of $\{\beta_r\}$
- inherits properties of GNVW index: quantized, additive, continuous, ...

If $O(\frac{1}{r^{1+\delta}})$-tails, can also compute as mutual information difference:

$$\operatorname{index}\alpha = \frac{1}{2}\big(I(L:R') - I(L':R)\big).$$



How to obtain time-dependent quasi-local Hamiltonians?



- Start with ALPU of index $\alpha = 0$.
- Approximate α by QCA β_1 of same index. Thus β_1 is circuit and can be implemented by time-dependent local Hamiltonian evolution.
- Repeat with $\beta_1^{-1}\alpha$.

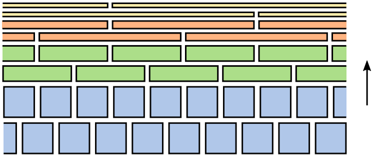
For an appropriate “schedule”, obtain time-dependent Hamiltonian

$$H(t) = \sum_X H_X(t)$$

that is piecewise constant and has geometrically local interactions

$$\|H_X(t)\| = O(f(k)\log k) \quad \text{with} \quad |X| = k \leqslant k(t).$$

How to obtain time-dependent quasi-local Hamiltonians?



- Start with ALPU of index $\alpha = 0$.
- Approximate α by QCA β_1 of same index. Thus β_1 is circuit and can be implemented by time-dependent local Hamiltonian evolution.
- Repeat with $\beta_1^{-1}\alpha$.

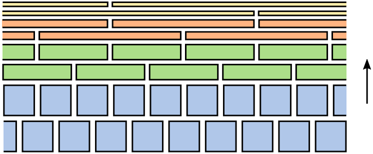
For an appropriate "schedule", obtain time-dependent Hamiltonian

$$H(t) = \sum_X H_X(t)$$

that is piecewise constant and has geometrically local interactions

$$\|H_X(t)\| = O(f(k) \log k) \quad \text{with} \quad |X| = k \leqslant k(t).$$

How to obtain time-dependent quasi-local Hamiltonians?



- Start with ALPU of index $\alpha = 0$.
- Approximate α by QCA β_1 of same index. Thus β_1 is circuit and can be implemented by time-dependent local Hamiltonian evolution.
- Repeat with $\beta_1^{-1}\alpha$.

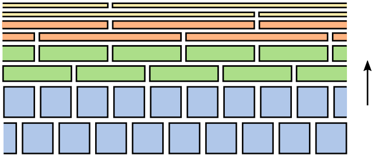
For an appropriate "schedule", obtain time-dependent Hamiltonian

$$H(t) = \sum_X H_X(t)$$

that is piecewise constant and has geometrically local interactions

$$\|H_X(t)\| = O(f(k) \log k) \quad \text{with} \quad |X| = k \leqslant k(t).$$

How to obtain time-dependent quasi-local Hamiltonians?



- Start with ALPU of index $\alpha = 0$.
- Approximate α by QCA β_1 of same index. Thus β_1 is circuit and can be implemented by time-dependent local Hamiltonian evolution.
- Repeat with $\beta_1^{-1}\alpha$.

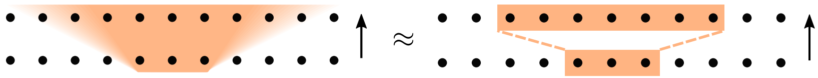
For an appropriate "schedule", obtain time-dependent Hamiltonian

$$H(t) = \sum_X H_X(t)$$

that is piecewise constant and has geometrically local interactions

$$\|H_X(t)\| = O(f(k) \log k) \quad \text{with} \quad |X| = k \leqslant k(t).$$

Summary and outlook



Approximately locality preserving unitaries (ALPUs) in 1D have structure & index theory generalizing the one of QCAs. In particular, implies a converse to Lieb-Robinson bounds. Main techniques are stability results for near inclusions of algebras. *Many open problems:*

- Periodic chain in 1D?
- Extension to high dimensions? 2D within reach...
- Beyond automorphisms: Is there a QCA near any "noisy" QCA?
- *Other applications of stability results in QI?*

Thank you for your attention!

Discussion slides