Multi-taper on domains
and
risk rates for the spectral norm

José Luis Romero

University of Vienna
and Austrian Academy of Sciences

universität wien

FWF
Der Wissenschaftsfonds.

Overview

- Sample complexity of Thomson's estimator
- MSE bounds and benchmarks for Gaussian series
- Number of tapers and bandwidth
- Multi-taper on domains
- Computation of the Slepian tapers versus computation of the multi-taper

Classical multi-taper

- Stationary (real) time-series: $X(k), k \in \mathbb{Z}$
- Observations: $X(0), \ldots, X(N-1)$

Classical multi-taper

- Stationary (real) time-series: $X(k), k \in \mathbb{Z}$
- Observations: $X(0), \ldots, X(N-1)$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k=0}^{N-1} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]$

Estimate for the spectral density $S(\xi) = \sum_{k \in \mathbb{Z}} \mathbb{E}\big[X(k)X(0)\big]\, e^{2\pi i k \xi}$

Classical multi-taper

- Stationary (real) time-series: $X(k), k \in \mathbb{Z}$
- Observations: $X(0), \ldots, X(N-1)$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k=0}^{N-1} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]$

Estimate for the spectral density $S(\xi) = \sum_{k\in\mathbb{Z}} \mathbb{E}\big[X(k)X(0)\big]\, e^{2\pi i k \xi}$

Parameters: K, W

Questions:
- Choice of parameters
- MSE and sample complexity
- Optimality?

Error metric

Error metric = uniform norm:

$$\|S - \widehat{S}\|_\infty = \max_{\xi \in \mathbb{R}} |S(\xi) - \widehat{S}(\xi)|$$

Error metric = uniform norm:

$$\|S - \widehat{S}\|_\infty = \max_{\xi \in \mathbb{R}} |S(\xi) - \widehat{S}(\xi)|$$

This is the spectral error for covariance estimation

$$\max\left\{|\lambda| : \lambda \in \sigma(\Sigma - \widehat{\Sigma})\right\} = \max_{\|a\|_2 \le 1} \left\|\Sigma a - \widehat{\Sigma} a\right\|_2$$

$\Sigma_{jk} = \mathbb{E}[X(k)X(0)]$ true covariance; $\widehat{\Sigma}_{jk}$ estimated covariance

Mean-squared error:

$$\mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right]$$

Error metric = uniform norm:

$$\|S - \widehat{S}\|_\infty = \max_{\xi \in \mathbb{R}} |S(\xi) - \widehat{S}(\xi)|$$

This is the spectral error for covariance estimation

$$\max\left\{|\lambda| : \lambda \in \sigma(\Sigma - \widehat{\Sigma})\right\} = \max_{\|a\|_2 \leq 1} \left\|\Sigma a - \widehat{\Sigma} a\right\|_2$$

$\Sigma_{jk} = \mathbb{E}[X(k)X(0)]$ true covariance; $\widehat{\Sigma}_{jk}$ estimated covariance

Mean-squared error:

$$\mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right]$$

As opposed to, say, $\sup_\xi \mathbb{E}[|S(\xi) - \widehat{S}(\xi)|^2]$

Error metric = uniform norm:

$$\|S - \widehat{S}\|_\infty = \max_{\xi \in \mathbb{R}} |S(\xi) - \widehat{S}(\xi)|$$

This is the spectral error for covariance estimation

$$\max \left\{ |\lambda| : \lambda \in \sigma(\Sigma - \widehat{\Sigma}) \right\} = \max_{\|a\|_2 \leq 1} \left\| \Sigma a - \widehat{\Sigma} a \right\|_2$$

$\Sigma_{jk} = \mathbb{E}[X(k)X(0)]$ true covariance; $\widehat{\Sigma}_{jk}$ estimated covariance

