Involutive Moving Frames

Örn Arnaldsson

University of Iceland

ornarnalds@hi.is

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Congruence in a Lie group

Let G be a Lie group with Maurer-Cartan form ω. Underpinning the moving frame is the following.

Two submanifolds $\iota : S \hookrightarrow G$ and $\bar{\iota} : \bar{S} \hookrightarrow G$ are congruent

$$\exists\, g \in G \;:\; \iota(S) \cdot g = \bar{\iota}(\bar{S})$$

if and only if

there is a map $\psi : S \to \bar{S}$ such that

$$\psi^* \bar{\iota}^* \omega = \iota^* \omega.$$

It turns out there is a Lie pseudo-group analog of this result.

For Lie pseduo-groups the coframe equivalence problem above,

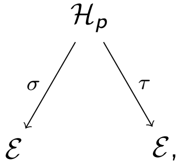
$$\psi^* \bar{\iota}^* \omega = \iota^* \omega,$$

is replaced with a G-structure equivalence problem and the equivariant moving frame becomes Cartan's equivalence method.

Congruence in a Lie pseudo-group

Let $\mathcal{H}$ be a Lie pseudo-group on $\mathcal{E}$ with groupoids

$$\begin{aligned}\mathcal{H}_p &= \{ j^p\varphi_{|_z} \mid z \in \mathcal{E},\ \varphi \in \mathcal{H} \} \\ &= \{ (z, Z, \ldots, Z^a_A, \ldots) \mid |A| \leq p \} = \{ (z, Z^{(p)}) \}.\end{aligned}$$



with $\sigma(j^p\varphi_{|_z}) = z$ and $\tau(j^p\varphi_{|_z}) = \varphi(z) = Z$.

For $g = j^p\varphi_{|_z}, h = j^p\psi_{|_z} \in \sigma^{-1}(z)$ we write

$$R_g \cdot h := j^p(\psi \circ \varphi^{-1})_{|_{\varphi(z)}}.$$

Let $\mu^a_A, |A| < p$ be the Maurer-Cartan forms of $\mathcal{H}_p$.

Two sections, s and $\bar{s}$, of $\mathcal{H}_p \to \mathcal{E}$, are "congruent",

$$\exists\ \varphi \in \mathcal{H}\ \ :\ \ R_{j^p\varphi} \cdot s = \bar{s}$$

if and only if

$$\exists\ \varphi \in \mathcal{H}\ \ :\ \ \varphi^* \bar{s}^* \mu^a_A = s^* \mu^a_A, \quad |A| < p.$$

See [Ö.A., 2020, Theorem 3.12].

Actually, we also need $\varphi^* \tau(\bar{s}) = \tau(s)$ so we restrict to s and $\bar{s}$ that have constant target coordiantes, $\tau(\bar{s}) = \tau(s) =$ constant.

Also, p must be at least as large as the order of $\mathcal{H}$.

Definition 1

The order of $\mathcal{H}$ is the smallest number, t such that

$$\mathcal{H}_t = \{F(z, Z^{(t)}) = 0\}$$

for some F and all other defining equations of $\mathcal{H}$ are consequences of $F = 0$.

So $F = 0$ is formally integrable/locally solvable and each local solution belongs to $\mathcal{H}$.

Example 2

Consider the Lie group action

$$(x, u) \mapsto (X, U) = (x + a, u + a_2 x^2 + a_1 x + a_0).$$

The defining equations are

$$X_u = 0, \quad X_x = U_u = 1, \quad U_{xxx} = 0$$

and it is easily seen that this Lie (pseudo) group has order 3.

As before, $\mathcal{H}$ is a Lie pseudo-group on $\mathcal{E}$, but let $\mathcal{E} \to \mathcal{X}$ be a fiber bundle with fibers $\mathcal{U}$ and let S and $\overline{S}$ be sections of $\mathcal{E}$. (We may assume $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ are open subsets.)

