Ideals of group algebras and other strongly Arens irregular algebras classified by their Arens regularity properties

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Joint work with Mahmoud Filali (University of Oulu, Finland) and Reza Esmailvandi (UJI, Spain)

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    - The first multiplication, say $\Box$, is defined in such a way that whenever $p = \lim_\alpha a_\alpha$, $q = \lim_\beta b_\beta$ (these are $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$-limits):

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- $(\mathfrak{A}^{**}, \Box)$ and $(\mathfrak{A}^{**}, \Diamond)$ are Banach algebras.
- For each $p \in \mathfrak{A}^{**}$, the maps $q \mapsto q\Box p$ and $q \mapsto p\Diamond q$ are weak*-continuous.

- Let G be a locally compact metric group (nondiscrete). If $\phi \in L^\infty(G)$ is not continuous at 1, we can find **bounded approximate identities** $\{e_{\alpha,1} \colon \alpha \in \Lambda\}$ and $\{e_{\alpha,2} \colon \alpha \in \Lambda\}$ such that

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- Let $\mathbf{e_i} \in L^1(G)^{**}$, $i = 1, 2$, be, respectively, weak*-accumulation points of $\{e_{\alpha,i} \colon \alpha \in \Lambda\}$. Then:

$$\langle e_1, \phi\rangle \neq \langle e_2, \phi\rangle \quad \Longrightarrow \quad e_1 \neq e_2,$$
$$p \Box e_i = p, \qquad e_i \Diamond p = p, \quad \text{for each } p \in L^1(G)^{**} \text{ and } i = 1, 2$$

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(e_1 and e_2 are $\Box$-right and $\Diamond$-left identities). Hence

$$\begin{aligned} (e_1 - e_2)\Box e_1 &= e_1 - e_2, & e_1 \Box (e_1 - e_2) &= 0, \\ e_1 \Diamond (e_1 - e_2) &= e_1 - e_2 & (e_1 - e_2)\Diamond e_1 &= 0. \end{aligned}$$

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We say that $\mathfrak{A}$ is **Arens regular** if for every $p, q \in \mathfrak{A}^{**}$, $\mathbf{p}\Box\mathbf{q} = \mathbf{p}\Diamond\mathbf{q}$.

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We define:

$$\mathcal{Z}_t^{(l)}(\mathfrak{A}^{**}) = \{p \in \mathfrak{A}^{**} : q \mapsto p \Box q \text{ is continuous}\} \qquad \text{(left topological center)}$$

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We say that $\mathfrak{A}$ is **strongly Arens irregular** (**SAI**) if $\mathfrak{A} = \mathcal{Z}_t^{(l)}\left(\mathfrak{A}^{**}\right) = \mathcal{Z}_t^{(r)}\left(\mathfrak{A}^{**}\right)$

$L^1(G)$ is (left) strongly Arens irregular

Theorem (Işik, Pym, Ülger, 1987)

Let G be a compact group and let e be a right identity of $L^1(G)^{**}$. For $m \in L^1(G)^{**}$

$$\boxed{\mathbf{m} = \mathbf{C}_\mu^{**}(\mathbf{e}) + \mathbf{r}} \qquad \text{where } \mu \in M(G) \text{ and } r \in C(G)^\perp.$$

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Proof, compact commutative case.

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Let $m \in \mathcal{Z}(\mathfrak{A}^{**})$. Then $m = C_\mu^{**}(e) + r$.

If $\mu \notin L^1(G)$, pick $\phi \in L^\infty(G)$ and $s \in C(G)^\perp$ with $0 \neq \langle s, \check{\mu} * \phi \rangle$. But

$$\langle s, \check{\mu} * \phi \rangle = \left\langle s, (C_\mu^{**}(e) + r).\phi \right\rangle = \langle s \Box m, \phi \rangle$$
$$= \langle m \Box s, \phi \rangle = 0.$$

If $\mu \in L^1(G)$, then $r \in \mathcal{Z}(\mathfrak{A}^{**})$. But $0 = e \Box r = r \Box e = r$ and $m \in L^1(G)$. □

WaSaBI $:= \{\mathfrak{A}$ Banach algebra $: \mathfrak{A}$ is WSC, has a BAI and $\mathfrak{A}$ is an Ideal of $\mathfrak{A}^{**}\}$.