Mean-squared error:

$$\mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right]$$

As opposed to, say, $\sup_\xi \mathbb{E}[|S(\xi) - \widehat{S}(\xi)|^2]$

Assumptions: X is Gaussian, S is C^2

Error metric = uniform norm:

$$\|S - \widehat{S}\|_\infty = \max_{\xi \in \mathbb{R}} |S(\xi) - \widehat{S}(\xi)|$$

This is the spectral error for covariance estimation

$$\max\left\{|\lambda| : \lambda \in \sigma(\Sigma - \widehat{\Sigma})\right\} = \max_{\|a\|_2 \le 1} \left\|\Sigma a - \widehat{\Sigma} a\right\|_2$$

$\Sigma_{jk} = \mathbb{E}[X(k)X(0)]$ true covariance; $\widehat{\Sigma}_{jk}$ estimated covariance

Mean-squared error:

$$\mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right]$$

As opposed to, say, $\sup_\xi \mathbb{E}[|S(\xi) - \widehat{S}(\xi)|^2]$

Assumptions: X is Gaussian, S is C^2

Budget: N observations

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Thomson's MT achieves the optimal rate; R-, Speckbacher, 2022)

$$\sup_{S} \mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

with $K = NW$ *and* $K \asymp N^{4/5} \log(N)^{1/5}$

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Thomson's MT achieves the optimal rate; R-, Speckbacher, 2022)

$$\sup_{S} \mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

with $K = NW$ *and* $K \asymp N^{4/5} \log(N)^{1/5}$

Explicit MSE bound with $K = NW$ and K arbitrary

– Choices $K < NW$?

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Thomson's MT achieves the optimal rate; R-, Speckbacher, 2022)

$$\sup_{S} \mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

with $K = NW$ *and* $K \asymp N^{4/5} \log(N)^{1/5}$

Explicit MSE bound with $K = NW$ and K arbitrary

– Choices $K < NW$? High dynamic ranges

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Thomson's MT achieves the optimal rate; R-, Speckbacher, 2022)

$$\sup_{S} \mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

with $K = NW$ *and* $K \asymp N^{4/5} \log(N)^{1/5}$

Explicit MSE bound with $K = NW$ and K arbitrary

– Choices $K < NW$? High dynamic ranges

Plenty of anecdotal evidence

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Thomson's MT achieves the optimal rate; R-, Speckbacher, 2022)

$$\sup_{S} \mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

with $K = NW$ *and* $K \asymp N^{4/5} \log(N)^{1/5}$

Explicit MSE bound with $K = NW$ and K arbitrary

– Choices $K < NW$? High dynamic ranges
Plenty of anecdotal evidence

– **Wish / challenge**: a formal model showing the optimality of a choice $K < NW$

Theorem (Benchmark; Cai, Ren, Zhou, 2013)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

Theorem (Thomson's MT achieves the optimal rate; R-, Speckbacher, 2022)

$$\sup_{S} \mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \asymp \left(\frac{\log N}{N}\right)^{4/5}, \qquad \|S\|_{C^2} \leq 1$$

with $K = NW$ *and* $K \asymp N^{4/5} \log(N)^{1/5}$

Explicit MSE bound with $K = NW$ and K arbitrary

– Choices $K < NW$? High dynamic ranges

Plenty of anecdotal evidence

– **Wish / challenge**: a formal model showing the optimality of a choice $K < NW$

– S. Karnik, J. Romberg, and M. A. Davenport,

"Thomson's multitaper method revisited." IEEE Trans. IT, to appear (see arxiv)

The bias of the MT is

$$\mathbb{E}[\widehat{S}^{\mathrm{mt}}(\xi)] - S(\xi) = \int S(\xi - \xi')\rho(\xi')\, d\xi'$$

with the spectral window $\rho(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k=0}^{N-1} S_j(k) e^{2\pi i k\xi}\right|^2$

The bias of the MT is

$$\mathbb{E}[\widehat{S}^{\mathrm{mt}}(\xi)] - S(\xi) = \int S(\xi - \xi')\rho(\xi')\, d\xi'$$

with the spectral window $\rho(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k=0}^{N-1} S_j(k)e^{2\pi i k\xi}\right|^2$