$\mathcal{H}$ acts on $J^\infty(\mathcal{E})$ by prolongation and we write

$$\varphi \cdot (x, u, \ldots, u^\alpha_J, \ldots) \mapsto (X, U, \ldots, \widehat{U}^\alpha_J, \ldots)$$

for the target variables.

Let s and $\bar{s}$ be sections of $\mathcal{H}_p \to \mathrm{im}(S)$ and $\mathcal{H}_p \to \mathrm{im}(\overline{S})$, respectively. (With equal and constant target coordinates.)

There is a section f_p of $\mathcal{H}_p \to \mathrm{im}(S)$ such that

$$R_{f_p} \cdot s = \bar{s},$$

$f_p^* \mu^a_A = 0$, $|A| < p$

if and only if

$f := \tau(f_p)$ satisfies

$$f^* \bar{s}^* \mu^a_A = s^* \mu^a_A, \quad |A| < p.$$

Said differently,

Theorem 3

There is a section f_p *of* $\mathcal{H}_p \to im(S)$ *that satisfies the contact condition*

$$f_p^* \mu_A^a = 0, \quad |A| < p$$

such that $f := \tau(f_p)$ *maps* S *to* $\overline{S}$

if and only if

There exist sections s *and* $\bar{s}$ *(with equal and constant target coordinates) of* $\mathcal{H}_p \to im(S)$ *and* $\mathcal{H}_p \to im(\overline{S})$ *such that*

$$f^* \bar{s}^* \mu_A^a = s^* \mu_A^a, \quad |A| < p.$$

See [Ö.A., 2021, Theorem 2.4].

But does f_p determine a bona-fide element of $\mathcal{H}$?

But does f_p determine a bona-fide element of $\mathcal{H}$?

Can we integrate f_p to the jet of a pseudo-group element $j^p\varphi$?

Locally, $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m$ in coordinates $z = (x, u) = (x^1, \ldots, x^n, u^1, \ldots, u^m)$.

Let $\mathcal{H}$ have order t. If every jet coordinate Z^a_{A,u^α}, $|A| = t - 1$, of order t that involves a u-derivative can be taken as a principal derivative in the defining equations for $\mathcal{H}$, we say $\mathcal{H}$ is *quasi-horizontal*.

For quasi-horizontal $\mathcal{H}$ on $\mathcal{E} \to \mathcal{X}$ the f_p from Theorem 3 uniquely determines a $\varphi \in \mathcal{H}$, [Ö.A., 2021, Theorem 2.12].

Eventually free Lie pseudo-groups are quasi-horizontal, [Ö.A., 2021, Theorem 3.6].

Example 4

Consider the pseudo-group $\mathcal{H}$ of transformations

$$(x, u, p, q) \mapsto (X, U, P, Q)$$

with defining equations

$$X_p = X_q = U_p = U_q = P_q = 0$$
$$P = \frac{U_x + pU_u}{X_x + pX_u}, \quad Q = \frac{P_x + pP_u + qP_p}{X_x + pX_u}$$

and the equivalence of sections $(x, u, p) \overset{S}{\mapsto} (x, u, p, q(x, u, p))$.

Obviously, any section of $\mathcal{H}_1 \to \mathrm{im}(S)$ annihilating contact forms will uniquely determine a pseudo-group element.

That is, any $(x, u, p) \mapsto (X, U, P)$ satisfying the blue will determine a solution to the red.

In such cases Theorem 3 is reduced to an equivalence problem for G-structures, [Ö.A., 2020, Section 4.2.1].

