Elliptic Darboux-Integrable Systems and their Extensions: Problems and Prospects

joint work with Mark Fels (Utah State)

Outline:

- Darboux Integrability
- Generalization to Elliptic Systems
- Adapted Coframes and the Vessiot Algebra
- Construction of Integrable Extensions
- Connections with Isometric Embedding

Conventions Let $\mathscr{I}$ denote an exterior differential system (EDS) on manifold M. (Recall, $\mathscr{I}$ is a graded ideal in $\Omega^*(M)$ closed under exterior derivative.)

For $k > 0$ let $\mathscr{I}^k$ denote kth graded piece (assume no 0-forms), which spans a sub-bundle $I^k \subset \Lambda^k T^* M$. For Pfaffian systems, write $I^1 = I$.

Submanifold $f : S \subset M$ is an *integral manifold* of $\mathscr{I}$ if $f^*\psi = 0$ for all $\psi \in \mathscr{I}$.

Defn[1] An EDS $\mathscr{I}$ is *decomposable* if it is generated algebraically by finitely many 1-forms and 2-forms on M, and there are sub-bundles $\widehat{V}, \check{V} \subset T^*M$ such that

- the 1-form generators of $\mathscr{I}$ are precisely the sections of $\widehat{V} \cap \check{V}$;
- the 2-form generators of $\mathscr{I}$ are either sections of $\Lambda^2 \widehat{V}$ or of $\Lambda^2 \check{V}$.

These $\widehat{V}, \check{V}$ are called the *singular systems* of $\mathscr{I}$.

[1] Anderson, Fels and Vassiliou (Advance in Mathematics 221 (2009), 1910–1963)

Example Solutions to a second-order PDE for $u(x,y)$ correspond to integral surfaces of an EDS $\mathscr{I}$ on $M^7 \subset J^2(\mathbb{R}^2,\mathbb{R})$. Use coordinates $x, y, u, u_i, u_{ij} = u_{ji}$ on jet space, assume the PDE is 'wavelike':

$$u_{12} = F(x, y, u, u_1, u_2).$$

and let M^7 is the submanifold defined by this equation. The integral surfaces of $\mathscr{I}$ are the *2-graphs* of solutions, defined by $u = f(x,y)$, $u_1 = f_x$, $u_2 = f_y$, etc.

Then $\mathscr{I}$ is generated by 1-forms

$$\theta_0 := du - u_1\, dx - u_2\, dy, \quad \theta_1 := du_1 - u_{11}\, dx - u_{12}\, dy, \quad \theta_2 := du_2 - u_{12}\, dx - u_{22}\, dy$$

and decomposable 2-forms

$$(du_{11} - D_x F\, dy) \wedge\, dx, \qquad (du_{22} - D_y F\, dx) \wedge\, dy.$$

where $D_x F = F_x + u_1 F_u + u_{11} F_{u_1} + u_{12} F_{u_2}$ and $D_y F = F_y + u_2 F_u + u_{12} F_{u_1} + u_{22} F_{u_2}$.

This is a decomposable EDS, with singular systems

$$\widehat{V} = \{\theta_0, \theta_1, \theta_2,\ dx, du_{11} - D_x F\, dy\},$$
$$\check{V} = \{\theta_0, \theta_1, \theta_2,\ dy, du_{22} - D_y F\, dx\}$$

A decomposable EDS $\mathscr{I}$ is 'integrable by the method of Darboux' if each singular system V has 'enough' independent first integrals, i.e., functions $f \in C^\infty(M)$ such that $df \in V$. Such differentials span the *terminal derived system* $V^{(\infty)} \subset V$. These first integrals are called the *Darboux invariants*.

Defn $\mathscr{I}$ is Darboux-integrable (DI) if

$$T^*M = \widehat{V}^{(\infty)} + \check{V} = \check{V}^{(\infty)} + \widehat{V} \text{ (non-direct sum)}, \quad \text{and } \widehat{V}^{(\infty)} \cap \check{V}^{(\infty)} = 0.$$

We'll say the integrability is *lean* if these sums are direct, i.e., $\widehat{V}^{(\infty)} \cap \check{V} = 0 = \check{V}^{(\infty)} \cap \widehat{V}$.