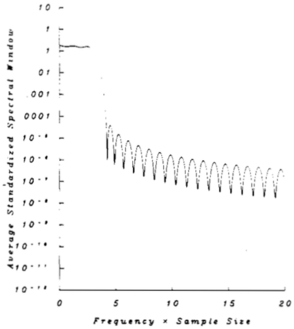


Figure: Thomson's 82

The bias of the MT is

$$\mathbb{E}[\widehat{S}^{\mathrm{mt}}(\xi)] - S(\xi) = \int S(\xi - \xi')\rho(\xi')\, d\xi'$$

with the spectral window $\rho(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k=0}^{N-1} S_j(k)e^{2\pi i k\xi}\right|^2$

Heuristic:

with $K = NW$: $\rho \approx \frac{1}{W}1_{[-W/2,W/2]}$

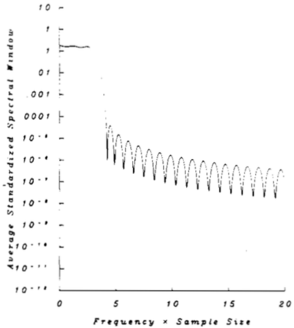


Figure: Thomson's 82

The bias of the MT is

$$\mathbb{E}[\widehat{S}^{\mathrm{mt}}(\xi)] - S(\xi) = \int S(\xi - \xi')\rho(\xi')\, d\xi'$$

with the spectral window $\rho(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k=0}^{N-1} S_j(k) e^{2\pi i k\xi}\right|^2$

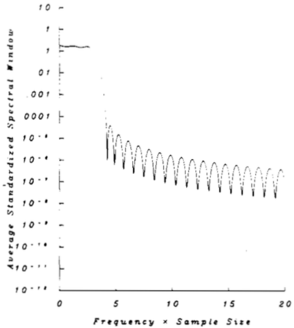


Figure: Thomson's 82

Heuristic:
with $K = NW$: $\rho \approx \frac{1}{W} 1_{[-W/2, W/2]}$

Theorem (Abreu, R-, 2017)

$$\int \left|\rho - \frac{1}{W} 1_{[-W/2,W/2]}\right| \lesssim \frac{\log(N)}{K}$$

The bias of the MT is

$$\mathbb{E}[\widehat{S}^{\mathrm{mt}}(\xi)] - S(\xi) = \int S(\xi - \xi')\rho(\xi')\, d\xi'$$

with the spectral window $\rho(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k=0}^{N-1} S_j(k)e^{2\pi i k\xi}\right|^2$

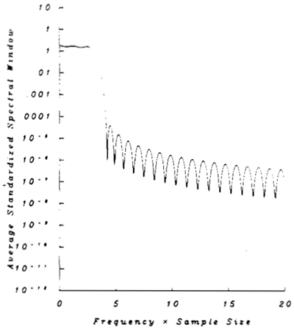


Figure: Thomson's 82

Heuristic:
with $K = NW$: $\rho \approx \frac{1}{W}1_{[-W/2,W/2]}$

Theorem (Abreu, R-, 2017)

$$\int \left|\rho - \frac{1}{W}1_{[-W/2,W/2]}\right| \lesssim \frac{\log(N)}{K}$$

Coro: weak MSE bounds:
$\sup_\xi \mathbb{E}|S(\xi) - \widehat{S}^{\mathrm{mt}}(\xi)|^2 \lesssim N^{-4/5}$

The bias of the MT is

$$\mathbb{E}[\widehat{S}^{\mathrm{mt}}(\xi)] - S(\xi) = \int S(\xi - \xi')\rho(\xi')\, d\xi'$$

with the spectral window $\rho(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k=0}^{N-1} S_j(k) e^{2\pi i k\xi}\right|^2$

