The Zariski dense orbit conjecture in positive characteristic

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Notation

- We let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers.
- For any self-map Φ on a variety X and for any integer $n \geq 0$, we let Φ^n be the n-th iterate of Φ (where Φ^0 is the identity map $\mathrm{id} := \mathrm{id}_X$, by definition)
- For a point $x \in X$ we denote by $\mathcal{O}_\Phi(x)$ the orbit of x under Φ, i.e., the set of all $\Phi^n(x)$ for $n \geq 0$.
- We say that x is preperiodic if its orbit $\mathcal{O}_\Phi(x)$ is finite.

The Zariski dense orbit conjecture in characteristic zero

Conjecture (Zhang, Medvedev-Scanlon, Amerik-Campana). Let X be a quasiprojective variety defined over an algebraically closed field K of characteristic 0 and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map. Then one of the following statements must hold:

1. There exists $\alpha \in X(K)$ whose orbit under Φ is well-defined and Zariski dense in X; or
2. There exists a non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.

The conjecture is known to hold in several cases:

- $\Phi : \mathbb{A}^N \longrightarrow \mathbb{A}^N$ is given by the coordinate-wise action of one variable polynomials $(x_1, \ldots, x_N) \mapsto (f_1(x_1), \ldots, f_N(x_N))$.
- Φ is a regular self-map of a semiabelian variety.
- Φ is a group endomorphism of a commutative linear algebraic group.

The conjecture in positive characteristic

If X is any variety defined over $\mathbb{F}_p$, then there exists no non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ invariant under the Frobenius endomorphism $F : X \longrightarrow X$ (corresponding to the field automorphism $x \mapsto x^p$); however, unless $\mathrm{trdeg}_{\mathbb{F}_p} K \geq \dim(X)$, there is no point in $X(K)$ with a Zariski dense orbit in X (each orbit of a point $\alpha \in X(K)$ lives in a subvariety $Y \subseteq X$ defined over $\mathbb{F}_p$ of dimension $\dim(Y) = \mathrm{trdeg}_{\mathbb{F}_p} L$, where L is the minimal field extension of $\mathbb{F}_p$ for which $\alpha \in X(L)$). So, none of the statements (A) or (B) will hold.

The above discussion motivates the next conjecture.

Conjecture 1: Let K be an algebraically closed field of positive transcendence degree over $\overline{\mathbb{F}_p}$, let X be a quasiprojective variety defined over K, and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map defined over K as well. Then at least one of the following three statements must hold:

(A) There exists $\alpha \in X(K)$ whose orbit $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in X.

(B) There exists a non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.

(C) There exist positive integers m and r, there exists a variety Y defined over a finite subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$ such that $\dim(Y) \geq \mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ and there exists a dominant rational map $\tau : X \dashrightarrow Y$ such that

$$\tau \circ \Phi^m = F^r \circ \tau,$$

where F is the Frobenius endomorphism of Y corresponding to the field $\mathbb{F}_q$.

Note that in Conjecture 1, the assumption that K has a positive characteristic over $\overline{\mathbb{F}_p}$ is crucial. Indeed, if X is any variety defined over $\mathbb{F}_q \subset \overline{\mathbb{F}_p}$, endowed with some endomorphism Φ which is also defined over $\mathbb{F}_q$, then each point $\alpha \in X(\overline{\mathbb{F}_p})$ would be preperiodic under the action of Φ and so, the trichotomy from Conjecture 1 cannot hold.

In the case of algebraic tori the following more precise statement of Conjecture 1 holds:

Theorem 1 (Dragos Ghioca and Sina Saleh): Let $N \in \mathbb{N}$ and let K be an algebraically closed field of characteristic p such that $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K \geq 1$. Let $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ be a dominant regular self-map defined over K. Then at least one of the following statements must hold:

(A) There exists $\alpha \in \mathbb{G}_m^N(K)$ whose orbit under Φ is Zariski dense in $\mathbb{G}_m^N$.

(B) There exists a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.

(C) There exist positive integers m and r, a connected algebraic subgroup Y of $\mathbb{G}_m^N$ (defined over a finite field $\mathbb{F}_q$) of dimension at least equal to $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ and a dominant regular map $\tau : \mathbb{G}_m^N \longrightarrow Y$ such that

$$\tau \circ \Phi^m = F^r \circ \tau, \tag{1}$$

where F is the usual Frobenius endomorphism of Y induced by the field automorphism $x \mapsto x^q$.