Theorem 3

There is a section f_p of $\mathcal{H}_p \to \mathrm{im}(S)$ that satisfies the contact condition

$$f_p^* \mu_A^a = 0, \quad |A| < p$$

such that $f := \tau(f_p)$ maps S to $\overline{S}$

if and only if

There exist sections s and $\bar{s}$ (with equal and constant target coordinates) of $\mathcal{H}_p \to \mathrm{im}(S)$ and $\mathcal{H}_p \to \mathrm{im}(\overline{S})$ such that

$$f^* \bar{s}^* \mu_A^a = s^* \mu_A^a, \quad |A| < p.$$

Continuing the second order ODE example, the pseudo-group is order 1. Let s and $\bar{s}$ have target coordinates $X = U = P = Q = 0$. The *recurrence formula* gives

$$\mu^x = -\omega^x, \;\; \mu^u = -\omega^u, \;\; \mu^p = -\omega^p, \;\; \mu^q = -\omega^q = -\widehat{Q}_X\omega^x - \widehat{Q}_U\omega^u - \widehat{Q}_P\omega^p$$

and so the equivalence problem is determined by $\omega^x, \omega^u, \omega^p$ and the first order invariants $\widehat{Q}_i$.

Now,

$$\omega^x = X_x dx + X_u du,$$
$$\omega^u = U_x dx + U_u du,$$
$$\omega^p = P_x dx + P_u du + P_p dp.$$

But after using $X = U = P = Q = 0$ and $P_p = \frac{U_u}{X_x + pX_u}$ we have

$$\begin{bmatrix} \omega^x \\ \omega^u \\ \omega^p \end{bmatrix} = \begin{bmatrix} X_x + pX_u & X_u & 0 \\ 0 & U_u & 0 \\ 0 & P_u & \frac{U_u}{X_x + pX_u} \end{bmatrix} \begin{bmatrix} dx \\ du - pdx \\ dp - qdx \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & \frac{a_3}{a_1} \end{bmatrix} \begin{bmatrix} dx \\ du - pdx \\ dp - qdx \end{bmatrix}.$$

Choosing s and $\bar{s}$ in Theorem 3 is the same as choosing a_1, a_2, a_3, a_4 in Cartan's equivalence method.

Theorem 3

There is a section f_p of $\mathcal{H}_p \to \mathrm{im}(S)$ that satisfies the contact condition

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such that $f := \tau(f_p)$ maps S to $\overline{S}$

if and only if

There exist sections s and $\bar{s}$ (with equal and constant target coordinates) of $\mathcal{H}_p \to \mathrm{im}(S)$ and $\mathcal{H}_p \to \mathrm{im}(\overline{S})$ such that

$$f^* \bar{s}^* \mu_A^a = s^* \mu_A^a, \quad |A| < p.$$

Solving $f^*\bar{s}^*\mu^a_A = s^*\mu^a_A$ for f will involve the structure equations

$$d\mu^a_A = \sum_b \omega^b \wedge \mu^a_{A,b} + \sum_{\substack{L+M=A \\ |M|\geq 1}} \binom{K}{L} \sum_b \mu^a_{L,b} \wedge \mu^b_M.$$

In our case, we obtain, essentially for free, the structure equations (in symbolic form)

$$\begin{aligned}
d\omega^x &= \mu^x_X \wedge \omega^x + \mu^x_U \wedge \omega^u, \\
d\omega^u &= \omega^x \wedge \omega^p + \mu^u_U \wedge \omega^u, \\
d\omega^p &= (-\widehat{Q}_i\omega^i) \wedge \omega^x + \mu^p_U \wedge \omega^u + (\mu^u_U - \mu^x_X) \wedge \omega^p.
\end{aligned}$$

The only structure functions appearing here are the $\widehat{Q}_i$ which we normalize to zero. Applying the recurrence formula to these indicates we can normalize them to zero and solve for P_{xx}, P_{ux} and X_{xx}.

$$\begin{aligned} d\omega^x &= \mu^x_X \wedge \omega^x + \mu^x_U \wedge \omega^u, \\ d\omega^u &= \omega^x \wedge \omega^p + \mu^u_U \wedge \omega^u, \\ d\omega^p &= \mu^p_U \wedge \omega^u + (\mu^u_U - \mu^x_X) \wedge \omega^p. \end{aligned}$$

Next, to check for involutivity we count the second order pseudo-group parameters not yet normalized ("order of indeterminancy"). These are $\{P_{uu}, U_{uu}, X_{uu}, X_{ux}\}$. The first reduced Cartan character is the maximum

rank of the set of one-forms

$$\left\{ \left(a\frac{\partial}{\partial \omega^x} + b\frac{\partial}{\partial \omega^u} + c\frac{\partial}{\partial \omega^p} \right) \lrcorner \; d\omega^i \mid i = x, u, p \right\}$$

and so on...