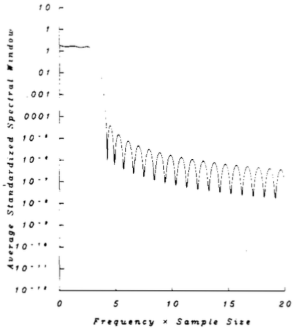


Figure: Thomson's 82

Heuristic:
with $K = NW$: $\rho \approx \frac{1}{W} 1_{[-W/2, W/2]}$

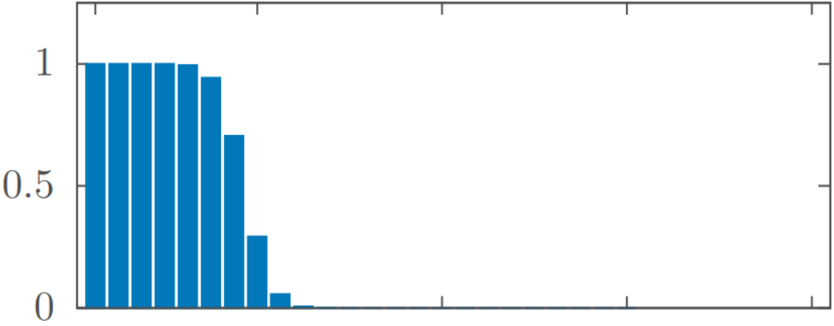
Theorem (Abreu, R-, 2017)

$$\int \left|\rho - \frac{1}{W} 1_{[-W/2,W/2]}\right| \lesssim \frac{\log(N)}{K}$$

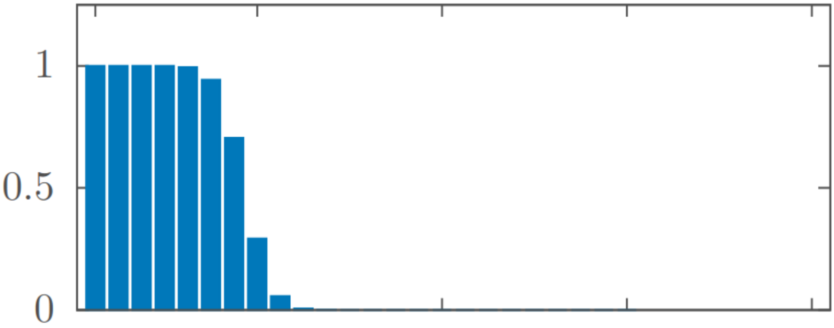
Coro: weak MSE bounds:
$\sup_\xi \mathbb{E}|S(\xi) - \widehat{S}^{\mathrm{mt}}(\xi)|^2 \lesssim N^{-4/5}$

Strong MSE bounds:
$\mathbb{E}\sup_\xi |S - \widehat{S}|_\infty^2 \lesssim N^{-4/5}\log(N)^{4/5}$
concentration for quadratic forms