Example Solutions of Liouville's equation $u_{xy} = e^u$, the singular systems are

$$\widehat{V} = \{\theta_0, \theta_1, \theta_2, dx, d(u_{11} - \tfrac{1}{2}(u_1)^2)\},$$
$$\check{V} = \{\theta_0, \theta_1, \theta_2, dy, d(u_{22} - \tfrac{1}{2}(u_2)^2)\},$$

For any solution there will be functions ϕ, ψ such that

$$u_{11} - \tfrac{1}{2}u_1^2 = \phi(x), \qquad u_{22} - \tfrac{1}{2}u_2^2 = \psi(y)$$

Moreover,'integrability' stems from the fact that imposing such functional dependencies among the invariants defines a submanifold $L \subset M$ to which $\mathscr{I}$ restricts to be a Frobenius system (in fact, a pair of equations of Lie type).

Singular systems for a decomposable EDS $\mathscr{I}$ are made up of factors of its real decomposable 2-form generators. We will refer to this as the *hyperbolic case*.

For elliptic PDE in the plane (e.g., Laplace's equation $\Delta u = 0$) decomposable 2-forms only arise via *complex* linear combinations of real generators.

Example Solutions of the elliptic version of Liouville's equation $u_{xx} + u_{yy} = 2e^u$ correspond to integral surfaces of EDS $\mathscr{I}$ on $M^7 \subset J^2(\mathbb{R}^2, \mathbb{R})$ generated by 1-forms

$$\theta_0 := du - p\,dx - q\,dy, \quad \theta_1 := dp - (e^u + r)\,dx - s\,dy, \quad \theta_2 := dq - s\,dx - (e^u - r)\,dy$$

(where $p = u_x$, $q = u_y$, $r = u_{xx} - u_{yy}$ and $s = u_{xy}$ on solutions) and their reduced exterior derivatives

$$d\theta_1 \equiv dx \wedge (dr - e^u q\,dy) + dy \wedge ds, \qquad d\theta_2 = dx \wedge ds - dy \wedge (dr - e^u p\,dx).$$

Taking complex linear combinations gives a pair of decomposable 2-forms,

$$(dx + \mathrm{i}dy) \wedge (dr - \mathrm{i}ds - \tfrac{1}{2}e^u(p - \mathrm{i}q)(dx - \mathrm{i}dy))$$

and its conjugate. We therefore define 'complexified' singular systems

$$\widehat{V} = \{\theta_0, \theta_1, \theta_2, dx + \mathrm{i}dy, dr - \mathrm{i}ds - \tfrac{1}{2}e^u(p - \mathrm{i}q)(dx - \mathrm{i}dy)\} \subset T^*M \otimes \mathbb{C}$$

and $\check{V} = \overline{\widehat{V}}$.

Defn Let $\mathscr{I}$ be an EDS on M and let $I^1 \subset T^*M$ be the span of the 1-forms of $\mathscr{I}$. Then $\mathscr{I}$ is *elliptic decomposable* if there is a splitting

$$(T^*M/I^1) \otimes \mathbb{C} = W \oplus \overline{W}$$

such that $\mathscr{I}$ is generated algebraically by sections of I^1 and sections of $\Lambda^2\widehat{V}$ and $\Lambda^2\check{V}$, where $\widehat{V}, \check{V}$ are pre-images of $W, \overline{W}$ under the $\mathbb{C}$-linear extension of the quotient map $T^*M \to T^*M/I^1$. Hence $\widehat{V} \cap \check{V} = I^1 \otimes \mathbb{C}$.

Rk Necessarily, $\mathcal{D} = \mathrm{ann}(I^1)$ must have even rank. The splitting corresponds to a complex structure on $\mathcal{D} \subset TM$ which doesn't necessarily extend to TM.

Defn Such a system is elliptic DI if

$$T^*M \otimes \mathbb{C} = \widehat{V}^{(\infty)} + \check{V} \text{ (non-direct sum)}, \quad \text{and } \widehat{V}^{(\infty)} \cap \check{V}^{(\infty)} = 0.$$

Example For elliptic Liouville $u_{xx} + u_{yy} = 2e^u$, compute that

$$\widehat{V}^{(\infty)} = \left\{ d(x + \mathrm{i}y), d\left(r - \mathrm{i}s - \tfrac{1}{4}(p - \mathrm{i}q)^2\right) \right\}.$$

Hence $T^*M \otimes \mathbb{C} = \widehat{V}^{(\infty)} \oplus \check{V}$ and the system is *leanly* Darboux-integrable.