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Theorem (Grosser 1979, Baker-Lau-Pym, 1998)

Let $\mathfrak{A} \in$ WaSaBI and let e be a mixed identity of $\mathfrak{A}^{**}$ ($p\Box e = e\Diamond p = p$ for every $p \in \mathfrak{A}^{**}$). For $p \in \mathfrak{A}^{**}$

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Proof with sequential BAI $(e_n)_n$.

Let $m \in \mathcal{Z}_t^{(l)}(\mathfrak{A}^{**})$. Then an accumulation point of $(m\Box e_n)_n$ will be of the form $m\Box e$ with e an accumulation point of $(e_n)_n$. But, if $m = \lim_\alpha a_\alpha$, $a_\alpha \in \mathfrak{A}$,
$m\Box e = \lim_\alpha \lim_\beta a_\alpha e_{n(\beta)} = \lim_\alpha a_\alpha = m$.
Hence $\lim_n m\Box e_n = m$ and WSC implies $m \in \mathfrak{A}$. □

$i\colon J \to \mathfrak{A}$ will be the inclusion map, $i^*: \mathfrak{A}^* \to J^*$, the restriction map. G will be a compact Abelian group and Γ a discrete amenable group.

- Let $J \trianglelefteq \mathfrak{A}$ be a closed ideal.

  Let $\tilde{w}$ be the topology that $e\Box\mathfrak{A}^{**}$ receives from $\sigma\left(WAP(\mathfrak{A})^*, WAP(\mathfrak{A})\right)$. Then

$$J^{**} = \overline{J}^{\tilde{w}} \oplus i^*\left(WAP(\mathfrak{A})^{\perp}\right).$$

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- (particular case) Let $E \subseteq \widehat{G}$, then

$$L^1_E(G)^{**} \cong M_E(G) \oplus i^* \left(C(G)\right),$$

where $M_E(G) = \{\mu \in M(G) \colon \widehat{\mu}(\chi) = 0, \text{ for every } \chi \notin E\}$.

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- (another particular case) Let $E \subseteq \Gamma$, then
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A subset $E \subseteq \widehat{G}$ is said to be a **Riesz set** if $M_E(G) = L^1_E(G)$.

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If $E \subseteq \widehat{G}$ is a Riesz set, then $L^1_E(G)$ is Arens regular.

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Classify ideals $L^1_E(G) \trianglelefteq L^1(G)$ ($J \trianglelefteq \mathfrak{A} \in$ WaSaBI) in terms of their Arens regularity properties.

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Big and small subsets of $\widehat{G}$

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Characterizations

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Let G be a compact Abelian group and let $E \subseteq \widehat{G}$. Then, $L^1_E(G)$ is Arens regular if and only if $i^*\left(M_{-E}(G) * L^\infty(G)\right) \subseteq \overline{i^*(C(G))}$.

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It is obvious here how the existence of a measure with $\widehat{\mathbf{1}_E} \in M(G)$ implies that $L^1_E(G)$ is SAI

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Corollary (sample)

Let G be a discrete amenable group and let $E \subset G$. Then $A_E(G)$ is SAI if and only if $\overline{\mathrm{co}}\left(i^*(B_E(G).VN(G))\right) = A_E(G)^*$.

The same can be done with $L^1(G)$, G compact and $E \subset \Sigma$, the dual object of G or, more generally with $\mathfrak{A} \in$ WaSaBI and $E \subset \Sigma$ a set of representations of $\mathfrak{A}$...

Small sets

G will be a compact Abelian group and $E \subseteq \widehat{G}$.

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We call the sets $\widehat{G} \setminus E$ with this property co-LP sets. If $L^\infty_{\widehat{G}\setminus E} \subset C(G)$, then we say that E is co-Rosenthal.

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$L^1_{E_1\cup E_2}(G)$ is SAI while $E_1 \cup E_2$ is not in the coset ring because E_2 being Sidon set then $\mathbf{1}_E \notin M(\mathbb{T})$.

