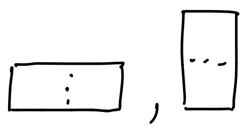
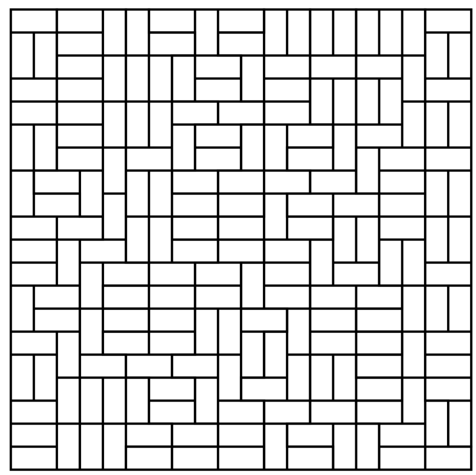
The multinomial tiling model

Richard Kenyon (Yale)
Cosmin Pohoata (Yale)

Random tilings

A set of prototiles T and a region R

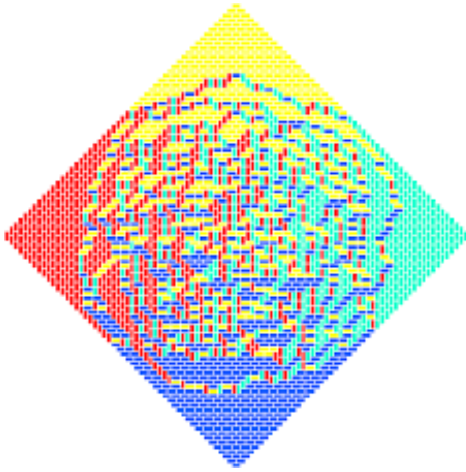


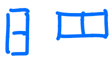


Q1 Can we tile R with (translated) copies of tiles from T? Hard

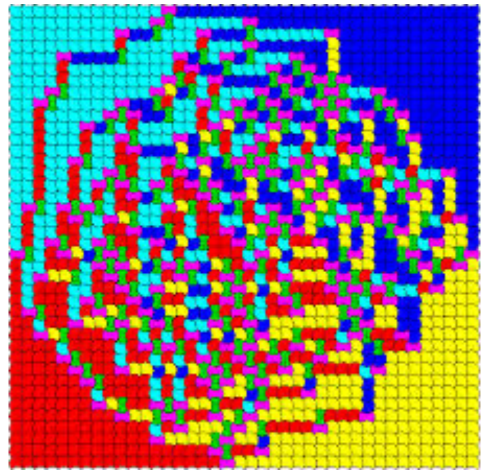
Q2 In how many ways? Harder

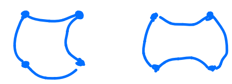
Q3 Can we describe the structure of a large (or infinite) random tiling? Hardest



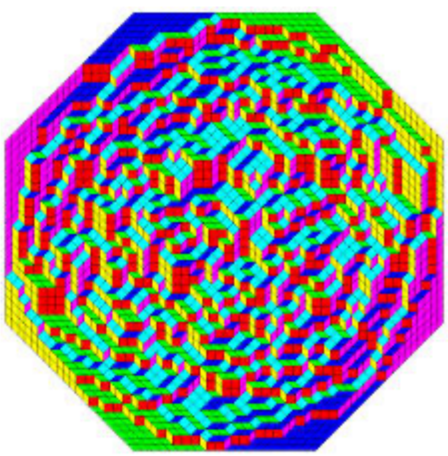
domino 

Integrable



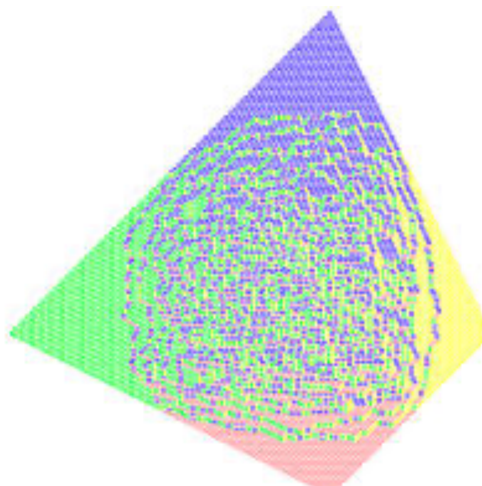
6 vertex
(square ice) 

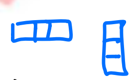
"integrable"



rhombus 

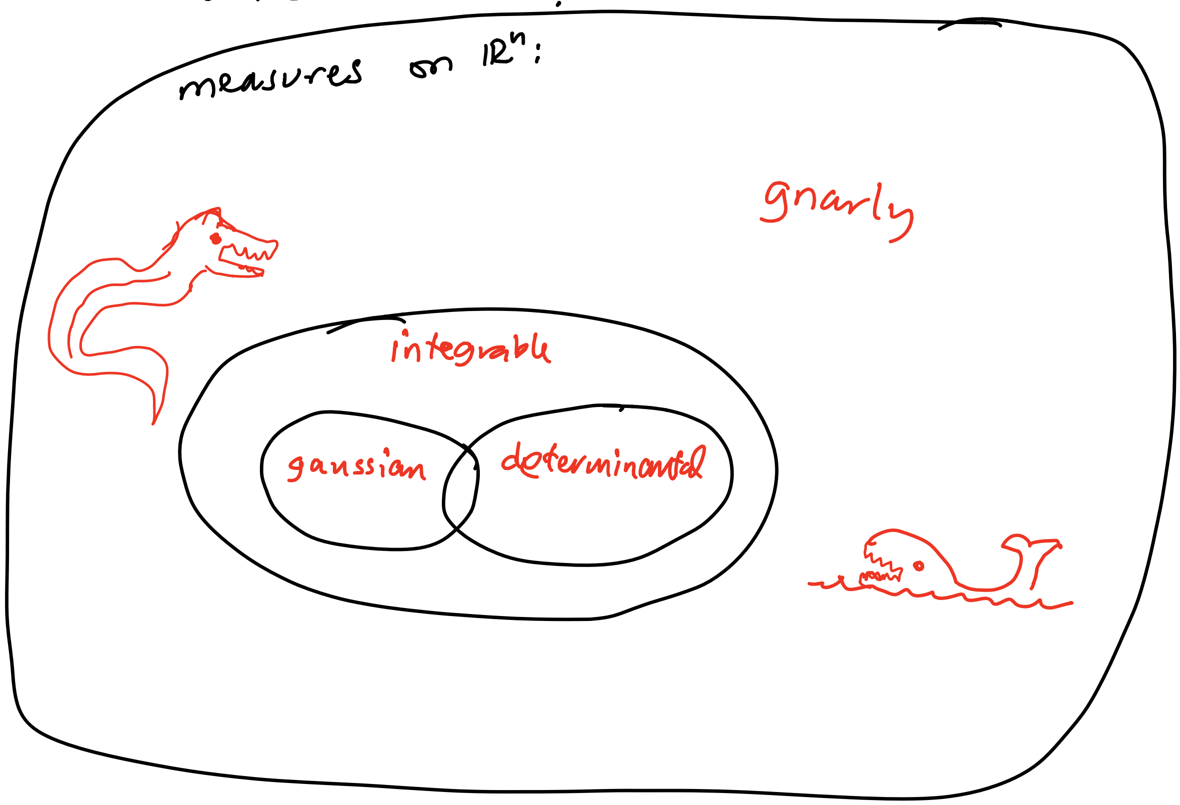
gnarly ?