Condition (C)

Let $d = \operatorname{trdeg}_{\overline{\mathbb{F}_p}} K$ and suppose that condition (C) holds for some $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ defined over K. We let Φ be the composition of a translation with the group endomorphism

$$\vec{x} \mapsto \vec{x}^A,$$

where $A \in M_{N,N}(\mathbb{Z})$. Then condition (C) is equivalent with saying that there exist $d+1$ distinct jordan blocks in the Jordan normal form of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{d+1}$ such that

$$\lambda_1^m = \lambda_2^m = \cdots = \lambda_{d+1}^m = p^r,$$

for some positive integers m, r.

Examples

Fix $K = \overline{\mathbb{F}_p(t)}$ and $p \geq 3$.

Example 1. Let $\Phi : \mathbb{G}_m^3 \longrightarrow \mathbb{G}_m^3$ be the endomorphism given by $(x, y) \mapsto (x^2, y^4)$. Then, the orbit of (t, t) under Φ consists of the points $\{(t^{2^n}, t^{4^n}) : n \in \mathbb{N}\}$ and must be Zariski dense. The reason is that the height of the second coordinate increases much rapidly than the first coordinate. So, condition (A) must hold. Also, note that none of the conditions (B) or (C) can hold in this case.

Example 2. Let $\Phi : \mathbb{G}_m^2 \longrightarrow \mathbb{G}_m^2$ be the endomorphism given by $(x, y) \mapsto (x, y^2)$. In this case, Φ leaves the projection map $\pi_1 : \mathbb{G}_m^2 \longrightarrow \mathbb{G}_m$ invariant. Hence, condition (B) must hold in this case. However, none of the conditions (A) or (C) can hold.

Example 3. Let $\Phi : \mathbb{G}_m^2 \longrightarrow \mathbb{G}_m^2$ be the endomorphism given by $(x, y, z) \mapsto (x^2, y^p, z^p)$. In this case, Φ induces the Frobenius on the last two coordinates. So, condition (C) must hold in this case. Also, none of the conditions (A) and (B) can hold.

Example 4. Let $\Phi : \mathbb{G}_m^3 \longrightarrow \mathbb{G}_m^3$ be the endomorphism given by $(x, y, z) \mapsto (x, y^{p^2}, z^{p^2})$. So, Φ induces the Frobenius on the last two coordinates. At the same time, Φ leaves the projection $\pi_1 : \mathbb{G}_m^3 \longrightarrow \mathbb{G}_m$ invariant. So, conditions (B) and (C) will hold simultaneously. This means that conditions (B) and (C) from Theorem 1 are not mutually exclusive.

Remark. Note that although conditions (B) and (C) are not mutually exclusive, condition (A) is mutually exclusive from (B) and (C).

Theorem 2 (Dragos Ghioca and Sina Saleh): Let K be an algebraically closed field of characteristic p such that $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K \geq 1$ and let G be a split semiabelian variety defined over $\overline{\mathbb{F}_p}$. Let $\Phi : G \longrightarrow G$ be a dominant regular self-map defined over K. Then at least one of the following statements must hold.

(A) There exists $\alpha \in G(K)$ whose orbit under Φ is Zariski dense in G.

(B) There exists a non-constant rational function $f : G \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.

(C) There exist positive integers m and r, a semiabelian variety Y defined over a finite subfield $\mathbb{F}_q$ of K of dimension at least equal to $\mathrm{trdeg}_{\overline{\mathbb{F}_p}} K + 1$ and a dominant regular map $\tau : G \longrightarrow Y$ such that

$$\tau \circ \Phi^m = F^r \circ \tau, \tag{2}$$

where F is the usual Frobenius endomorphism of Y induced by the field automorphism $x \mapsto x^q$.

Some useful definitions

Definition (NFP Matrices). Let C be a simple semiabelian variety (meaning that it is isomorphic to $\mathbb{G}_m$ or a simple abelian variety) defined over $\mathbb{F}_q \subset \overline{\mathbb{F}_p}$. Let $L_C := \mathrm{End}(C) \otimes \mathbb{Q}$ and let F_C be the image of the Frobenius endomorphism of C in E_C. For any $n \in \mathbb{N}$, a matrix $A \in M_{n,n}(L_C)$ is called an NFP (No Frobenius Power) matrix whenever the minimal polynomial $P(x)$ of A over $\mathbb{Q}(F_C)$ has no roots that are multiplicatively dependent with respect to F_C (i.e., no root λ of $P(x)$ satisfies $\lambda^m = F_C^k$ for some integers m and k, not both equal to 0).

Definition (Reduced semiabelian varieties). We define a split semiabelian variety G to be **reduced** if G is isomorphic to

$$\prod_{i=1}^{r} C_i^{k_i}, \tag{3}$$

where $k_1, \ldots, k_r \in \mathbb{N}$ and $C_1, \ldots, C_r$ are simple semiabelian varieties that are pairwise non-isogenous.