Examples of Darboux-Integrable Elliptic PDE

(based on Goursat-Vessiot classification)

Here, u is a real function of $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$.

$$\begin{aligned}
(z+\overline{z})u_{z\overline{z}} &= 2\sqrt{u_z u_{\overline{z}}} \\
u u_{z\overline{z}} &= \sqrt{1+u_z^2}\sqrt{1+u_{\overline{z}}^2} \\
(\sin u) u_{z\overline{z}} &= \sqrt{1+u_z^2}\sqrt{1+u_{\overline{z}}^2} \\
u u_{z\overline{z}} &= \pm\phi(u_z)\phi(u_{\overline{z}})
\end{aligned}$$

where $\phi(t)$ is a solution of the ODE $df/dt \pm t/f = c$ for a nonzero real constant c,

$$(z+\overline{z})u_{z\overline{z}} = \gamma(u_z)\gamma(u_{\overline{z}})$$

where γ is implicitly defined by $\gamma(t) - 1 = \exp(t - \gamma(t))$,

$$\begin{aligned}
u_{z\overline{z}} &= e^u, \\
u_{z\overline{z}} &= \left(\frac{1}{u+z} + \frac{1}{u+\overline{z}}\right) u_z u_{\overline{z}}.
\end{aligned}$$

Construction of solutions for hyperbolic DI systems depends on the action of the *Vessiot group*. Its existence for any DI system $\mathscr{I}$ is revealed through careful coframe adaptations. Its action lets us construct an integrable extension of $\mathscr{I}$ that splits into simpler systems.

Thm (Anderson-Fels-Vassiliou)
Let $\mathscr{I}$ be a lean DI hyperbolic decomposable system on manifold M. Let $n = \operatorname{rk} I^1$, $p = \operatorname{rk} \widehat{V} - n$, $q = \operatorname{rk} \check{V} - n$, and let $1 \le i, j, k \le n$, $1 \le a, b \le p$, $1 \le \alpha, \beta \le q$. Near any point there exists 1-forms $\theta^i_X, \theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha$ such that $(\boldsymbol{\theta}_X, \hat{\boldsymbol{\pi}}, \check{\boldsymbol{\pi}})$ and $(\boldsymbol{\theta}_Y, \hat{\boldsymbol{\pi}}, \check{\boldsymbol{\pi}})$ are each coframes such that

$$I^1 = \{\boldsymbol{\theta}_X\} = \{\boldsymbol{\theta}_Y\}, \qquad \widehat{V}^{(\infty)} = \{\hat{\boldsymbol{\pi}}\}, \qquad \check{V}^{(\infty)} = \{\check{\boldsymbol{\pi}}\},$$

and which satisfy structure equations

$$\begin{aligned}
&d\hat{\pi}^a = 0, \qquad d\check{\pi}^\alpha = 0,\\
&d\theta^i_X = \tfrac{1}{2}A^i_{ab}\hat{\pi}^a \wedge \hat{\pi}^b + \tfrac{1}{2}B^i_{\alpha\beta}\check{\pi}^\alpha \wedge \check{\pi}^\beta + \tfrac{1}{2}C^i_{jk}\theta^j_X \wedge \theta^k_X + M^i_{aj}\theta^j_X \wedge \hat{\pi}^a,\\
&d\theta^i_Y = \tfrac{1}{2}E^i_{ab}\hat{\pi}^a \wedge \hat{\pi}^b + \tfrac{1}{2}F^i_{\alpha\beta}\check{\pi}^\alpha \wedge \check{\pi}^\beta - \tfrac{1}{2}C^i_{jk}\theta^j_Y \wedge \theta^k_Y + N^i_{\alpha j}\theta^j_Y \wedge \check{\pi}^\alpha,
\end{aligned}$$

where C^i_{jk} are structure constants for a real Lie algebra $\mathfrak{g}$, the Vessiot algebra of $\mathscr{I}$.