3×1 bars 
(M. Tassy)

gnarly ?

The fundamental problem is to describe a probability measure in high dimensions, like $\{0,1\}^n$ or $\mathbb{R}^n$.



(random tiling)
These questions are hard, but <u>in the cases we understand</u>,

they lead to interesting mathematics:

- phase transitions
- conformal invariance
- (discrete) geometry
- probability: SLE_κ, CLE_κ, "quantum gravity"
- dynamics: aperiodicity, quasicrystals
- integrability

We'll treat an easier version of the tiling problem, working on arbitrary graphs, leading to:

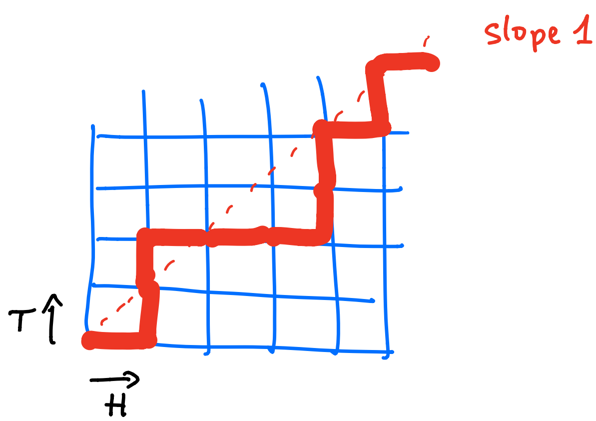
- exact counting
- limit shapes
- phase transitions
- crystals & quasi crystals
- conformal scaling limits

Idea: add a "multiplicity" N; $N \to \infty$ turns the computation of the partition sum $Z = \sum_{\cdots} \cdots$ into a maximization problem.

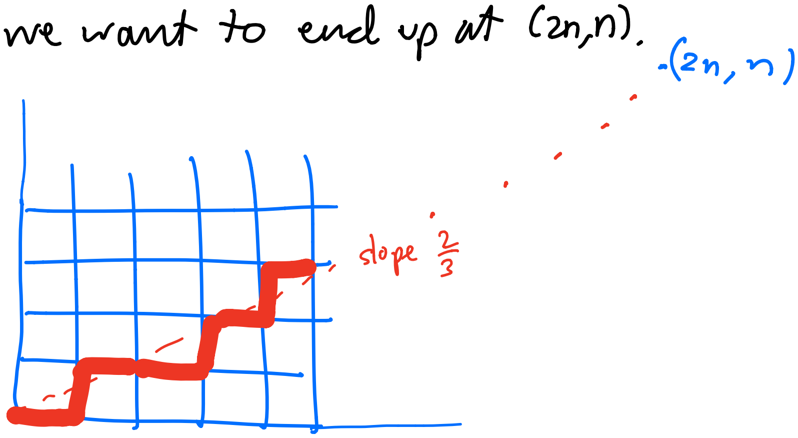
But first a little probability fact

"exponential tilting"

flip a fair coin



But suppose we want to end up at $(2n, n)$.



If you condition on the endpoint, the distribution of flips acts as if the coin were biased.

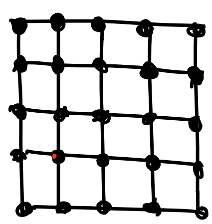
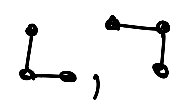
The multinomial tiling model

Let G be a finite graph.

$T = \{t_1, \dots, t_k\}$ a set of subsets (called <u>tiles</u>)

$w: T \to \mathbb{R}_{\geq 0}$ weight on tiles

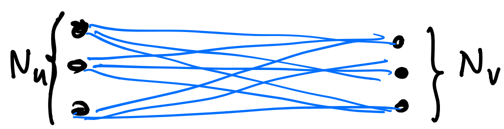
<u>EXAMPLE</u>

$G =$  $\quad T = \{$ all translates of  $\}$

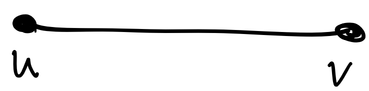
$\vec{N} = \{N_v\}_{v \in V}$, $N_v \in \mathbb{N}$, <u>multiplicities</u> at each vertex.

Let $G_{\vec{N}}$ be the $\vec{N}$-fold "blow up" of G:

$G_{\vec{N}}$



G



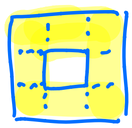
$G_{\vec{N}}$ = replace each vtx v with N_v vertices
replace each edge with K_{N_v, N_w} ← complete bipartite graph

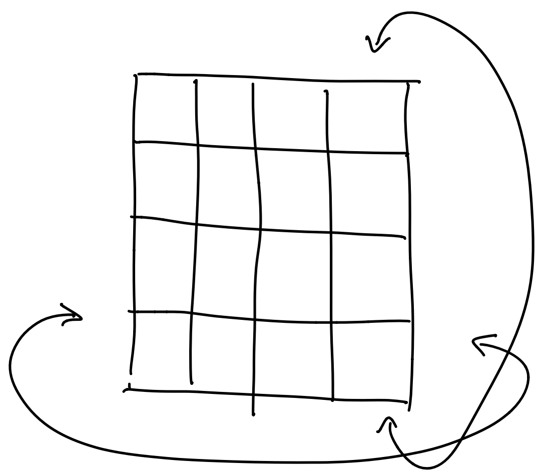
Each tile t_i can lift to $G_{\vec{N}}$ (in many ways)

<u>Def</u> An <u>$\vec{N}$-fold tiling</u> is a tiling of $G_{\vec{N}}$.