Some useful definitions

Definition (F-sets). Let K be a field of positive characteristic, let G be a semiabelian variety defined over $\mathbb{F}_q \subset \overline{\mathbb{F}_p}$, let F be the Frobenius endomorphism of G, and let $\Gamma \subseteq G(K)$ be a finitely generated $\mathbb{Z}[F]$-module.

(a) By a *sum of F-orbits in* Γ we mean a set of the form

$$C(\gamma, \alpha_1, \dots, \alpha_m; k_1, \dots, k_m) := \left\{ \gamma + \sum_{j=1}^{m} F^{k_j n_j}(\alpha_j) \colon n_j \in \mathbb{N}_0 \right\} \subseteq \Gamma$$

where $\gamma, \alpha_1, \dots, \alpha_m$ are some given points in $G(K)$ and $k_1, \dots, k_m$ are some given positive integers.

(b) An *F-set* in Γ is a set of the form $C + \Gamma'$ where C is a sum of F-orbits in Γ, and $\Gamma' \subseteq \Gamma$ is a submodule, while in general, for two sets $A, B \subset G(K)$, $A + B$ is simply the set $\{a + b \colon a \in A,\ b \in B\}$.

The F-structure Theorem

Theorem (Rahim Moosa, Thomas Scanlon). Let G be a semiabelian variety defined over $\mathbb{F}_q$, let $\mathbb{F}_q \subset K$ be an algebraically closed field, let $V \subset G$ be a subvariety defined over K and let $\Gamma \subset G(K)$ be a finitely generated $Z[F]$-submodule. Then $V(K) \cap \Gamma$ is a finite union of F-sets contained in Γ.

Useful reductions in our proofs

- It suffices to prove conjectures 1 and 2 after replacing Φ by an iterate.
- We can also replace Φ by a conjugate of the form $\Psi \circ \Phi \circ \Psi^{-1}$ for some automorphism Ψ of X. In the case of semiabelian varieties, any endomorphism is of the form $\tau_{\vec{\beta}} \circ \varphi$ where $\tau_{\vec{\beta}}$ is a translation and φ is a group endomorphism. In this case it is useful to let the automorphism Ψ be a suitable translation to simplify the translation $\tau_{\vec{\beta}}$ of Φ as much as possible.

General strategy for our proof

The proofs for theorems 1 and 2 are similar. In both cases, there are three extreme cases to consider. We will show that a given endomorphism Φ decomposes as product of these extreme cases. However, in contrast to the case of $\mathbb{G}_m^N$, choosing a point with a Zariski dense orbit becomes significantly harder due to the fact that the split semiabelian variety G can have distinct simple semiabelian varieties as its components.

More precisely, we can show that it suffices to show the theorem in the case where Φ is an endomorphism of a reduced semiabelian variety and it decomposes as the product of the following three extreme cases (In all of the following cases C is a simple semiabelian variety defined over $\mathbb{F}_q \subset \overline{\mathbb{F}_p}$):

Case 1. $\Phi : C^k \longrightarrow C^k$ is an endomorphism corresponding to an NFP matrix in $M_{k,k}(\mathrm{End}(C))$.

General strategy for our proof

Case 2. $\Phi : C^k \longrightarrow C^k$ is a group endomorphism corresponding to a matrix of the form

$$J_{F_C^{n_1},i_1} \bigoplus J_{F_C^{n_2},i_2-i_1} \bigoplus \cdots \bigoplus J_{F_C^{n_\ell},i_\ell-i_{\ell-1}}$$

in $M_{k,k}(\mathrm{End}(C))$.

Case 3. $\Phi : C^k \longrightarrow C^k$ is the composition of a translation with a unipotent group endomorphism of C^k.

After reducing the problem to these cases we assume that none of the conditions (B) or (C) hold and we will prove the existence of a Zariski dense orbit. We will first discuss the general strategy in dealing with Case 1.

The general strategy for Case 1

Let $\Phi : C^k \longrightarrow C^k$ be a finite-to-finite map corresponding to an NFP matrix in $M_{k,k}(\mathrm{End}(C))$. Then, for any

$$\vec{x} = (x_1, \ldots, x_k) \in C^k,$$

where $x_1, \ldots, x_k$ are linearly independent over $\mathrm{End}(C)$, any orbit of $\vec{x}$ under Φ must be Zariski dense in C^k. To prove this fact we need to use the Moosa-Scanlon F-structure theorem. Using the F-structure theorem, the proof of the aforementioned claim reduces to a finite number of polynomial-exponential equations of the form

$$\lambda^n = \sum_{i=1}^{k} a_i F_C^{n_i},$$

where λ is multiplicatively independent with respect to F_C. These equations are a special case of Laurent's famous theorem.

