A transcendental dynamical degree

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Degrees of one variable rational maps

Let $f(z) = P(z)/Q(z)$ be a rational map of a complex variable z.

- The *degree* of f is the integer $\deg f := \max\{\deg P, \deg Q\}$;
- Alternatively, $\deg(f) = \#f^{-1}(z)$, for any point $z \in \mathbb{P}^1$ in the Riemann sphere.
- Hence $\deg(f^n) = (\deg f)^n$ for all $n \in \mathbb{Z}_{>0}$

Rational maps of more variables

A rational map $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ is one given (in affine coordinates) by $f(x_1, \dots, x_k) = (f_1, \dots, f_k)$, where each component f_j is rational in $x_1, \dots, x_k$.

- Alternatively, f is given in homogeneous coordinates by $f = [F_0, \dots, F_k]$ where F_j are homogeneous polynomials in $x_0, \dots, x_k$ with $\deg F_j$ independent of j.
- Set $\deg(f) := \deg F_j =$ the (first) degree of f.
- Warning 1: $\deg(f)$ is not the topological degree of f. Instead $\deg(f) = \deg f^{-1}(H_1)$, where $H_1 \subset \mathbb{P}^k$ is a hyperplane.
- Warning 2: unlike $k = 1$, rational maps need not be open or even well-defined everywhere when $k \geq 2$. Let $I(f)$ denote the (codimension ≥ 2) 'indeterminate set' of f.

The first dynamical degree

Proposition

If $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ *is rational, then* $\deg(f^{n+m}) \leq \deg(f^m)\deg(f^n)$*.*
Hence the **first dynamical degree**

$$\lambda(f) := \lim \deg(f^n)^{1/n} \in [1, \deg(f)].$$

exists.

Strict inequality can hold in this proposition and $\lambda(f)$ need *not* be an integer.

Problem

How do we compute $\lambda(f)$*? What are it's possible values?*

Why care?

$\lambda(f)$ is a measure of the complexity of iterates of f. If $p \in \mathbb{P}^k$ is a rational point, i.e. in homogeneous coordinates $p = [x_0, x_1, x_2]$, where $x_0, x_1, x_2 \in \mathbb{Z}$ have no common prime factors, and we let $\|p\| := \max |x_j|$ then

$$\lambda_{arith}(f, p) := \limsup(\log \|f^n(p)\|)^{1/n} \leq \lambda(f).$$

Might hope that equality holds for typical p and f.

This idea is sometimes used in e.g.s to compute $\lambda(f)$.

$\lambda(f)$ and other dynamical degrees also control the topological entropy of f.

Bonus slide: Results from smooth dynamics

Let $f : M \to M$ be a C^∞ map on a compact smooth manifold. Topological entropy $h_{top}(f) \geq$ logarithm of:

- (Misiurewicz-Przytycki) the topological degree $d_{top}(f)$ of f.
- (Manning) the spectral radius $\rho(f_*)$ of $f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$.
- (Yomdin) the spectral radius of $f_* : H_*(M, \mathbb{R}) \to H_*(M, \mathbb{R})$ acting on *all* homology groups of M.

In any case, $e^{h_{top}(f)}$ is bounded below by the magnitude of an eigenvalue of an integer matrix.

Bonus slide: All the dynamical degrees

The *jth degree* of a rational map $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ is the quantity

$$d_j(f) := (f^{-1}(H_j) \cdot H_{n-j}),$$

where $H_j \subset \mathbb{P}^k$ is a general codimension j subspace. The *jth dynamical degree* is $\lambda_j(f) := \lim d_j(f^n)^{1/n}$.

Theorem (Gromov,Dinh-Sibony)

$h_{top}(f) \leq \log \max_{1 \leq j \leq k} \lambda_j(f)$.

Remark: equality is known to hold in many cases.

Cases

Endomorphisms: $f : \mathbb{P}^k \to \mathbb{P}^k$ with $I(f) = \emptyset$. Then $\lambda(f) = \deg(f)$.

Monomial maps: (Lin, Favre-Wulcan) $x \mapsto x^A := (x^{A_1}, \ldots, x^{A_k})$ where $A \in \mathrm{Mat}_{k\times k}(\mathbb{Z})$ is a matrix with rows A_j. Then

$$\lambda = \rho(A),$$

where $\rho(A) = |\text{leading eigenvalue of } A|$ is the spectral radius.