Let $\Omega_{\vec{N}} = \{ \vec{N}\text{-fold tilings} \}$

Note that there may be N-fold tilings even if there are no 1-fold tilings:

Ex.  can 8-fold tile an $n \times n$ torus



multiplicity of t

$m \in \Omega_{\vec{N}}$ has weight $w(m) = \prod_{t \in T} w(t)^{m(t)}$

$$\boxed{Z(w,\vec{N}) = \sum_{m \in \Omega_{\vec{N}}} w(m)}$$

partition function.

We associate a variable x_v to vertex $v \in G$.

Define a polynomial:

$$\boxed{P := \sum_{t \in T} w_t \prod_{v \in t} x_v}$$

Thm $Z(w) := \sum_{\vec{N} \geq 0} Z(w,\vec{N}) \frac{\vec{x}^{\vec{N}}}{\vec{N}!} = \exp(P)$

$\vec{x}^{\vec{N}} = \prod x_v^{N_v}$, $\quad \vec{N}! = \prod N_v!$

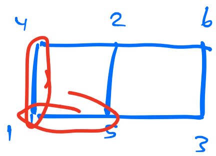
$\exp(P) = \sum \frac{P^k}{k!}$

use k tile

Pf: "easy combinatorics" $\square$

Re-Interpretation: $\frac{p^k}{k!} \longleftrightarrow$ "pick tiles independently, then ignore order"

Example

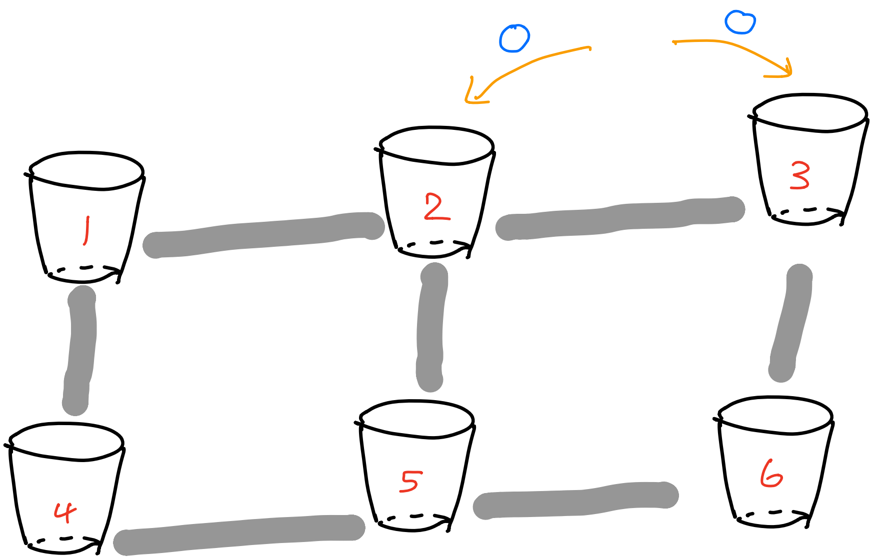


$T = \{\text{edges}\}$

$N_v \equiv N$

$w \equiv 1$

$$P = x_1x_4 + x_1x_5 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_5 + x_3x_6$$



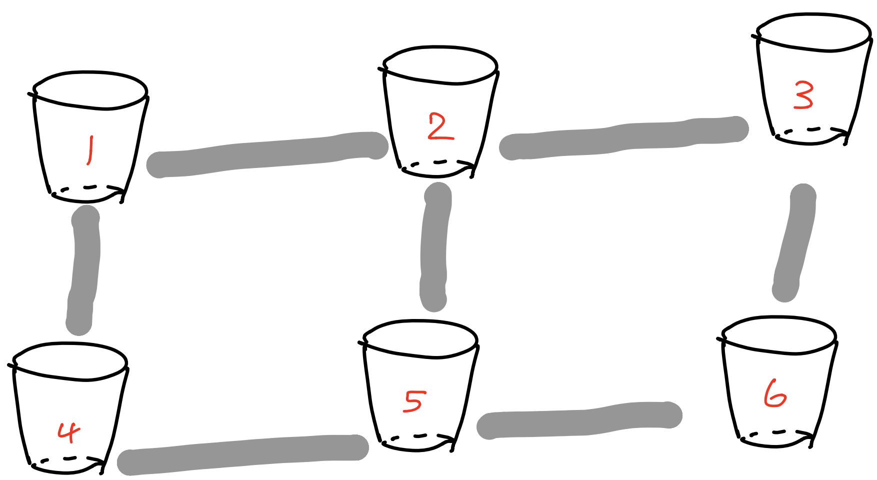
pick two adjacent buckets; put one ball in each.

Repeat until each bucket has exactly N balls.

This is a random walk in $\mathbb{Z}_+^6$,
conditioned to end at $(N, \dots, N)$.

If we pick each edge at random ($\frac{1}{7}$)
buckets 2 & 5 fill up more quickly than
the others.

We need to $\boxed{\text{tilt}}$ (bias) the walk by
giving buckets 1, 3, 4, 6 higher weight.



Find $x_1, \dots, x_6$ so that

(rate of bucket 1 =) $\dfrac{x_1 x_4 + x_1 x_2}{P} = \dfrac{1}{3}$

(rate of bucket 2 =) $\dfrac{x_1 x_2 + x_5 x_2 + x_3 x_2}{P} = \dfrac{1}{3}$

...

