Cutoff profile for random transpositions

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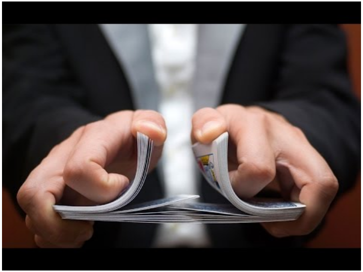
♠ : Different ways to shuffle a deck of cards

Spacial motion (Diaconis, Pal, 2017)



Dovetail shuffle (Bayer, Diaconis, 1992)

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- Pick two cards uniformly and independently;
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Interpretation :

- Random walk on $\mathfrak{S}_n$ with

$$P(\sigma, \sigma\tau) = \mu_n(\tau) = \begin{cases} 1/n & \text{if } \tau = id \\ 2/n^2 & \text{if } \tau \text{ is a transp.} \end{cases}$$

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μ_n : increment measure.

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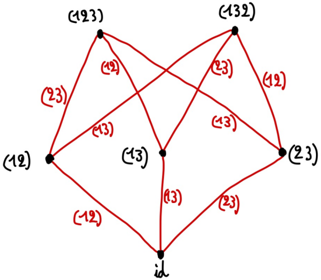

Cayley graph for $n = 3$

♠ : Distance to stationarity

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$\nu_n(t)$: distribution of the walk after t steps.

Définition

Distance to stationarity after t steps :

$$\mathrm{d}_n(t) := \mathrm{d}_{\mathrm{TV}}(\nu_n(t), \mathrm{Unif}_n).$$

where for probability measures μ and ν on $\mathfrak{S}_n$,

$$\mathrm{d}_{\mathrm{TV}}(\mu,\nu) = \frac{1}{2}\sum_{\sigma\in\mathfrak{S}_n} |\mu(\sigma)-\nu(\sigma)| = \frac{1}{2}\,\mathrm{d}_1(\mu,\nu).$$

♠ : Cutoff for random transpositions

Theorem (Diaconis and Shahshahani, 1981)

It takes $\frac{1}{2}n\ln(n)$ *steps to mix a deck of n cards by random transpositions. For every* $0 < \epsilon < 1$*,*

$$d_n\left((1-\epsilon)\frac{1}{2}n\ln(n)\right) \xrightarrow[n\to+\infty]{} 1 \quad \& \quad d_n\left((1+\epsilon)\frac{1}{2}n\ln(n)\right) \xrightarrow[n\to+\infty]{} 0$$

That is what is called the **cutoff phenomenon**.

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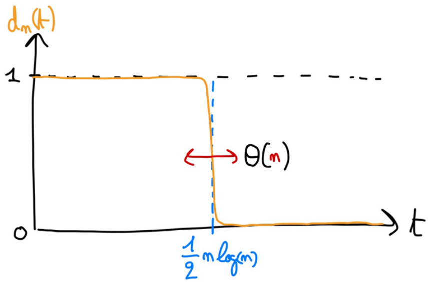
More precisely, it takes $\frac{1}{2} n \ln(n) + \Theta(n)$ steps to mix.



♠ : Cutoff profile for random transpositions

Theorem (T., 2020)

For random transpositions, we have for every $c \in \mathbb{R}$,

$$d_n\left(\frac{1}{2}n\ln(n)+cn\right)\xrightarrow[n\to+\infty]{} f(c):=\mathrm{d_{TV}}\left(\mathrm{Poiss}\left(1+e^{-2c}\right),\mathrm{Poiss}(1)\right).$$

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Comments :

- We call f the **cutoff profile**.
- Question asked by N. Berestycki at an AIM workshop in 2016.
- The other known profiles can be written in a similar form.

♠ : Some related results

- **On random transpositions themselves**
  Cutoff result : Diaconis, Shahshahani, 1981, *PTRF*
  Precise lower bound : Matthews, 1988, *J. of Th. Prob.*
  Phase transition result : N. Beresticky, Durrett, 2006, *PTRF*
  Probability of long cycles : Alon, Kozma, 2013, *Duke*
  Strong stationary time : White, 2019
  Cutoff profile : T., 2020, *Ann. Prob.*

- **Generalisations to other conjugacy classes :**
  Cutoff for k-cycles : N. Beresticky, Schramm, Zeitouni, 2011, *Ann. Prob.*
  Cutoff for conjugacy-invariant walks on $\mathfrak{S}_n$: N. Berestycki, Şengül, 2014, *PTRF*
  Profile for k-cycles : Nestoridi, Olesker-Taylor, 2021, *PTRF*

- **Other generalisations :**
  Biaised random transpositions : Matheau-Raven, 2020
  Quantum random transpositions : Freslon, T., Wang, 2021

♡ : The non-commutative Fourier transform

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Inverse Fourier transform, isometry between Hilbert spaces, Parseval identity.