Rk There exist local coordinates x^a, y^α, t^k such that $\hat{\pi}^a = dx^a$ and $\check{\pi}^\alpha = dy^\alpha$.

'Corollary' There exist 1-forms

$$\hat{\theta}^i = \hat{R}(x)^i_j \theta^j_X + \hat{S}(x)^i_a \hat{\pi}^a, \qquad \check{\theta}^i = \check{R}(y)^i_j \theta^j_Y + \check{S}(y)^i_\alpha \check{\pi}^\alpha$$

such that

$$d\hat{\theta}^i = \tfrac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k + (*) \check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}}, \qquad d\check{\theta}^i = -\tfrac{1}{2} C^i_{jk} \check{\theta}^j \wedge \check{\theta}^k + (*) \hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}}.$$

Moreover, if $\theta^i_X = Q^i_j \theta^j_Y$ and $\boldsymbol{\lambda} = \hat{\boldsymbol{R}} \boldsymbol{Q} \check{\boldsymbol{R}}^{-1}$ then the 1-forms

$$\hat{\omega}^i = \hat{\theta}^i + \lambda^i_j \check{S}(y)^j_\alpha \check{\pi}^\alpha, \qquad \check{\omega}^i = \check{\theta}^i + (\lambda^{-1})^i_j \hat{S}(x)^j_a \hat{\pi}^a$$

span a Frobenius system $\{\hat{\omega}^i\} = \{\check{\omega}^i\}$ and satisfy Maurer-Cartan equations

$$d\hat{\omega}^i = \tfrac{1}{2} C^i_{jk} \hat{\omega}^j \wedge \hat{\omega}^k, \qquad d\check{\omega}^i = -\tfrac{1}{2} C^i_{jk} \check{\omega}^j \wedge \check{\omega}^k.$$

Define vector fields $\hat{X}_i$ on M such that $\hat{X}_i \lrcorner\, \hat{\omega}^j = \delta^i_j$ and $\hat{X}_i \lrcorner\, \hat{\pi}^a = \hat{X}_i \lrcorner\, \check{\pi}^\alpha = 0$. These generate a locally free G-action on M. Let $S \subset M$ be a integral submanifold of $\{\hat{\omega}^i\} = \{\check{\omega}^i\}$ which is a *slice* for this action. Then the action lets us (locally) identify

$$\Phi : S \times G \to M,$$

so that $\Phi^* \hat{\omega}^i = \tau^i$ left-invariant and $\Phi^* \check{\omega}^i = \mu^i$ right-invariant Maurer-Cartan forms on G. (The x^a and y^α pull back to be coordinates on S.)

Theorem Let G_1, G_2 be copies of G. On $N = S \times G_1 \times G_2$ define a 'superposition map' $\Sigma : N \to S \times G \cong M$ by

$$\Sigma(s, c_1, c_2) = (s, c_1 c_2^{-1}).$$

Note that this is the quotient map for the right diagonal G-action defined by

$$(c_1, c_2) \cdot g = (c_1 g, c_2 g).$$

Let Λ^i_j be the function on G such that $\tau^i = \Lambda^i_j \mu^j$ (hence $\Phi^* \lambda^i_j = \Lambda^i_j$).
Let τ^i_1, μ^i_1 denote the left- and right-invariant Maurer-Cartan forms respectively on G_1, and similarly τ^i_2, μ^i_2 on G_2. Then the Pfaffian system $\mathscr{E}$ on N generated by

$$E = \{\tau^i_1 - \Lambda(c_1)^i_j \check{S}(y)^j_\alpha dy^\alpha,\ \mu^i_2 + \hat{S}(x)^i_a dx^a\}$$

is an integrable extension of $\mathscr{I}$.

In other words, $\mathscr{E}$ is generated algebraically by $\Sigma^* \mathscr{I}$ and sections of E. Moreover, any integral manifold of $\mathscr{I}$ is the image of an integral manifold of $\mathscr{E}$.