$\vdots$

In general we need to solve

$$(*) \qquad \boxed{\frac{x_v P_{x_v}}{P} = \alpha_v} \leftarrow \text{desired rate of bucket } v$$

Thm For any feasible rates $\vec{\alpha}$ there is an essentially unique solution to $(*)$ leading to a unique "gauge equivalent" weight function w' with the property that

$$\sum_{t \ni v} w'(t) = \alpha_v$$

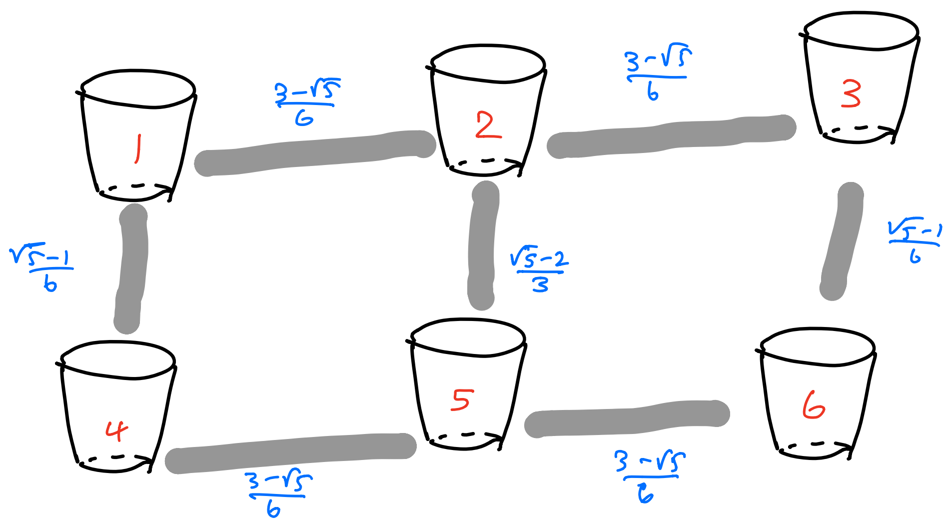
$\Rightarrow$ exponential growth rate

$$\sigma(w, \vec{\alpha}) := \lim_{K \to \infty} \frac{1}{K} \log\left(\frac{K!}{\vec{N}!} Z(w, \vec{N}) \right)$$

$$\boxed{\sigma(w, \vec{\alpha}) = \log P(\vec{x}_0) - \sum_v \alpha_v \log x_{0,v}}$$

$$X_1 = X_3 = X_4 = X_6 = \frac{1+\sqrt{5}}{2}$$

$$X_2 = X_5 = 1$$

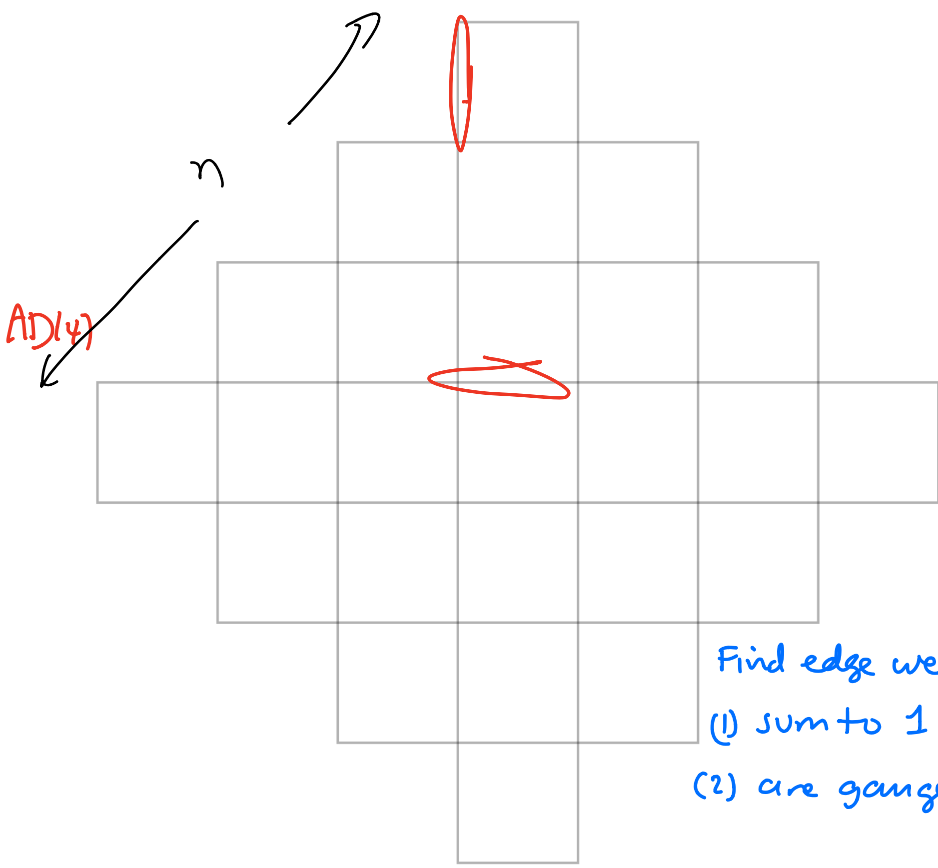


$$\sigma = \frac{1}{3}\log\left(\frac{11+5\sqrt{5}}{54}\right)$$

EX "Aztec diamond" $w \equiv 1$ $N_v \equiv N$

Tiles = dimers = edges

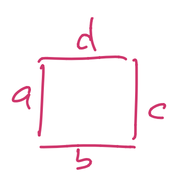
What is the "critical gauge" w' ?