Pierre-Loïc Méliot, *Representation Theory of Symmetric Groups*, chap. 1.

♡ : A method to prove a cutoff phenomenon on finite groups

- **Lower bound of the cutoff time** (not a problem)

- **The Diaconis-Shahshahani upper bound lemma:**
  Note that $\nu_n(t) = \mu_n^{*t}$, and let $f = \mu_n^{*t} - \mathrm{Unif}_n$.

$$4\mathrm{d}_n(t)^2 = \left(\sum_{\sigma\in\mathfrak{S}_n} |f(\sigma)|\right)^2 \overset{CS}{\leq} n! \sum_{\sigma\in\mathfrak{S}_n} f(\sigma)^2 \overset{Pars.}{=} \sum_{\lambda\in\widehat{\mathfrak{S}_n}} \mathrm{d}_\lambda \mathrm{Tr}\left(\widehat{f}(\lambda)\widehat{f}(\lambda)^*\right)$$

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- For random transpositions : the increment measure μ_n is **conjugacy stable** (i.e. $\mu_n(\sigma) = \mu_n(\eta\sigma\eta^{-1})$), so each $\widehat{\mu_n}(\lambda)$ is a **multiple of the identity matrix** : $\widehat{\mu_n}(\lambda) =: s_\lambda \mathrm{I}_{\mathrm{d}_\lambda}$. We deduce :

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- s_λ : eigenvalue of multiplicity d_λ^2 (of the transition matrix).

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Method:

- Apply the inverse Fourier transform on the **finite group** $\mathfrak{S}_n$ to the function $f = \mu_n^{*t} - \mathrm{Unif}_n$:

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- Understand which λ matter, and the (complicated) terms $\mathrm{ch}^\lambda(\sigma)$.
- Generalised to reversible Markov C. (Nestordi, Olesker-Taylor, 2021).

♡ : Representations of the symmetric group

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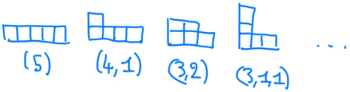
Example: $n = 5$.



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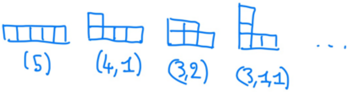
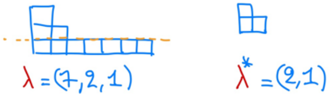


- **Definition:** Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of n. *Associated truncated partition :* Partition $\lambda^* := (\lambda_2, \lambda_3, \ldots)$. **Example:**



♡ : Ideas of the proof for random transpositions

- **Starting point :**

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- **Eigenvalues** (known by D& S). For λ such that $\lambda_1 = n - j$,

$$s_\lambda = \frac{n + 2\mathrm{Diag}(\lambda)}{n^2} = 1 - \frac{2j}{n} + O\left(\frac{1}{n^2}\right). \tag{1}$$

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- **Multiplicities** of eigenvalues (bound given by D& S: $\mathrm{d}_\lambda \leq \binom{n}{\lambda_1} \mathrm{d}_{\lambda^*}$), **(Hook-length formula)**,

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- **"Eigenvectors"** $\mathrm{ch}^\lambda(\sigma)$: complicated. **(Murnagham-Nakayama formula)**.

♡ : The polynomial convergence lemma

For $j \geq 1$, let us define the *weighted average of characters*

$$A_j(\sigma) := \sum_{\lambda \in \widehat{\mathfrak{S}_n} \,:\, \lambda_1 = n-j} \frac{\mathrm{d}_{\lambda^*}}{j!} \mathrm{ch}^{\lambda}(\sigma). \tag{2}$$

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Fix $j \geq 1$. Then for "almost all" permutations $\sigma \in \mathfrak{S}_n$,

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"almost all" permutations = permutations with a cycle of length $\geq j$.