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- A subset E of $\mathbb{Z}$ such that E is co-Rosenthal but $\mathbb{Z} \setminus E$ is not Sidon: $E = \bigcup_{n=1}^{\infty} \{(2n)!j \colon 1 \leq j \leq 2n\}$ (**Rosenthal, 1967**).

Ideals in other algebras

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- Co-Rosenthal ideals: We say that J is co-Rosenthal, if $J^\perp \subseteq WAP(J)$. If J is co-Rosenthal and is not Riesz, then J is not Arens regular. If J is co-Rosenthal and $M(\mathfrak{A}) = \mathfrak{A} \oplus_1 M_s$, then J cannot be Riesz.

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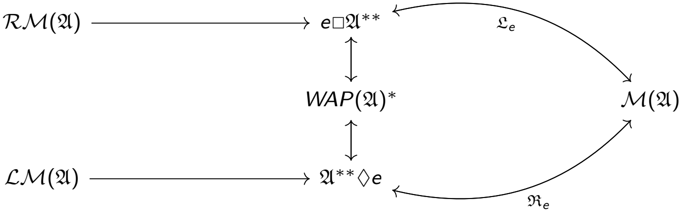
- If $E \subseteq \mathbb{Z}$ is co-Rosenthal (or co-LP), must L^1_E be SAI?

- (Ülger 2011) If $L^1_E(\mathbb{T})$ is Arens regular, must $E \subseteq \mathbb{Z}$ be Riesz?
- If $E \subseteq \mathbb{Z}$ is small-1-1, must $L^1_E(\mathbb{T})$ be Arens regular?
- If $E \subseteq \mathbb{Z}$ is co-Rosenthal (or co-LP), must L^1_E be SAI?
- Same questions for algebras in WaSaBI. Can any of the answers be different? Special attention to $L^1_E(G)$, G compact not Abelian, and $A_E(G)$, G discrete and amenable. **Interesting:** if $G = SU(2)$, no infinite subset of $\widehat{G}$ is a $\Lambda(1)$-set.

THANKS FOR YOUR ATTENTION

Multipliers in the second dual $\widehat{G}$



$$\begin{aligned}\mathfrak{A}^{**} &= e\square\mathfrak{A}^{**} \oplus WAP(\mathfrak{A})^{\perp} = \mathfrak{A}^{**}\lozenge e \oplus WAP(\mathfrak{A})^{\perp} \\ &\cong M(\mathfrak{A}) \oplus WAP(\mathfrak{A})^{\perp} \cong WAP(\mathfrak{A})^* \oplus \left(\frac{\mathfrak{A}^*}{WAP(\mathfrak{A})}\right)^* .\end{aligned}$$

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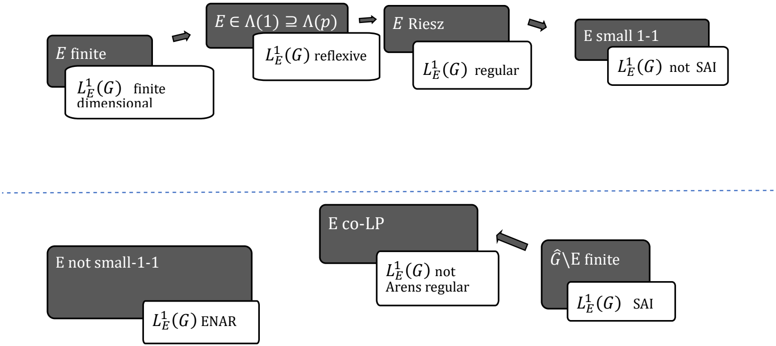
E finite
$L^1_E(G)$ finite dimensional
$E \in \Lambda(1) \supseteq \Lambda(p)$
$L^1_E(G)$ reflexive
E Riesz
$L^1_E(G)$ regular
E small 1-1
$L^1_E(G)$ not SAI
E not small-1-1
$L^1_E(G)$ ENAR
E co-LP
$L^1_E(G)$ not Arens regular
$\hat{G} \setminus$E finite
$L^1_E(G)$ SAI

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