Turns out this is not involutive and we move from $\mathcal{H}_1$ to $\mathcal{H}_2$ and include the first order Maurer-Cartan forms μ^a_A, $|A| = 1$ in the equations $f^*\bar{s}^*\mu^a_A = s^*\mu^a_A$, compute structure equations, normalize structure functions and so on...

Note that we are constructing an equivariant moving frame, or rather, a partial moving frame.

This process will branch according to various relative invariants and on each branch will terminate at involution or an invariant coframe.

However, contrary to the original equivariant moving frame, we only normalize invariants appearing in the structure equations at each step, we do not prolong the action of $\mathcal{H}$ to $J^k(\mathcal{E})$ and attemp to normalize *all* k-order lifted invariants, saving time.

See [Ö.A., 2020] for more.

Recall $p \geq$ order of $\mathcal{H}$.

Theorem 3

There is a section f_p of $\mathcal{H}_p \to \mathrm{im}(S)$ that satisfies the contact condition

$$f_p^* \mu_A^a = 0, \quad |A| < p$$

such that $f := \tau(f_p)$ maps S to $\overline{S}$

if and only if

There exist sections s and $\bar{s}$ (with equal and constant target coordinates) of $\mathcal{H}_p \to \mathrm{im}(S)$ and $\mathcal{H}_p \to \mathrm{im}(\overline{S})$ such that

$$f^* \bar{s}^* \mu_A^a = s^* \mu_A^a, \quad |A| < p.$$

Let $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m$.

Theorem 5

[Ö.A., 2021] Let $\mathcal{H}$ have order t. Assume that at some point during the equivalence method above, we mangage to normalize all pseudo-group parameters of order $\leq t$ to obtain an equivariant moving frame $\rho : \mathcal{S}_q \to \mathcal{H}_t$ with some domain of definition $\mathcal{S}_q \subset J^q(\mathcal{E})$. Write

$$\rho^* \mu_A^a = \sum_{i=1}^{n} I_{A,i} \omega^i + \textit{contact forms on } J^\infty(\mathcal{E}).$$

Then a generating set for the algebra of invariants is

$$\{I_{A,i}\}_{|A|<t, 1\leq i\leq n}.$$

A closer look at the recurrence formula reveals that

an upper bound on the size of this set is

$$n \cdot (\dim \mathcal{H}_{t-1} - n)$$

and if $\mathcal{H}$ acts transitively on $J^q(\mathcal{E})$ then we have the upper bound

$$n \cdot (\dim \mathcal{H}_{t-1} - \dim J^{q-1}(\mathcal{E}))$$

If $\mathcal{H} = G$ is a Lie group acting transitively on $\mathcal{E}$, then $\dim \mathcal{H}_{t-1} = \dim G - \dim \mathcal{E}$.

Not everything is quasi-horizontal

Find the internal symmetries of

$$v_x = u_{xx}^2.$$

This is a matter of completing a system of differential equations to involution...

Cartan's equivalence method is applicable but not equivariant moving frame methods...

Beyond symbolic formulas

But hang on! Cartan's equivalence method solves constant coefficient linear equations when normalizing structure functions (torsion coefficients). And Cartan's method provides formulas for the invariants.

On the other hand, the equivariant moving frame can very easily provide the structure of the invariant algebra through its recurrence formula.

However, finding the non-linear formulas of invariants while working in the standard pseudogroup jet-coordintes Z^a_A is much less convenient than in Cartan's prolongation procedure.