Rk System $\mathscr{E}$ is the product of Pfaffian systems on $G_1 \times$ (y-variables) and on $G_2 \times$ (x-variables), each of which is of Lie type. To see why it's an extension, compute the pullbacks of generators of $\mathscr{I}$:

$$\Sigma^*(\hat{R}^i_j \theta^j_X) = -(\mu^i_2 + \hat{S}(x)^i_a dx^a) + \Lambda^{-1}(c_2)^i_j \left(\tau^j_1 - \Lambda^j_k(c_1)\check{S}(y)^k_\alpha dy^\alpha\right).$$

Coframes for Lean DI Elliptic Systems

Locally on M there are complex-valued 1-forms $\theta^i, \hat{\pi}^a$ such that

$$I^1 \otimes \mathbb{C} = \{\theta^i\} = \{\overline{\theta^i}\}, \qquad \widehat{V}^{(\infty)} = \{\hat{\pi}^a\}$$

and $\theta^i, \hat{\pi}^a, \check{\pi}^a = \overline{\hat{\pi}^a}$ are a local coframe.

Thm These can be chosen to satisfy structure equations $d\hat{\pi}^a = 0$ and

$$d\theta^i = \tfrac{1}{2}A^i_{ab}\hat{\pi}^a \wedge \hat{\pi}^b + \tfrac{1}{2}B^i_{\alpha\beta}\check{\pi}^\alpha \wedge \check{\pi}^\beta + \tfrac{1}{2}C^i_{jk}\theta^j \wedge \theta^k + M^i_{aj}\theta^j \wedge \hat{\pi}^a,$$

with complex coefficients. Call this an *adapted coframe*.

Rk If we choose local coordinates such that $\hat{\pi}^a = dz^a$, then A^i_{ab}, C^i_{jk}, M^i_{aj} are holomorphic functions of the $z^1, \ldots z^p$.

Conjecture Based on examples we've calculated, these coframes can always be chosen so that the C^i_{jk} are real and constant. Consequently, they are structure constants of the *Vessiot algebra* $\mathfrak{g}$. Say an adapted coframe is of *Vessiot type* if this is the case.

Corollary Let $\theta^i, \hat{\pi}^a, \check{\pi}^a$ be Vessiot adapted coframe. Then there exist 1-forms

$$\hat{\theta}^i = \hat{R}(z)^i_j \theta^j + \hat{S}(z)^i_a \hat{\pi}^a$$

such that $d\hat{\theta}^i = \frac{1}{2}C^i_{jk}\hat{\theta}^j \wedge \hat{\theta}^k + (*)\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}}$. Moreover if $\theta^i = Q^i_j\overline{\theta^j}$ and $\boldsymbol{\lambda} = \hat{\boldsymbol{R}}\boldsymbol{Q}\overline{\hat{\boldsymbol{R}}}^{-1}$ then the 1-forms

$$\hat{\omega}^i = \hat{\theta}^i + \lambda^i_j \overline{\hat{S}(z)^j_a}\check{\pi}^a$$

satisfy $d\hat{\omega}^i = \frac{1}{2}C^i_{jk}\hat{\omega}^j \wedge \hat{\omega}^k$ and span a *real* Frobenius system.

These enable us to identify M with the product of a homogeneous space with a maximal integral submanifold of the $\{\hat{\boldsymbol{\omega}}\}$.

Canonical Extension for DI Elliptic Systems *(A Conjectural Picture)*

Given the 1-forms $\hat{\omega}^i$, $\hat{\pi}^a$ on M, define complex vector fields $\hat{Z}_j$ such that

$$\hat{Z}_j \lrcorner \hat{\omega}^i = \delta^i_j, \qquad \hat{Z}_j \lrcorner \hat{\pi}^a = \hat{Z}_j \lrcorner \check{\pi}^a = 0.$$

By the structure equations, $[\hat{Z}_j, \overline{\hat{Z}_k}] = 0$, so the real and imaginary parts

$$X_j = \hat{Z}_j + \overline{\hat{Z}_j}, \qquad Y_j = \mathrm{i}(\hat{Z}_j - \overline{\hat{Z}_j})$$

form a Lie algebra $\mathfrak{k}$ of real vector fields on M, such that $\mathfrak{k} \cong \mathfrak{g} \otimes \mathbb{C}$ where $\mathfrak{g}$ is the Vessiot algebra.

These generate a (non-free) action of K (the complexification of the Vessiot group) on M. Given $p \in M$ we can adapt the $\hat{\omega}^i$ so that the X_i generate the isotropy subgroup G at p.

