SYZ MIRROR SYMMETRY AND ITS APPLICATIONS

CHEOL-HYUN CHO, YOOSIK KIM, SIU-CHEONG LAU, YAN-LUNG LI, YU-SHEN LIN, YAT-HIN SUEN, XIAO ZHENG

1. Mirror construction of quiver stacks and applications

A quiver algebra is a quotient of the path algebra of a directed graph by a two-sided ideal of relations. It has very interesting and deep applications in gauge theory, representation theory and resolution of singularities. In [JL07], Jeffreys-Lau developed the notion of quiver near-algebras and their moduli spaces of framed representations, motivated from the applications in machine learning. There are three main innovations. First, *near-rings and their noncommutative differential forms* were constructed to incorporate non-linear operations that do not occur in usual quiver theory. Second, *canonical metrics in quiver algebras* were found which are crucial in real applications. Third, *toric and tropical geometry was employed to prove the universal approximation theorem* for the non-linear softmax function.

In the workshop, we have studied mirror construction of quiver algebras. In [CHL21], Cho-Hong-Lau constructed quiver algebras as noncommuative mirrors of a symplectic manifold by using nc Maurer-Cartan spaces of Lagrangian immersions. In the recent work of Lau-Nan-Tan [LNT22], a general construction scheme of mirror functors for algebroid stacks of quiver algebras were developed. Such a construction is very useful in building the *connection between quiver mirrors and SYZ mirrors*. In particular, the notion of *local charts of a quiver algebra* was found, and an A_∞ natural equivalence were built between the two mirror functors.

As an application, Siu-Cheong Lau and Xiao Zheng have discussed about *quiver mirrors of affine ADE surfaces*. In the workshop, we have focused on the $\hat{D}_4$-type Lagrangian immersion. Our discussion was very fruitful. We have found its quiver mirror by solving the Maurer-Cartan equation for nc deformations. The key non-trivial ingredients are constant disc bubbles at the immersed sectors, which were computed by our previous technique using wall-crossing in [HKL].

Furthermore, by inventing the notion of *higher-rank localization of quiver algebras*, we have successfully constructed local charts of the quiver mirror, which glue up to the resolution of a $\hat{D}_4$-singularity. It will be very interesting to combine with existing techniques of cluster algebras, so that we can construct its SYZ mirror by starting with a single Lagrangian immersion.

Siu-Cheong Lau has also introduced his ongoing work on the *triality between framed Fukaya category, framed torsion-free sheaves and stable quiver representations*. The classical work of Kronheimer-Nakajima has found the deep relation between Yang-Mills instantons, framed sheaves and quiver representations over ALE gravitational instantons. In this ongoing work, Lau-Tan has found mirror functors that transform framed Lagrangians to framed sheaves and quiver representations respectively.

2. Fukaya-Seidel category without Morsification

Cheol-Hyun Cho has introduced his recent work on *orbifold Fukaya-Seidel category*. Given a polynomial W, one can consider a Fukaya-Seidel category of W. This is defined by considering $W : \mathbb{C}^n \to \mathbb{C}$ as a fibration, and with a suitable perturbation W_ϵ of W so that W_ϵ is complex Morse, one can consider a direct Fukaya category of vanishing cycles in the Milnor fiber of W_ϵ which depends on the choice of vanishing paths. A derived category of this Fukaya category is independent of the Morsification as well as the choice of vanishing paths. This construction has been very powerful to study symplectic geometry of singularities.

There is a special class of polynomials, called invertible polynomials (roughly, the matrix of exponents of monomials in a polynomial is invertible). There is a mirror symmetry between symplectic geometry of an invertible polynomial and a complex geometry of a transpose invertible polynomial, so called Berglund Hübsch mirror symmetry. In fact, the precise mirror symmetry should necessarily involve an "orbifold" version of Fukaya-Seidel category, which was ill-defined as the Morsification W_ϵ no longer have the symmetry that W has. We overcome this difficulty by constructing a different Fukaya category associated to W without Morsification, but rather using the monodromy representation and the non-compact Lagrangians. This may be viewed as a categorification of variation operator.

To define the new Fukaya category, one has to first define a symplectic cohomology class representing the monodromy, which we call the monodromy orbit. We consider the quantum cap action of the monodromy orbit on the wrapped Fukaya category of the Milnor fiber of W, which is a Liouville manifold. The cone of the quantum cap action categorifies the variation operator on middle dimensional relative homology group of the Milnor fiber. This associates a certain cohomology groups for each pair of Lagrangians. Then we need to define an A_∞ structure, including products. This can be achieved by introducing pseudo-holomorphic popsicles, with insertions of monodromy orbits in the interior.