Energy of Slepian tapers on bandwidth interval

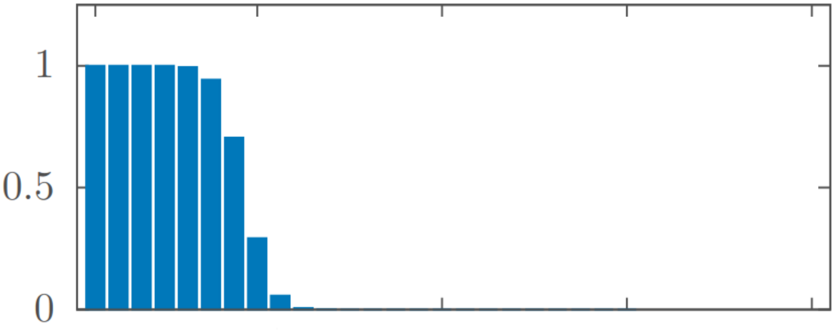


Energy of Slepian tapers on bandwidth interval



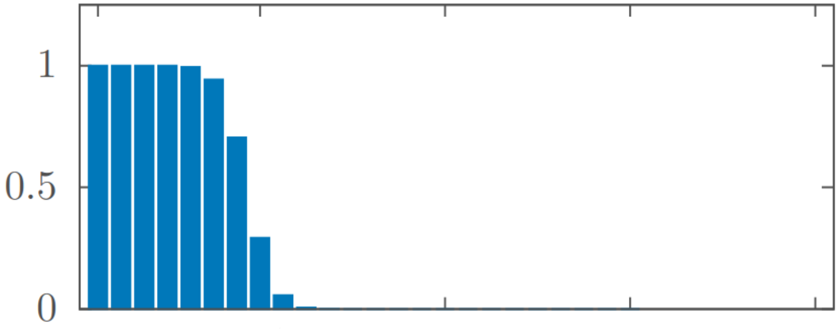
Lii and Rosenblatt 2008: Bias for individual tapered periodograms

Energy of Slepian tapers on bandwidth interval



Lii and Rosenblatt 2008: Bias for individual tapered periodograms
Heuristic explanation with Szegő theorem - no estimates

Energy of Slepian tapers on bandwidth interval



Lii and Rosenblatt 2008: Bias for individual tapered periodograms
Heuristic explanation with Szegő theorem - no estimates

Aggregated bias is much simpler!

$$\rho(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \left| \sum_{k=0}^{N-1} S_j(k) e^{2\pi i k \xi} \right|^2$$

Multi-taper on domains

- Stationary (real) process: $X(k), k \in \mathbb{Z}^d$
- Observations: $X(k)$, $k \in \Omega$

Multi-taper on domains

- Stationary (real) process: $X(k), k \in \mathbb{Z}^d$
- Observations: $X(k)$, $k \in \Omega$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k\in\Omega} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]^d$

Multi-taper on domains

- Stationary (real) process: $X(k), k \in \mathbb{Z}^d$
- Observations: $X(k)$, $k \in \Omega$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k \in \Omega} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]^d$

Parameters: K, W - natural choice: $K = N \cdot W^d$

Multi-taper on domains

- Stationary (real) process: $X(k), k \in \mathbb{Z}^d$
- Observations: $X(k)$, $k \in \Omega$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k \in \Omega} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]^d$

Parameters: K, W - natural choice: $K = N \cdot W^d$

Questions:
− Choice of parameters; MSE and sample complexity; Optimality?

Multi-taper on domains

- Stationary (real) process: $X(k), k \in \mathbb{Z}^d$
- Observations: $X(k)$, $k \in \Omega$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k \in \Omega} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]^d$

Parameters: K, W - natural choice: $K = N \cdot W^d$

Questions:

– Choice of parameters; MSE and sample complexity; Optimality?

– **Numerical stability of the calculation of the tapers**

Multi-taper on domains

- Stationary (real) process: $X(k), k \in \mathbb{Z}^d$
- Observations: $X(k)$, $k \in \Omega$

Estimator

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k \in \Omega} X(k)\, S_j(k)\, e^{2\pi i k \xi} \Big|^2$$

S_j Slepian tapers with bandwidth $[-W/2, W/2]^d$

Parameters: K, W - natural choice: $K = N \cdot W^d$

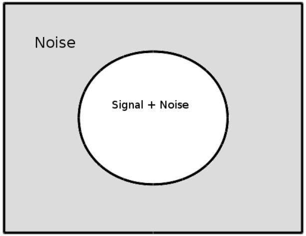
Questions:

– Choice of parameters; MSE and sample complexity; Optimality?