Let S be the maximal integral manifold $\{\hat{\omega}^i\}$ through p. (The z^a restrict to give complex coords on S.) Since this is a slice for the K-action, we have an identification

$$\Phi : S \times K/G \to M.$$

Left and Right Actions

There is a natural projection $\pi : K \to K/G$ which we extend to the product with S. Then π gives the quotient modulo the G-action on K by right multiplication, but $\Phi \circ \pi$ is also equivariant for the K-actions on M and on $S \times K$ via left-multiplication.

$$\begin{array}{ccc} K \circlearrowleft S \times K & & \\ \pi \big\downarrow & & \\ S \times K/G & \xrightarrow{\Phi} & M \end{array}$$

On K the right-invariant vector fields X_j^R, Y_j^R satisfy

$$[X_j^R, X_k^R] = C_{jk}^\ell X_\ell^R, \qquad [X_j^R, Y_k^R] = C_{jk}^\ell Y_\ell^R, \qquad [Y_j^R, Y_k^R] = -C_{jk}^\ell X_\ell^R.$$

Since they generate left-multiplication on K, these push forward under π to give well-defined vector fields on K/G. Moreover, if we form the $(1,0)$-vector fields

$$Z_j^R = \tfrac{1}{2}(X_j^R - \mathrm{i} Y_j^R)$$

then $\pi_* Z_j^R$ spans the tangent spaces on K/G and $(\Phi \circ \pi)_* Z_j^R = \hat{Z}_j$.

On K define right-invariant $(1,0)$-forms $\mu^j_{(1,0)}$ and left-invariant (1,0)-forms $\tau^j_{(1,0)}$ which are dual to respectively to the Z^R_j and the

$$Z^L_j = \tfrac{1}{2}(X^L_j - \mathrm{i}Y^L_j).$$

Define functions Λ^i_j on K such that $\tau^i_{(1,0)} = \Lambda^i_j \mu^j_{(1,0)}$.

Then we *conjecture* that an integrable extension of $\mathscr{I}$ is given by the Pfaffian system $\mathscr{E}$ on $N = S \times K$ generated by

$$E = \{\tau^i_{(1,0)} - \Lambda^i_j \hat{S}^j_a(z)dz^a\} + \text{(complex conjugates)}.$$

Integral submanifolds of this extension are solutions of a Lie equation on K given by *holomorphic* functions of the z^a.

Thus, we can use this to obtain solutions of the original system $\mathscr{I}$ in terms of holomorphic data.

Potential Applications to Isometric Embedding

Let B^2 be a Riemannian surface with orthonormal frame field $\mathsf{v}_1, \mathsf{v}_2$.
Define dual 1-forms η^1, η^2 on M and connection form η^1_2:

$$\nabla_{\mathsf{w}} \mathsf{v}_2 = \eta^1_2(\mathsf{w})\mathsf{v}_1, \qquad \forall \mathsf{w} \in TB.$$

Let $\mathcal{F}$ be the Euclidean frame bundle of $\mathbb{R}^3$:

$$\mathcal{F}^6 = \{(\mathsf{p}, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3) \mid \mathsf{e}_i \text{ oriented o.n. basis for } T_{\mathsf{p}}\mathbb{R}^3\},$$

with canonical forms ω^a and connection forms $\omega^a_b = -\omega^b_a$ defined by

$$d\mathsf{p} = \mathsf{e}_a \otimes \omega^a, \qquad d\mathsf{e}_a = \mathsf{e}_b \otimes \omega^b_a.$$

Prop An isometric $f : B \to \mathbb{R}^3$ induces lift $\widehat{f} : B \to \mathcal{F}$ by $\mathsf{e}_i = f_* \mathsf{v}_i$, $\mathsf{e}_3 = \mathsf{e}_1 \times \mathsf{e}_2$.
The graph S of $\widehat{f}$ is an integral surface of 1-forms

$$\theta_1 := \omega^1 - \eta^1, \quad \theta_2 := \omega^2 - \eta^2, \quad \theta_3 := \omega^3, \quad \theta_4 := \omega^1_2 - \eta^1_2.$$

Conversely, any integral surface $S \subset B \times \mathcal{F}$ that submerses onto B defines an isometric immersion.