For example, while the recurrence formula gives that, modulo horizontal forms ω^i,

$$d\widehat{Q}_X \equiv \mu^p_{XX} + \widehat{Q}_{PX}\mu^u_X$$

and so P_{xx} can be normalized from $\widehat{Q}_X = 0$, we have

$$\widehat{Q}_X = -\frac{2qp^2 U_u X_{uu}}{a_1^4} - \frac{4qpU_u\ X_{ux}}{a_1^4} - \frac{2qU_u X_{xx}}{a_1^4} + \frac{p^2 P_{uu}}{a_1^2} + \frac{2pP_{ux}}{a_1^2} + \frac{P_{xx}}{a_1^2} + \frac{3qa_1 P_u + qq_p\ U_u + pq_u U_u + q_x U_u}{a_1^3}$$

Not constant coefficient in the top order terms and generally unpleasant even in this smallish equivalence problem... imagine what $\widehat{Q}_{XX}$ will look like!

Cartan-esque coordinates

In fact, hiding in the formulas for the pseudo-group Maurer-Cartan forms μ^a_A are expressions that render the above expressions constant coefficient at top order just like in Cartan's original method.

We have, by definition,

$$\omega^a = Z^a_b \omega^b, \quad \text{and} \quad \mu^a_b = (dZ^a_t)(Z^{-1})^t_b - Z^a_{t_1 t_2}(Z^{-1})^{t_1}_b (Z^{-1})^{t_2}_t \omega^t$$

where $(Z^{-1})^a_b$ are the entries of the inverse Jacobian matrix with entries Z^a_b.

Define $\alpha^a_b := (dZ^a_t)(Z^{-1})^t_b$ and the second order Cartan-coordiantes

$$r^a_{bc} := Z^a_{t_1 t_2}(Z^{-1})^{t_1}_b (Z^{-1})^{t_2}_c$$

so that

$$\mu^a_b = \alpha^a_b - r^a_{bt}\omega^t.$$

There are higher order Cartan-coordinates,

$$r^a_{(a_1,\ldots,a_k)} := Z^a_{(t_1,\ldots,t_k)}(Z^{-1})^{t_1}_{a_1}\cdots(Z^{-1})^{t_k}_{a_k},$$

where we sum over all possible $(t_1,\ldots,t_k)$. (This is symmetric in the $(a_1,\ldots,a_k)$ and can be expressed using multi-indices.)

These parametrize, along with z, Z and Z^a_b, the groupoid spaces $\mathcal{H}_p$. We have, for $|A| \geq 2$,

$$\mu^a_A = dr^a_A - r^a_{A,t}\omega^t + \text{low order stuff}.$$

Now, the recurrence formula says

$$0 = dQ = \widehat{Q}_X \omega^x + \widehat{Q}_U \omega^u + \widehat{Q}_P \omega^p + \mu_X^p$$
$$= \widehat{Q}_X \omega^x + \widehat{Q}_U \omega^u + \widehat{Q}_P \omega^p + \alpha_x^p - r_{xt}^p \omega^t$$

Copying Cartan, we simply compute α_x^p and then compare coefficients in the above. We can already see this will be constant coefficient in the r's.

We had

$$\widehat{Q}_X = -\frac{2qp^2 U_u X_{uu}}{a_1^4} - \frac{4qpU_u\ X_{ux}}{a_1^4} - \frac{2qU_u X_{xx}}{a_1^4} + \frac{p^2 P_{uu}}{a_1^2} + \frac{2pP_{ux}}{a_1^2} + \frac{P_{xx}}{a_1^2} + \frac{3qa_1 P_u + qq_p\ U_u + pq_u U_u + q_x U_u}{a_1^3}.$$

Turns out this is equal to

$$\widehat{Q}_X = r^p_{xx} + \frac{qa_1 P_u + qq_p U_u + pq_u U_u + q_x U_u}{a_1^3}.$$

References

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Involutive Moving Frames
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