Plane polynomial maps: (Favre-Jonsson) $f = (f_1, f_2)$, where f_j are polynomials. Then $\lambda(f)$ is a quadratic integer. Recently generalized to higher dimensions by Dang-Favre.

Plane birational maps: (D-Favre) f is invertible. Then $\lambda(f)$ is an algebraic integer.

More about dynamical degrees

Let $X \to \mathbb{P}^k$ be obtained by blowing up. Can lift $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ to get $f : X \dashrightarrow X$. Say f is *algebraically stable* on X if $\forall n \in \mathbb{Z}_{\geq 0}$,

$$(f^*)^n = (f^n)^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X).$$

Proposition

If f is algebraically stable on X, then $\lambda = \rho(f^*)$.

Proposition (Fornæss-Sibony)

When $k = 2$, f fails to be algebraically stable on X iff there exists a complex curve $C \subset X$ such that $f^n(C) \in I(f)$ for some $n > 0$.

Blowing up your problems

Theorem (D-Favre)

If $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is birational, then there exists a blowup $X \to \mathbb{P}^2$ such that $f : X \dashrightarrow X$ is algebraically stable.

$\exists$ more recent proofs by Lonjou-Urech and Birkett.

Theorem (Bonifant-Fornæss)

The set of all possible first dynamical degrees λ is **countable**.

Monomial maps again

Theorem (Favre)

If $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ has eigenvalues $\zeta, \bar{\zeta} \in \mathbb{C}$ such that $\frac{\arg \zeta}{2\pi} \notin \mathbb{Q}$, then $x \mapsto x^A$ is not algebraically stable on any blowup $X \to \mathbb{P}^2$.

Idea of the proof: $x \mapsto x^A$ restricts to an endomorphism of $\mathbb{C}^* \times \mathbb{C}^*$. Suffices to consider 'toric' blowups $X \to \mathbb{P}^2$.

'Poles' $C_\sigma \subset X \setminus (\mathbb{C}^*)^2$ are indexed by rational rays $\sigma \subset \mathbb{R}^2$ and map forward according to

$$C_\sigma \mapsto C_{A\sigma}.$$

.

...plus an involution

All monomial maps preserve the rational two form $\frac{dx_1 \wedge dx_2}{x_1 x_2}$. Converse isn't true. E.g. the involution

$$g : (x_1, x_2) \mapsto \left(x_1 \frac{x_1 - x_2 - 1}{1 - x_1 - x_2}, x_2 \frac{x_2 - x_1 - 1}{1 - x_1 - x_2} \right)$$

Theorem (D-Lin)

If $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ preserves $\frac{dx_1 \wedge dx_2}{x_1 x_2}$, then there exists a homogeneous, piecewise linear covering $A_f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ such that if $X \to \mathbb{P}^2$ is toric and $C_\sigma \subset X$ is a pole, then $f(C_\sigma) = C_{A_f(\sigma)}$.

Can check $A_g = \mathrm{id}$. Hence for $f(x) := g(x^A)$, we have $A_f = A$; thus f is 'unstabilizable' whenever x^A is.

A power series formula

Theorem (Bell-D-Jonsson)

Let ζ *be a Gaussian integer with* $\frac{\arg\zeta}{2\pi} \notin \mathbb{Q}$, $A = \begin{pmatrix} \operatorname{Re}\zeta & -\operatorname{Im}\zeta \\ \operatorname{Im}\zeta & \operatorname{Re}\zeta \end{pmatrix}$, *and* $f(x) := g(x^A)$. *Then* $t = \lambda(f)^{-1}$ *is the unique positive solution of*

$$\Delta(t) := \sum_{n=1}^{\infty} \deg(x^{A^n}) t^n = 1.$$

Hence $\lambda(f) > \rho(A)$ is no longer an eigenvalue of A.

Proposition

In the theorem,

$$\deg(x^{A^n}) = \max_{\gamma \in \Gamma} \operatorname{Re} \gamma \zeta^n,$$

where $\Gamma = \{-2, \pm 2i, 1 + \pm 2i\}$ *is independent of* A.

The transcendence result

Theorem (Bell-D-Jonsson)

If $t > 0$ is within the radius of convergence of $\Delta(t)$, then $\Delta(t)$ and t can't both be algebraic.