Abouzaid and Seidel used popsicle maps for various constructions in symplectic geometry. The popsicle maps that we introduce here needs different compactifications due to interior insertions. *This compactification involves a new codimension-one sphere bubbling phenomenon* due to the alignment of conformal structures of the popsicles. We prove that such sphere bubblings of codimension one does not occur for log Fano or log Calabi-Yau cases, or for two real dimensional Milnor fibers. Using the new Fukaya category, we can prove *Berglund Hübsch mirror symmetry of an invertible curve singularity in full generality.*

3. Scattering diagram and Lagrangian Floer theory

Yu-Shen Lin has introduced his recent work with S. Bardwell-Evans, M.-W. Cheung and H. Hong on *scattering diagrams for log Calabi-Yau surfaces* [BCHL].

Given a Looijenga pair (Y, D), we are interested in how to understand the scattering diagram constructed by Gross-Hacking-Keel [GK15] via Lagrangian Floer theory. Furthermore, we want to compute the superpotential for (Y, D). From Gross-Hacking-Keel [GK15], (Y, D) has a toric model $(\bar{Y}, \bar{D})$ after suitably blow-up on the corners of D. The Floer theory of Lagrangians in Y may be complicated due to the sphere bubbles when D contains a component with a negative Chern number. So the naive idea is to consider $X = Y \setminus D$ instead and then special Lagrangians in

X would only bound Maslov index zero discs, which are more well-studied nowadays. However, X is non-compact and the geometry is not even complete. Thus, the compactness of the relevant moduli spaces of pseudo-holomorphic discs becomes an issue. The solution is to consider admissible Lagrangians which are preimages of moment map fibres in the toric model. Using an observation of Hong-Lin-Zhao [HLZ] and algebraic geometry of complex surfaces, one can show that the relevant moduli spaces of holomorphic discs with boundary on admissible Lagrangians are always compact and thus Floer theory still apply to admissible Lagrangians. Here using the integrable complex structure is crucial to the argument. It is proved by Lin [Lin] that Floer theory of special Lagrangian fibrations in Calabi-Yau surfaces carry a natural scattering diagram. It suffices to match the initial rays and their wall functions with those of the canonical scattering diagram of Gross-Hacking-Keel. One can easily use the local model of Auroux [Aur] to create an initial ray for each of the non-toric blow-up $(Y,D) \to (\bar{Y},\bar{D})$. However, since the Floer theory only applies to admissible Lagrangians which do not cover the whole X, a priori there might be additional initial walls entering the base parametrized the admissible Lagrangians. The key idea is to prove a holomorphic to tropical correspondence: every holomorphic disc has a corresponding tropical disc which the unbounded edges reflect the exceptional divisors intersecting with the holomorphic discs. Such holomorphic to tropical correspondence provides certain strict obstructions to the existence of holomorphic discs with boundaries on admissible Lagrangians. This implies that the scattering diagram from the Maslov index zero discs coincides with the canonical scattering diagram of Gross-Hacking-Keel. As a consequence, one also gets certain folklore conjecture that the open Gromov-Witten invariants for Lagrangian boundary conditions near the boundary divisor coincide with the logarithmic Gromov-Witten invariants of $\mathbb{A}^1$-curves of (Y,D) via tropical geometry. In the case when Y is semi-Fano, the scattering diagram of the Maslov index zero discs determines the superpotential of Y.

4. Mirror construction for quadrics

Yoosik Kim has introduced the *SYZ construction of Lie-theoretical mirrors*. He has focused on quadrics in the workshop.

Rietsch constructed a Lie-theoretical mirror of homogeneous space based on Peterson's discovery that the quantum cohomology ring is realized as a coordinate ring of a certain subvariety of the Langlands dual flag variety. When $G/P \simeq \mathrm{SO}_n(\mathbb{C})/P$ is a quadrics, Pech–Rietsch–Williams found a malleable expression of the Lie theoretical mirror and described its cluster structure. A natural question is how to construct PRW mirror via SYZ mirror symmetry. A joint project with Cho, Hong and Lau in progress aims at constructing the PRW mirror by using Lagrangian Floer theory.

We have constructed a Lagrangian torus fibration on quadrics whose SYZ mirrors are isomorphic to the PRW mirrors. To illustrate the construction, let us focus on the case where $X = \mathcal{Q}_5$, which has an affine chart of the form

$$x_1 x_{\underline{1}} - x_2 x_{\underline{2}} + x_3 x_{\underline{3}} - 1 = 0.$$

It has a Hamiltonian T^3-action given by $(\theta_i \star \mathbf{x} \mapsto e^{\sqrt{-1}\theta_i} x_i, e^{-\sqrt{-1}\theta_i} x_{\underline{i}})$.

The Lagrangian fibration that we construct has discriminant loci located at $x_1 x_{\underline{1}} = 0$, $x_2 x_{\underline{2}} = 0$, and $x_3 x_{\underline{3}} = 0$. The expected theorem of our project is that the mirror of a product of immersed sphere and torus occurring at the intersection between $x_1x_{\underline{1}} = 0$ and $x_2x_{\underline{2}} = 0$ is the PRW mirror of the quadrics. Moreover, the Lagrangian tori in the four chambers around the intersection produce the cluster charts of the PRW mirror.