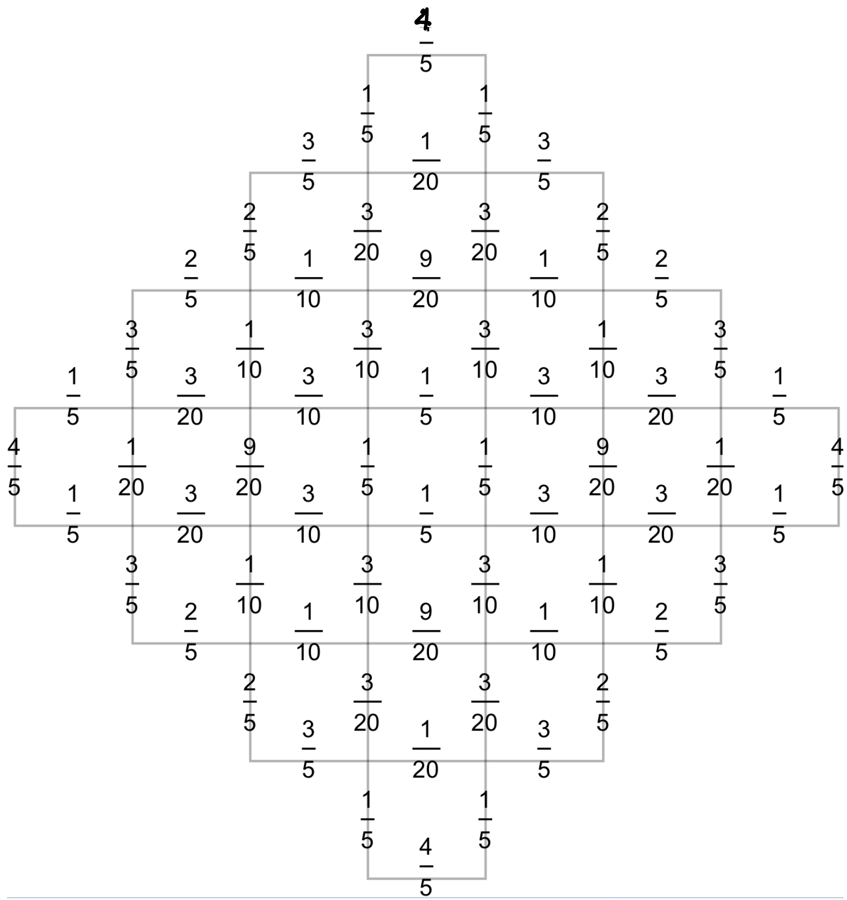
Find edge weights which

(1) sum to 1 at each vertex

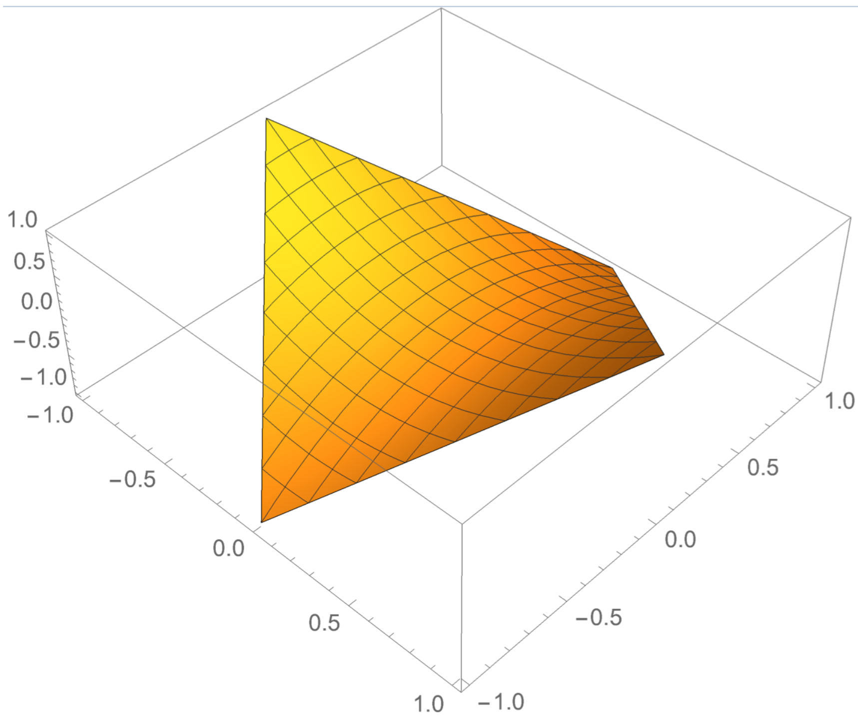
(2) are gauge equiv. to $w \equiv 1$



$\Longleftrightarrow \quad \frac{ac}{bd} = 1$


4/5
1/5 1/5
3/5 1/20 3/5
2/5 3/20 3/20 2/5
2/5 1/10 9/20 1/10 2/5
3/5 1/10 3/10 3/10 1/10 3/5
1/5 3/20 3/10 1/5 3/10 3/20 1/5
4/5 1/20 9/20 1/5 1/5 9/20 1/20 4/5
1/5 3/20 3/10 1/5 3/10 3/20 1/5
3/5 1/10 3/10 3/10 1/10 3/5
2/5 1/10 9/20 1/10 2/5
2/5 3/20 3/20 2/5
3/5 1/20 3/5
1/5 1/5
4/5

The (rescaled) height function associated to an $\vec{N}$-fold tiling of $AD(n)$ is, in $n \to \infty$ limit,



$$\boxed{h(x,y) = x^2 - y^2}$$

Density fluctuations

How many times do we place each tile?

After placing K tiles, each tile t is placed a binomial # of times X_t, with mean $K w_t$. For large K, this tends to a Gaussian random variable. We then condition this Gaussian to lie on subspace $\left\{ \sum_{t \ni v} X_t = N \right\}$

Thm: $\hat{X}_t := \dfrac{X_t - K w_t}{\sqrt{K}}$ converges to a multidimensional Gaussian RV with covariance

$$\mathrm{Cov}(\hat{X}_s, \hat{X}_t) = w_s I_{s=t} - w_s w_t (D^* \Delta^{-1} D)_{s,t}$$

$$= w_s \mathbb{P}_{s,t}$$

a projection mtx to ker D

Here $D : \mathbb{R}^T \to \mathbb{R}^V$ is the incidence matrix tiles $\to$ verts

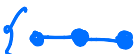
$$D_{t,v} = \begin{cases} 1 & \text{if } v \in t \\ 0 & \text{else} \end{cases}$$

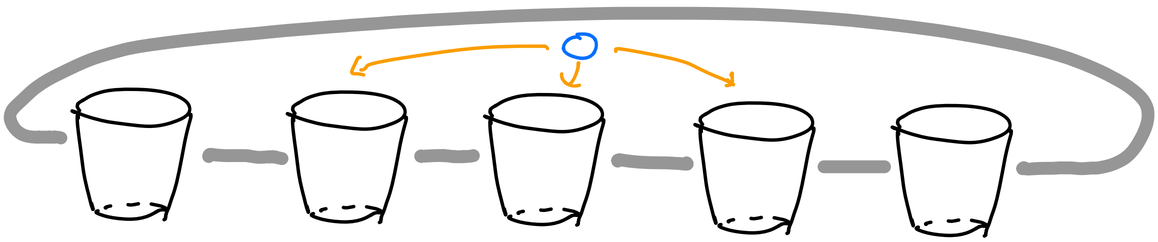
and

$\Delta : \mathbb{R}^V \to \mathbb{R}^V$ is the tiling Laplacian

$$\Delta = D \underbrace{W}_{\mathrm{diag}(\{w_t\})} D^*$$

$$\boxed{\Delta_{uv} = \sum_{t \ni u,v} w_t}$$

EXAMPLE 1: tiling $\mathbb{Z}/n\mathbb{Z}$ with translates of $\{$  $\}$



$$X_{i-1} + X_i + X_{i+1} = N \qquad \Longrightarrow \qquad X_i = X_{i+3}$$

multiplicity of tile at i

<u>case 1.</u> $3 \nmid n$. Then $X_i \equiv N/3$ so $\mathrm{Cov} \equiv 0$.