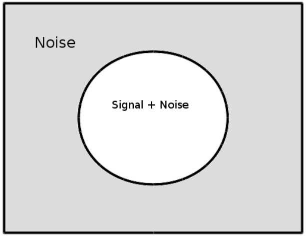
– **Numerical stability of the calculation of the tapers**

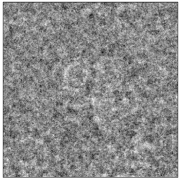
Motivation: Geophysics (F. Simons)

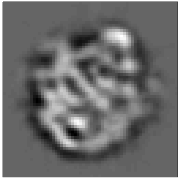
From: Joakim Andén



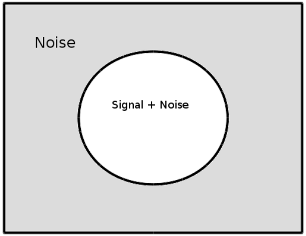
From: Joakim Andén

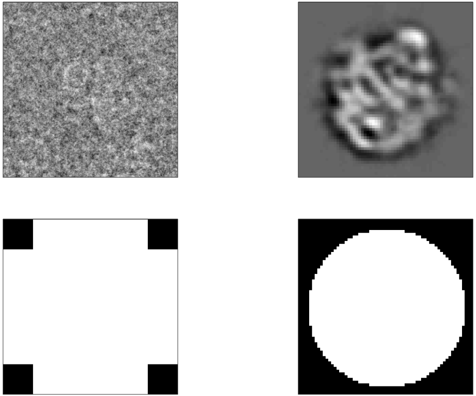


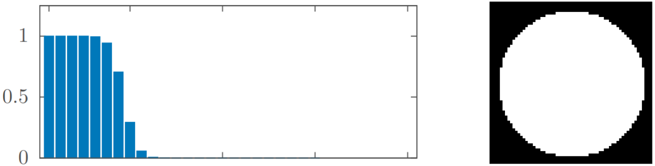


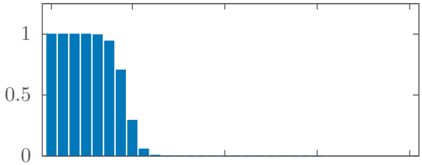


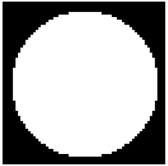
From: Joakim Andén






1
0.5
0

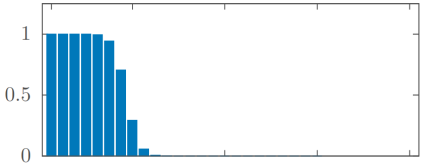


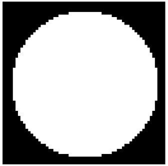


(Proxy tapers; Andén, R-, 2020)

If $\{S_1^*, \ldots, S_K^*\}$ *is an o.n.b. for the span of the Slepian tapers, then*

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k\in\Omega} X(k)\, S_j^*(k)\, e^{2\pi i k \xi} \Big|^2$$



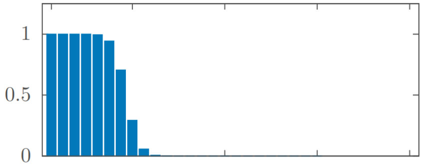


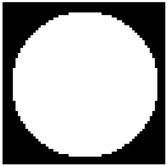
(Proxy tapers; Andén, R-, 2020)

If $\{S_1^*, \ldots, S_K^*\}$ *is an o.n.b. for the span of the Slepian tapers, then*

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k \in \Omega} X(k)\, S_j^*(k)\, e^{2\pi i k \xi} \Big|^2$$

- Individual tapered periodograms = ill-conditioned
- Average tapered periodograms = well-conditioned



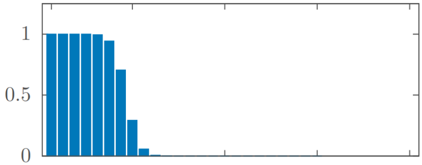


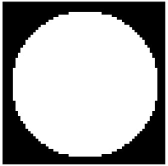
(Proxy tapers; Andén, R-, 2020)

If $\{S_1^*, \ldots, S_K^*\}$ *is an o.n.b. for the span of the Slepian tapers, then*

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K} \sum_{j=0}^{K-1} \Big| \sum_{k \in \Omega} X(k)\, S_j^*(k)\, e^{2\pi i k \xi} \Big|^2$$

- Individual tapered periodograms = ill-conditioned
- Average tapered periodograms = well-conditioned
- SVD-based computations work even if they show warnings