5. Mirrors of Lagrangian multi-sections

Yat-Hin Suen has talked about his recent work with Yong-Guen Oh on the *Lagrangian realization problem*, namely, whether mirrors of toric vector bundles can be realized by Lagrangian multi-sections.

Suen demonstrated a sketch of proof of the realization problem for rank 2 toric vector bundles on complex projective plane and certain objects inside the (smooth locus of the) moduli space of toric vector bundles over smooth complete toric surfaces with equivariant Chern classes determined by that of the the toric tangent sheaf. One future research direction is hinted by the second case. On a toric surface, the moduli space of rank 2 toric vector bundles admits a natural compactification by toric-equivariant torsion free sheaves. It is then very natural to ask what Lagrangian objects should be included in the mirror side in order to compactify the moduli of Lagrangian multi-sections with fixed asymptotic conditions so that the two compactified moduli spaces are isomorphic under the mirror transformation.

Suen also discussed with Yu-Shen Lin about the relation between (deformed) Hermitian-Yang-Mills metrics and special Lagrangians. Although the correct metric equation for general holomorphic vector bundles is still unknown, the explicit construction of the mirror Lagrangian of $T_{\mathbb{P}^2}$ may give us a hope to understand the tempting relation of instantons and the Fubini-Study metric. We also discussed mirror symmetry of del Pezzo surfaces and geometric quantization.

6. NC SYZ mirrors for multiplicative hypertoric varieties

Xiao Zheng gave a talk about *NC mirrors of multiplicative hypertoric varieties.* It is a joint work in progress with Siu-Cheong Lau and Ziming Ma.

A multiplicative hypertoric variety is given by a quasi-Hamiltonian reduction of $(T^*\mathbb{C}^n)^\circ = \{(z, w) \in \mathbb{C}^{2n} | z_i w_i + 1 \neq 0\}$ by a subtorus K in $\mathrm{T} = (\mathbb{C}^\times)^n$. Namely, for a pair $(\alpha, \beta) \in \mathfrak{k}_{\mathbb{R}} \times \mathrm{K}$, the multiplicative hypertoric variety $\mathfrak{M}(\alpha, \beta)$ is the GIT quotient

$$\mathfrak{M}(\alpha, \beta) := \mu_K^{-1}(\beta) /\!\!/_\alpha \mathrm{K},$$

where μ_K is the torus-valued moment map. $\mathfrak{M}(0, 0)$ is a symplectic singularity, while $\mathfrak{M}(\alpha, 0)$ and $\mathfrak{M}(0, \beta)$ are resolution and smoothing of $\mathfrak{M}(0, 0)$, respectively, for generic α and β.

The residual torus-action gives us a natural Lagrangian torus fibration on $\mathfrak{M}(0, \beta)$ with the central fiber $\mathbb{L}$ a Lagrangian immersion which is in particular a skeleton. Using the recipe of [CHL21], the nc deformation space of $\mathbb{L}$ gives a quiver algebra $\mathbb{A}$ which is a nc resolution of $\mathfrak{M}(0, 0)$.

The quiver relations for $\mathbb{A}$ are very interesting and are deeply related with the holomorphic moment map of the quiver. We have discussed how to deduce these relations by using quasi-isomorphisms between the immersion $\mathbb{L}$ and smooth SYZ Lagrangian tori. This gives a nice application of the mirror construction of quiver stacks in [LNT22].

7. Lagrangian correspondence for weak MC deformations

Yan-Lung Li has talked about *Lagrangian correspondence* (also known as canonical relation). It is a seminal concept in symplectic geometry that serves as the correct notion of "morphism" between symplectic manifolds with different dimensions. It provides a way of formulating functoriality of invariants associated to symplectic manifolds and their Lagrangian submanifolds, which by now have wide-ranging applications in symplectic geometry and symplectic topology.

In the workshop, Yan-Lung Li presented the functoriality of weak Maurer-Cartan spaces of Lagrangian correspondences under geometric composition. This generalizes a result of Fukaya [Fuk] in case of (strict) Maurer-Cartan spaces. More precisely, we constructed a composition map between the weak Maurer-Cartan spaces. In particular, the disc potentials involved satisfy an identity.

Siu-Cheong Lau and Yan-Lung Li have discussed *Lagrangian correspondence in the equivariant setting*. The aim is to explain the relation of SYZ mirrors of a symplectic manifold and its symplectic quotient. This plays a crucial role in Hori-Vafa mirror [HV00] and the proposal of Teleman [Tel14]. In particular, we have discussed how to *extract the mirror map from equivariant Lagrangian correspondence*. First, we will classify holomorphic discs bounded by Lagrangian correspondences in the toric setup. Second, we will prove weak unobstructedness of Lagrangian correspondence. Third, we find possible Maslov-zero stable discs in the equivariant space that contribute to non-trivial mirror maps for toric semi-Fano manifolds.

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