<u>case 2.</u> $3 \mid n$.

$$\mathrm{Cov} = \frac{1}{n} \begin{pmatrix} 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 \\ & & & & \ddots \end{pmatrix}$$

a "crystal"

2. Bars of length 3 and singletons on $\mathbb{Z}/n\mathbb{Z}$.

Let ε = fraction of singleton tiles

$\frac{1}{n^2} \ll \varepsilon \ll 1$

$$\mathrm{Cov}(\hat{X}_0,\hat{X}_s) = \frac{1}{n}\sum_{\substack{z^n=1\\ z\neq 1}} \frac{z^s}{\varepsilon 1 + (1+z+z^2)(1+z^{-1}+z^{-2})}$$

$$\mathrm{Cov}(\hat{X}_0,\hat{X}_s) \simeq \frac{\cos\left(\frac{2\pi}{3}s\right)}{2\sqrt{3\varepsilon}}\, e^{-|s|\sqrt{\frac{\varepsilon}{3}}}$$

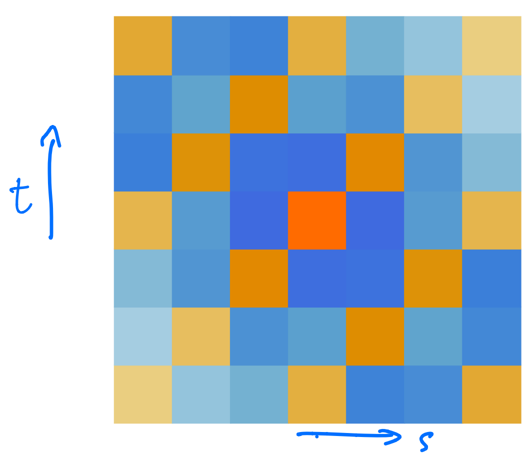
3. L-triomino and monomers on $\mathbb{Z}^2$

$\varepsilon \ll 1$ density of monomers

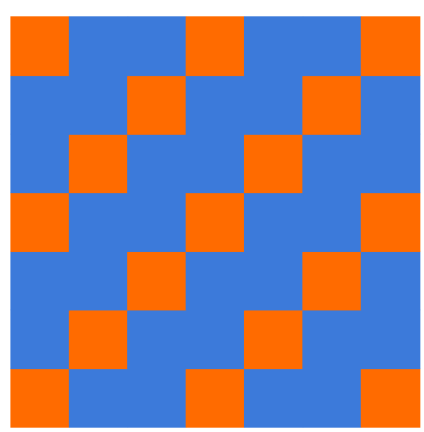
$$* \quad \boxed{\mathrm{Cov}(\hat X_{0,0}, \hat X_{s,t}) = \frac{1}{(2\pi i)^2} \iint_{S^1\times S^1} \frac{z^s u^t}{\varepsilon + (1+z+u)(1+\frac{1}{z}+\frac{1}{u})} \frac{dz}{z}\frac{du}{u}}$$

$$= \frac{\cos\left(\frac{2\pi}{3}(s-t)\right)}{\pi\sqrt{3}} \underbrace{B}_{\text{Bessel-K}}\left(\frac{2}{\sqrt{3}}\sqrt{\varepsilon(s^2+st+t^2)}\right)$$

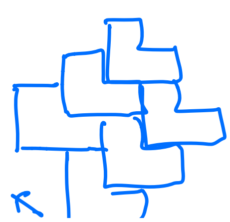
$$= \frac{\cos\left(\frac{2\pi}{3}(s-t)\right)}{\pi\sqrt{3}} \left(\log\frac{1}{\varepsilon} - \log\frac{s^2+st+t^2}{3} - 2\gamma_E + O(\varepsilon)\right)$$



$\varepsilon = .001$



$\varepsilon \to 0$



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4. Generally for a polyomino t in $\mathbb{Z}^2$ and a small density ε of monomers, we have

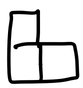
Thm $\operatorname{Cov}(\hat{X}_{00}, \hat{X}_{st}) =$ Fourier coefficients:

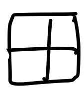
$$= \left(\frac{1}{2\pi i}\right)^2 \iint_{S^1 \times S^1} \frac{z^s w^t}{\varepsilon + |p|^2} \frac{dz}{z} \frac{dw}{w}$$

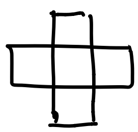
where $p(z,w) = \sum_{(i,j) \in t} z^i w^j$ the "characteristic polynomial"

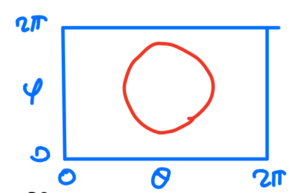
The leading asymptotics here depends

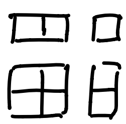
On <u>type of zeros</u> of $p(z,w)$ on $\mathbb{T}^2$.

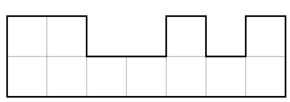
 : $p(z,w) = 1+z+w$ "simple" zeros, roots of 1

 : $p(z,w) = (1+z)(1+w)$ line of zeros

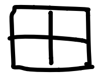
 : $p(z,w) = 1+z+\frac{1}{z}+w+\frac{1}{w}$ curve of zeros

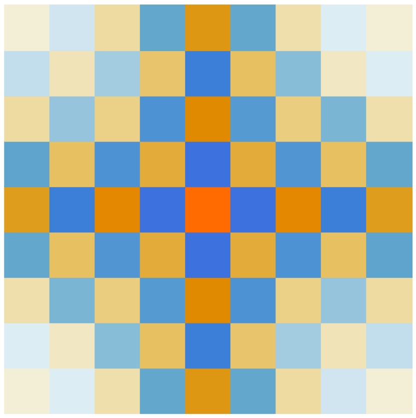


 : $p(z,w) = (1+z+z^3)(1+w+w^3)$ no zeros

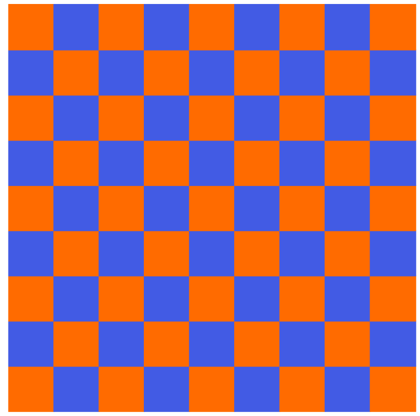
 simple zeros, irrational args

?? other behavior.. ??