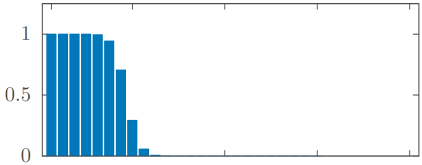
(Proxy tapers; Andén, R-, 2020)

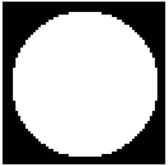
If $\{S_1^*, \dots, S_K^*\}$ *is an o.n.b. for the span of the Slepian tapers, then*

$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\Big|\sum_{k\in\Omega} X(k)\, S_j^*(k)\, e^{2\pi i k\xi}\Big|^2$$

- Individual tapered periodograms = ill-conditioned
- Average tapered periodograms = well-conditioned
- SVD-based computations work even if they show warnings
- Other more direct approaches also work:

– Draw $\approx K$ random vectors
– Apply the Toeplitz matrix (band-time limiting) 2/3 times
– Orthogonalize the output





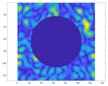
(Proxy tapers; Andén, R-, 2020)

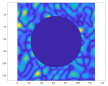
If $\{S_1^*, \ldots, S_K^*\}$ *is an o.n.b. for the span of the Slepian tapers, then*

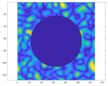
$$\widehat{S}^{\mathrm{mt}}(\xi) = \frac{1}{K}\sum_{j=0}^{K-1}\left|\sum_{k\in\Omega} X(k)\, S_j^*(k)\, e^{2\pi i k\xi}\right|^2$$

- Individual tapered periodograms = ill-conditioned
- Average tapered periodograms = well-conditioned
- SVD-based computations work even if they show warnings
- Other more direct approaches also work:

– Draw $\approx K$ random vectors
– Apply the Toeplitz matrix (band-time limiting) 2/3 times
– Orthogonalize the output







$\Omega \subset \mathbb{Z}^d$ $N = \#\Omega,$ $\#\partial\Omega$

$\Omega \subset \mathbb{Z}^d \qquad N = \#\Omega, \qquad \#\partial\Omega$

Theorem (dimension $d \geq 2$; R-, Speckbacher, 2022)

Assuming $\#\partial\Omega \lesssim N^{1-1/d}$. *For* $K = N \cdot W^d$ *and* $K = \left(\log(\mathrm{diam}(\Omega))^d \cdot N^2\right)^{1/3}$:

$$\mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \lesssim \left(\frac{\log(\mathrm{diam}\Omega)}{N^{1/d}}\right)^{4/3}$$

$\Omega \subset \mathbb{Z}^d \qquad N = \#\Omega, \qquad \#\partial\Omega$

Theorem (dimension $d \geq 2$; R-, Speckbacher, 2022)

Assuming $\#\partial\Omega \lesssim N^{1-1/d}$. For $K = N \cdot W^d$ and $K = \left(\log(\mathrm{diam}(\Omega))^d \cdot N^2\right)^{1/3}$:

$$\mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \lesssim \left(\frac{\log(\mathrm{diam}\Omega)}{N^{1/d}}\right)^{4/3}$$

Follows from bound for general K

$\Omega \subset \mathbb{Z}^d \qquad N = \#\Omega, \qquad \#\partial\Omega$

Theorem (dimension $d \geq 2$; R-, Speckbacher, 2022)

Assuming $\#\partial\Omega \lesssim N^{1-1/d}$. *For* $K = N \cdot W^d$ *and* $K = \big(\log(\mathrm{diam}(\Omega))^d \cdot N^2\big)^{1/3}$:

$$\mathbb{E}\big[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\big] \lesssim \left(\frac{\log(\mathrm{diam}\Omega)}{N^{1/d}}\right)^{4/3}$$

Follows from bound for general K

Theorem (some benchmarks $d \geq 2$; R-, Speckbacher, 2022)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\big[\|S - \widehat{S}\|_\infty^2\big] \gtrsim \left(\frac{\log N}{[\mathrm{diam}\Omega]^d}\right)^{\frac{4}{4+d}}, \qquad \|S\|_{C^2} \leq 1$$