♡ : Consequence on the profile

We now can deal with $\mathrm{d}_\lambda, s_\lambda$ **and** $\mathrm{ch}^\lambda(\sigma)$.

we have for each j at time $t_c = \frac{1}{2} n \log(n) + cn$,

$$\sum_{\lambda \in \widehat{\mathfrak{S}_n} \,:\, \lambda_1 = n-j} \mathrm{d}_\lambda s_\lambda^{t_c} \mathrm{ch}^\lambda(\sigma) \approx \sum_{\lambda \in \widehat{\mathfrak{S}_n} \,:\, \lambda_1 = n-j} \frac{n^j}{j!} \mathrm{d}_{\lambda^*} \frac{e^{-2jc}}{n^j} \mathrm{ch}^\lambda(\sigma)$$

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so we can rewrite

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Then simple computations and the fact that $\mathrm{Fix}(\cdot) \to \mathrm{Poiss}(1)$ lead to

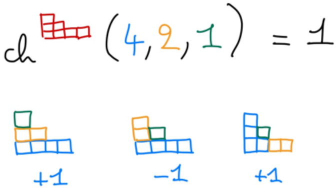
$$\mathrm{d}_n(t_c) \xrightarrow[n\to+\infty]{} f(c) := \mathrm{d}_{\mathrm{TV}}\left(\mathrm{Poiss}\left(1+e^{-2c}\right), \mathrm{Poiss}(1)\right).$$

♡ : The Murnagham-Nakayama formula

Murnagham-Nakayama formula :

Gives an exact formula for the characters $\text{ch}^{\lambda}(\sigma)$: we count how many times we can cover the diagram λ with ribbons corresponding to the cycle structure of σ.

Example 1:



♡ : Playing with the Murnagham-Nakayama formula

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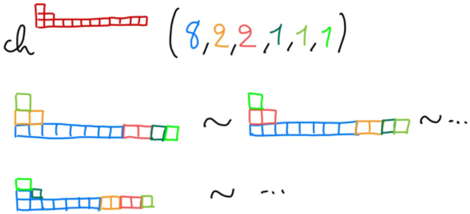

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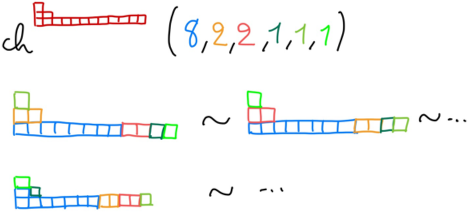

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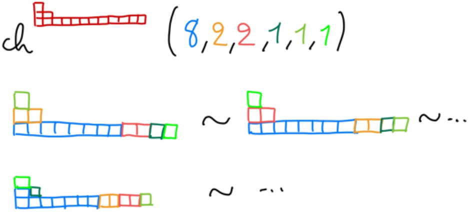

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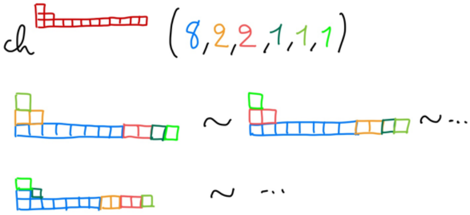

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Then we need to understand how to cover the truncated diagram λ^* of size j with small ribbons → Murnagham-Nakayama for diagrams of size j → characters of $\mathfrak{S}_j$.

♡ : Playing with the Murnagham-Nakayama formula

Case $j = 2$.

$$2! \cdot A_2(\sigma) = \mathrm{ch}^{(n-2,2)}(\sigma) + \mathrm{ch}^{(n-2,1,1)}(\sigma)$$
$$= \left(N_2 + \binom{N_1}{2} - N_1\right) + \left(-N_2 + \binom{N_1}{2} - N_1 + 1\right)$$
$$= 2\binom{\mathrm{Fix}(\sigma)}{2} - 2\mathrm{Fix}(\sigma) + 1.$$

For larger j's, the formulas become rapidly more voluminous. For example

$$\mathrm{ch}^{(n-4,1,1,1,1)}(\sigma) = +\left(\binom{N_1}{4} + N_3N_1 + \binom{N_2}{2} - N_4\right)$$
$$-\left(\binom{N_1}{3} - N_2N_1 + N_3\right)$$
$$+\left(\binom{N_1}{2} - N_2\right)$$
$$-N_1$$
$$+1.$$

$\diamondsuit$: Thank you for your attention!

$$\mathrm{d_{TV}}\left(\mathrm{Poiss}\left(1+e^{-c}\right),\mathrm{Poiss}(1)\right)$$

$$\mathrm{d_{TV}}\left(\mathcal{N}\left(e^{-c},1\right),\mathcal{N}(0,1)\right)$$

$$\mathrm{d_{TV}}\left(\mathrm{Meix}^{+}\left(-e^{-c},0\right)*\delta_{e^{-c}},\mathrm{Meix}^{+}(0,0)\right)$$