The *isometric immersion system* $\mathscr{I}$ is the Pfaffian EDS on $M = B \times \mathcal{F}$ generated by $\theta_1, \theta_2, \theta_3, \theta_4$.

This system satisfies structure equations

$$\left.\begin{aligned} &d\theta_1 \equiv 0, \quad d\theta_2 \equiv 0, \\ &d\theta_3 \equiv \omega^3_1 \wedge \eta^1 + \omega^3_2 \wedge \eta^2, \\ &d\theta_4 \equiv \omega^3_1 \wedge \omega^3_2 - K\eta^1 \wedge \eta^2 \end{aligned}\right\} \quad \mathrm{mod}\ \theta_1 \ldots, \theta_4,$$

where K is the Gauss curvature of (B, g).

Consider the 'elliptic case' where $K > 0$. If we let $k = \sqrt{K}$, then there are decomposable 2-form generators

$$(\omega^3_1 \mp \mathrm{i}k\eta^2) \wedge (\omega^3_2 \pm \mathrm{i}k\eta^1).$$

Thus, the isometric system $\mathscr{I}$ is elliptic decomposable with singular systems

$$\widehat{V} = \{\theta_1, \ldots, \theta_4, \omega^3_1 - \mathrm{i}k\eta^2, \omega^3_2 + \mathrm{i}k\eta^1\}, \qquad \check{V} = \overline{\widehat{V}}.$$

<u>Thm</u> (Clelland, Vassiliou, I-) There are exactly three surface metrics (up to scale) of positive curvature for which $\mathscr{I}$ is Darboux-integrable. There are local coordinates u, v on B in which the metrics take the form

$$\begin{aligned} g_1 &= \cosh^4 u(du)^2 + \sinh^2 u(dv)^2, \\ g_{-1} &= \sinh^4 u(du)^2 + \cosh^2 u(dv)^2, \\ g_0 &= u^2((du)^2 + (dv)^2), \end{aligned}$$

Each metric has an intrinsic Killing field $\partial/\partial v$.

<u>Questions</u> What are the Vessiot groups for the isometric embedding systems associated to these metrics? What are the canonical extensions? How do we express the isometric embeddings in terms of holomorphic data?

Each of the above metrics g_C can be expressed as $(ds)^2 + q'(s)^2(dv)^2$ where $q = K^{-3/4}$. In each case, q satisfies an ODE $(\frac{1}{3}q')^2 - q^{2/3} = C$. The solution plays an important role in a kind of *Weierstrass representation* for the isometric embeddings:

<u>Defn</u> Define a 'generalized Gauss map' $\Psi : M \to \mathbb{C}^3$ by

$$\Psi : (b, p, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3) \mapsto q^{1/3}\mathsf{e}_3 - \tfrac{1}{3}\mathrm{i}q'\mathsf{e}_2.$$

This lands in the set $\mathring{Q}_C$ of non-real points on the quadric $Q_C \subset \mathbb{C}^3$ defined by $z_1^2 + z_2^2 + z_3^2 = C$.

<u>Thm B</u> (I–, McKay) If S is any solution surface, then $\Psi|_S$ is holomorphic with respect to the complex structure defined by $\widehat{V}|_S$. Moreover, the immersion is determined by integrating the Weierstrass-type formula

$$dp|_S = \mathsf{X} \times d\mathsf{Y}$$

where $\mathsf{X} = \mathrm{Re}\,\Psi$, $\mathsf{Y} = \mathrm{Im}\,\Psi$.

<u>Rk</u> In fact, the image of an isometric immersion of any of these metrics is an *affine minimal surface* in $\mathbb{R}^3$, and this is just a special case of Blaschke's Weierstrass-type representation for such surfaces.

Questions What is the relationship between the local extension space N and the classifying quadric Q? What is the relationship between the complexified Vessiot group K, the internal symmetry of (B, g), and the extrinsic symmetries of $\mathscr{I}$?

$$
\begin{array}{ccccc}
S \times K = N^{12} & & & & \\
\downarrow \pi & & & & \\
S \times K/G & \xrightarrow{\Phi} & M^8 & \xrightarrow{\Psi} & Q_C \subset \mathbb{C}^3 \\
 & \swarrow & & \searrow & \\
 (B, g_C) & & & & \mathcal{F}
\end{array}
$$

Thank You!