Corollary

$f = g(x^A)$ has transcendental first dynamical degree.

Note that f is *not* invertible; its topological degree is $|\det A| \neq 1$.

Proof of transcendence

For $j \in \mathbb{N}$, let $\gamma(j) \in \Gamma$ be the element that maximizes $\text{Re}\,\gamma\zeta^j$. If ζ^n is nearly real, then $\gamma(j)$ is nearly n-periodic.

For real $t \in (0, |\zeta|^{-1})$ write $\alpha = t\zeta^{-1}$. Then $\Delta(\alpha) = \text{Re}\,\Phi(\alpha)$, where

$$\Phi(z) = \sum \gamma(j) z^j$$

is very well-approximated by $\Phi_n(z) = (1 - z^n)^{-1} \sum_{j=1}^n \gamma(j) z^j$. Indeed

$$|1 - z^n|^2 \,\text{Re}(\Phi(z) - \Phi_n(z)) = \sum_{j>n} (\gamma(j) - \gamma(j-n)) z^j,$$

where most $j > n$ are *n-regular*, i.e. $\gamma(j) = \gamma(j - n)$.

Proof of transcendence

Lemma

Suppose that the continued fraction expansion of $\frac{\arg \zeta}{2\pi}$ *has unbounded coefficients. Then for any* $C > 1$, *there exist arbitrarily large* $n > 0$ *such that all* $j \in (n, Cn]$ *are* n*-regular. So*

$$0 < |1 - \alpha^n|^2 \operatorname{Re}(\Phi(\alpha) - \Phi_n(\alpha)) \lesssim \alpha^{Cn}$$

Now assume to get a contradiction that α and $\Delta(\alpha) = \operatorname{Re}\Phi(\alpha)$ lie in a number field $K \hookrightarrow \bar{\mathbb{Q}}$ (also assumed to contain e.g. Γ).

Theorem (Evertse. Sort of.)

Fix a finite subset $\Gamma \subset K$*, positive integers* ℓ, D *and* $\epsilon > 0$*. If* $x_1, \dots, x_\ell$ *are polynomials in* $\alpha, \bar{\alpha}$ *with coefficients in* Γ *and total degree* $D = \sum \deg x_j$*, then there exist* $B', r > 0$ *such that*

$$|x_1 + \cdots + x_\ell| \geq B' r^{D+\epsilon} \max |x_j|$$

provided no subsum on the left side vanishes

Applying the theorem with $\ell = 2$, $x_1 = |1 - \alpha^{2n}| \operatorname{Re} \Phi(\alpha)$, $x_2 = |1 - \alpha^n|^2 \operatorname{Re} \Phi_n(\alpha)$, we get

$$|1 - \alpha|^2 \operatorname{Re}(\Phi(\alpha) - \Phi_n(\alpha)) \gtrsim r^{2n+\epsilon}.$$

which contradicts the previous estimate when C is large.

QED, except $\arg \zeta / 2\pi$ might (for all we know) be of bounded type.

Bellon-Viallet conjecture fails

Theorem (Bell-D-Jonsson-Krieger)

For all $k \geq 3$, there exist $A \in SL_k(\mathbb{Z})$ and a birational involution $g_k : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ such that $\lambda(g_k(x^A))$ is transcendental.

As before:

- $g(x^A)$ is now birational.
- Get a power series formula for λ similar to the one in the $k = 2$ theorem.
- Again use Evertse's Theorem, continued fraction arguments etc to prove transcendence.

But!

The matrix A is *much* harder to pin down. One needs

- (to deal with bounded type issues) no 'angular resonances' among roots of the characteristic polynomial for A;
- and 'discordance' which requires replacing initial choice of $A \in SL_k(\mathbb{Z})$ by a conjugate.
- (for the power series formula) A-orbits of finitely many integer vectors avoid finitely many rational hyperplanes, which requires replacing A by a power.

At least when $k = 3$, we can use Mathematica to verify all of these things for a particular matrix A with entries bounded by 20.

Further questions

- Just what does the set of all possible dynamical degrees look like?
- Do the 'arithmetic degree's of these rational maps equal the first dynamical degrees? I.e. can they also be transcendental?
- Are there Gaussian integers with irrational arguments of unbounded type?
- Can we somehow circumscribe rational maps that can't be stabilized by e.g. blowing up?

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