$\Omega \subset \mathbb{Z}^d \qquad N = \#\Omega, \qquad \#\partial\Omega$

Theorem (dimension $d \geq 2$; R-, Speckbacher, 2022)

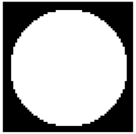
Assuming $\#\partial\Omega \lesssim N^{1-1/d}$. *For* $K = N \cdot W^d$ *and* $K = \left(\log(\mathrm{diam}(\Omega))^d \cdot N^2\right)^{1/3}$:

$$\mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \lesssim \left(\frac{\log(\mathrm{diam}\Omega)}{N^{1/d}}\right)^{4/3}$$

Follows from bound for general K

Theorem (some benchmarks $d \geq 2$; R-, Speckbacher, 2022)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \gtrsim \left(\frac{\log N}{[\mathrm{diam}\Omega]^d}\right)^{\frac{4}{4+d}}, \qquad \|S\|_{C^2} \leq 1$$



$\Omega \subset \mathbb{Z}^d \qquad N = \#\Omega, \qquad \#\partial\Omega$

Theorem (dimension $d \geq 2$; R-, Speckbacher, 2022)

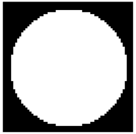
Assuming $\#\partial\Omega \lesssim N^{1-1/d}$. *For* $K = N \cdot W^d$ *and* $K = \left(\log(\mathrm{diam}(\Omega))^d \cdot N^2\right)^{1/3}$:

$$\mathbb{E}\left[\|S - \widehat{S}^{\mathrm{mt}}\|_\infty^2\right] \lesssim \left(\frac{\log(\mathrm{diam}\Omega)}{N^{1/d}}\right)^{4/3}$$

Follows from bound for general K

Theorem (some benchmarks $d \geq 2$; R-, Speckbacher, 2022)

$$\inf_{\widehat{S}} \sup_{S} \mathbb{E}\left[\|S - \widehat{S}\|_\infty^2\right] \gtrsim \left(\frac{\log N}{[\mathrm{diam}\Omega]^d}\right)^{\frac{4}{4+d}}, \qquad \|S\|_{C^2} \leq 1$$



(Optimality of MT for some acquisition geometries)

If $d = 2$ *and* $\mathrm{diam}\,\Omega \asymp N^{1/2}$ *then*

Benchmark MSE $= N^{-2/3}(\log N)^{2/3}$

Multi-taper MSE $= N^{-2/3}(\log N)^{4/3}$

The spectral window

$$\rho(\xi) := \frac{1}{K} \sum_{j=0}^{K-1} \left| \sum_{k \in \Omega} S_j(k) e^{2\pi i k \xi} \right|^2$$

Theorem (Andén, R-, 2020)

With $K = N \cdot W^d$*:*

$$\int \left| \rho - \tfrac{1}{W^d} 1_{[-W/2, W/2]^d} \right| \lesssim \frac{\#\partial\Omega \cdot W^{d-1}}{K} \left[1 + \log\left(\tfrac{N}{\#\partial\Omega} \right) \right]$$

$K = N \cdot W^d$ or $K < N \cdot W^d$?

$K = N \cdot W^d$ or $K < N \cdot W^d$?

- $K = N \cdot W^d$: no instabilities in computation of MT (proxy tapers work)

$K = N \cdot W^d$ or $K < N \cdot W^d$?

- $K = N \cdot W^d$: no instabilities in computation of MT (proxy tapers work)
- Need to understand better advantages $K < N \cdot W^d$

$K = N \cdot W^d$ or $K < N \cdot W^d$?

- $K = N \cdot W^d$: no instabilities in computation of MT (proxy tapers work)

- Need to understand better advantages $K < N \cdot W^d$

- Can we just use $K < N \cdot W^d$ proxy tapers?