The general strategy for Case 2

Using the F-structure theorem, Case 2 will reduce to an equation of the form

$$F_C^{n\cdot\gamma_1} R_1(n) + \cdots + F_C^{n\cdot\gamma_k} R_k(n) = c + \sum_{i=1}^{t} b_i F_C^{\delta_i n_i},$$

such that for some subset S of $\mathbb{N}$ of positive upper asymptotic density, for every $n \in S$ the above equation has a solution. However, due to an upper bound on the number of solutions to these equations proven by Dragos Ghioca, Alina Ostafe, Sina Saleh, and Igor Shparlinski this implies that the polynomials $R_1, \ldots, R_k$ must be constant. This result further reduces the problem to the case where $\Phi : \prod_{j=1}^{r} C^{k_j} \longrightarrow \prod_{j=1}^{r} C^{k_j}$ is given by

$$(x_1, x_2, \ldots, x_r) \mapsto (F_{C^{k_1}}^{\ell_1}(x_1), F_{C^{k_2}}^{\ell_2}(x_2), \ldots, F_{C^{k_r}}^{\ell_r}(x_r)),$$

where $\ell_1, \ldots, \ell_r$ are distinct positive integers and $x_j \in C^{k_j}$ for $j = 1, \ldots, r$.

The general strategy for Case 2

The strategy for finding a Zariski dense orbit in this case is as follows. Let α_j be a generic point of C^{k_j} over $\overline{\mathbb{F}_p}$ which is defined over K for every $j = 1, \ldots, r$. Then, one can show that the orbit of $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ under Φ is Zariski dense in $\prod_{j=1}^r C^{k_j}$. Here are some examples.

Fix $K = \overline{\mathbb{F}_p(t_1, t_2)}$.

Example 5. Let $\Phi : \mathbb{G}_m^4 \longrightarrow \mathbb{G}_m^4$ be the endomorphism given by $(x, y, z, t) \mapsto (x^p, y^p, z^{p^2}, t^{p^2})$. Then, the orbit of (t_1, t_2, t_1, t_2) under Φ will be Zariski dense.

Example 6. Let $\Phi : \mathbb{G}_m^4 \longrightarrow \mathbb{G}_m^4$ be the endomorphism given by $(x, y, z, t) \mapsto (x^p, y^p, z^{p^2}, t^{p^4})$. Then, the orbit of (t_1, t_2, t_1, t_1) under Φ will be Zariski dense.

The unipotent case

Fix $K = \overline{\mathbb{F}_p(t)}$ and a semiabelian variety C defined over $\overline{\mathbb{F}_p}$.

Example 7. Let $\Phi : C^4 \longrightarrow C^4$ be the endomorphism given by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, \beta_1 + x_2, x_3 + x_4, \beta_2 + x_4).$$

Then, one of the following two statements must hold:

(*i*) β_1 and β_2 are linearly independent over $\mathrm{End}(C)$. In this case, if we choose $\alpha_1, \ldots, \alpha_4$ such that the elements $\alpha_1, \ldots, \alpha_4, \beta_1, \beta_2$ are linearly independent over $\mathrm{End}(C)$, then the orbit of $(\alpha_1, \ldots, \alpha_4)$ under Φ will be Zariski dense in C^4.

(*ii*) β_1 and β_2 are linearly dependent over $\mathrm{End}(C)$. So, there exist $\sigma_1, \sigma_2 \in \mathrm{End}(C)$ such that $\sigma_1(\beta_1) + \sigma_2(\beta_2) = 0$. Then, Φ leaves invariant the endomorphism $C^4 \longrightarrow C$ given by

$$(x_1, x_2, x_3, x_4) \mapsto \sigma_1(x_2) + \sigma_2(x_4),$$

and therefore, there does not exist any Zariski dense orbit.

The non-isotrivial case

We expect our methods to extend to the case of the non-isotrivial abelian varieties as well. However, the strengthening of the Moosa-Scanlon F-structure theorem which describes the intersection of a non-isotrivial abelian variety with a finitely generated group is more complicated and therefore, the non-isotrivial case will introduce some extra complications in the proof.

We also expect that extending our methods to the general case of semiabelian will be significantly more difficult than the split case. This is because in the case of split semiabelian varieties it is easier to understand the dynamics of the endomorphisms.

Finally, The general case of Conjecture 1 for an arbitrary quasi-projective variety X is expected to be as difficult as the general case of the Zariski dense orbit conjecture in characteristic zero.

Thank You!