Square  ,  monomers

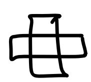


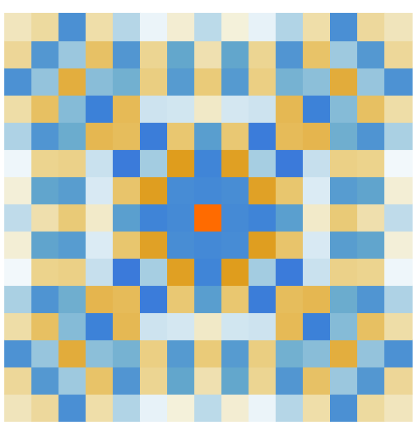
$\varepsilon = .001$



$\varepsilon \to 0$

$$\mathrm{Cov}(\hat{X}_{00}, \hat{X}_{st}) = \frac{(-1)^{s+t}}{\sqrt{\varepsilon}} \log \frac{1}{\sqrt{\varepsilon}} + \cdots$$

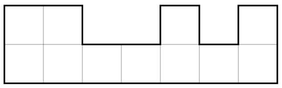
"plus" polyomino 

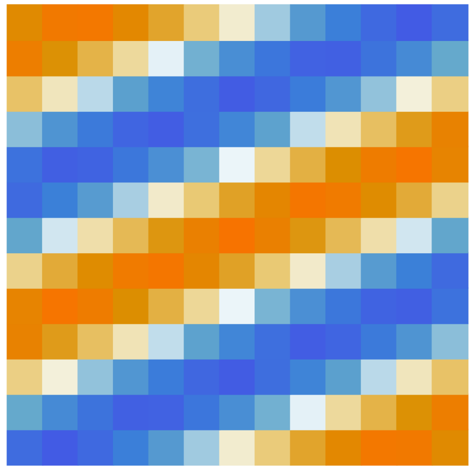


$\varepsilon \to 0$

$$\mathrm{Cov}(\hat{X}_{00}, \hat{X}_{st}) = \Theta\left(\frac{1}{\varepsilon^{\frac{1}{2}}(s^2+t^2)^{\frac{1}{4}}}\right)$$

"key" polyomino





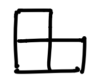
$\varepsilon \to 0$

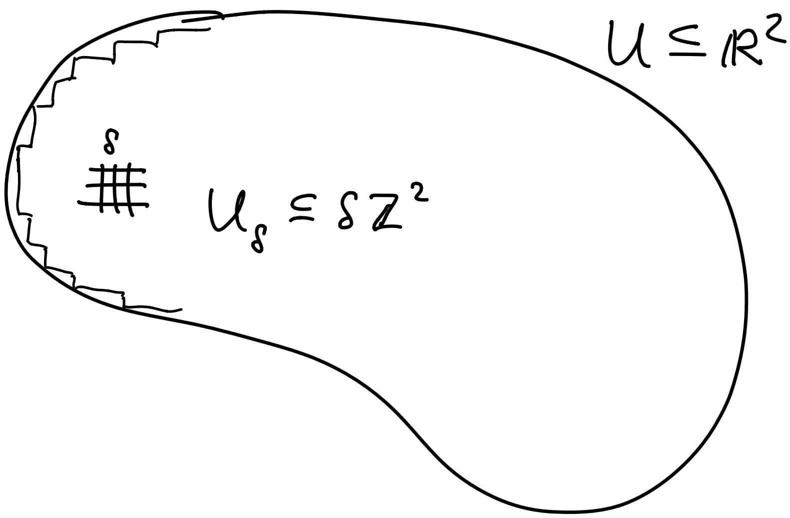
$$\mathrm{Cor}(\hat{X}_{00}, \hat{X}_{st}) = \cos(s\theta_0 + t\varphi_0) \log \tfrac{1}{\varepsilon} + \dots$$

Since $\frac{\theta_0}{\pi}, \frac{\varphi_0}{\pi}, \frac{\theta_0}{\varphi_0} \notin \mathbb{Q}$ this is a

<u>quasicrystal</u> : long range correlations but not periodic.

Many multinomial tiling models have conformally invariant scaling limits

$T = \{$ translates of  $\}$ $\quad N_v \equiv N$, $w \equiv 1$.



We take free boundary conditions:

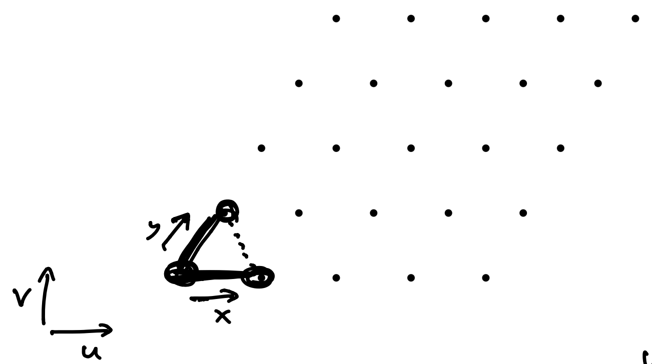
(i.e. no multiplicity condition on tiles outside U_δ)

$\mathrm{Cov}(\hat{X}_{\infty}, \hat{X}_{st}) =$ projection to $\ker D$ $= ?$

Recall $D: \mathbb{R}^T \to \mathbb{R}^V$

Apply a linear change of coordinates:

L



$\omega = e^{\frac{2\pi i}{3}}$

IDEA: $\ker D = \left\{ f(x,y) = \mathrm{Re}\left(\omega^{x-y} F(x,y)\right) \text{ where } F \text{ is "discrete analytic"} \right\}$

$$Df(x,y) = \mathrm{Re}\left[\omega^{x-y}\left(F(x,y) + \omega F(x+\delta,y) + \omega^{-1} F(x,y+\delta)\right)\right]$$
$$= \delta\,\mathrm{Re}\left[c\,\omega^{x-y}(F_u + iF_v) + O(\delta)\right]$$

Thm: On U_δ,

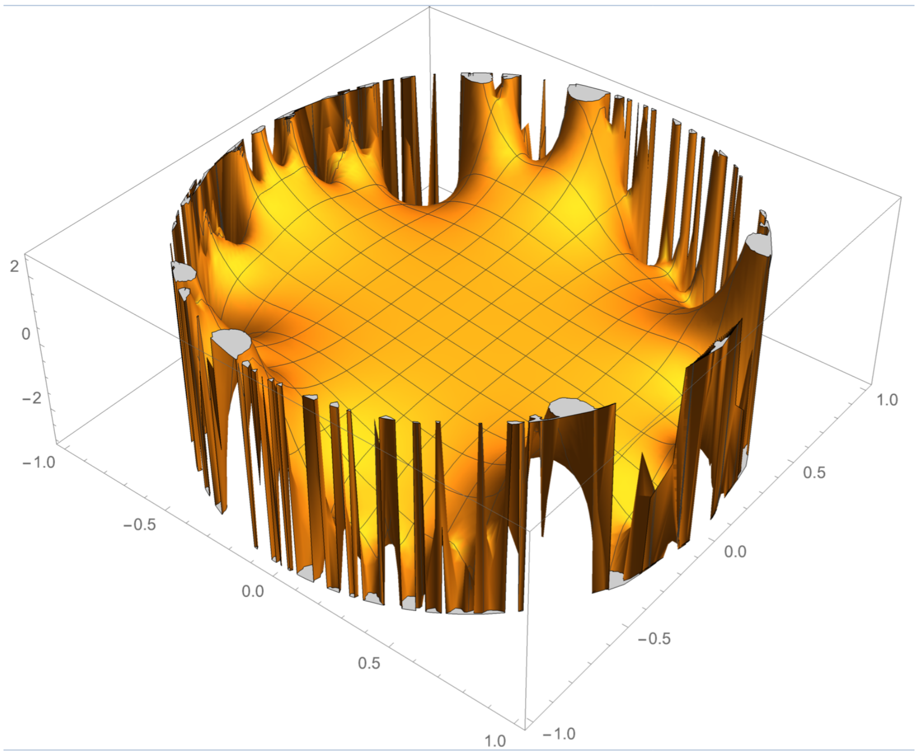
$$\mathrm{Cov}(X_{z_1}, X_{z_2}) = c\,\mathrm{Re}\left(\omega^{(x_1-y_1)+(x_2-y_2)}\,\mathbb{B}(z_1,z_2)\right) + o_\delta(1)$$

where $\mathbb{B}$ is the Bergman kernel

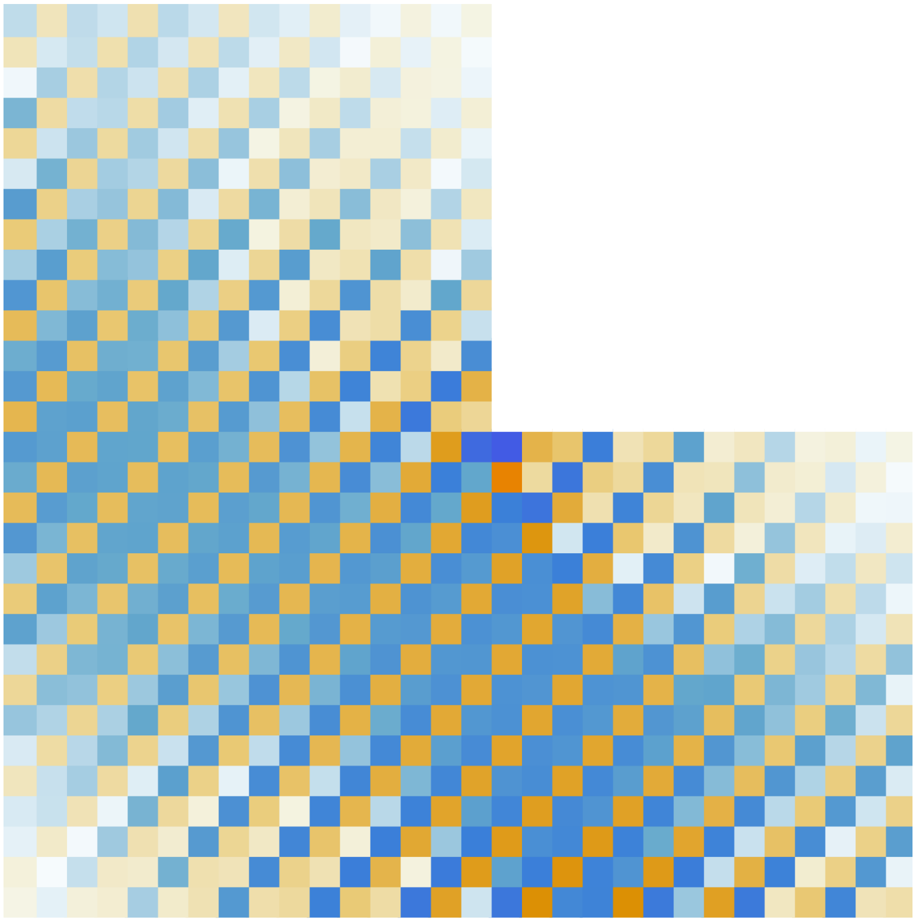
($\mathbb{B}$ is the projection kernel from $L^2(U)$ to $L^{2,h}(U)$)

e.g. unit disk $\mathbb{B}(z_1,z_2) = \dfrac{1}{\pi(1 - z_1\bar{z}_2)^2}$

Bergman-Gaussian on $\mathbb{D}$: $\mathrm{Re}\left(\sum_{j=0}^{\infty} X_j z^j\right)$ iid $N_{\mathbb{C}}(0,1)$



Density fluctuations of tiles on one of the three sublattices.



correlation function of ⊞ in L-shaped domain.

$\mathrm{Cov}(X_u, X_v)$ for fixed v.

Thank you!