Analytic and Probabilistic Combinatorics
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1 Overview of the Field

In analytic combinatorics, the objects of study often come from enumerative or algebraic combinatorics. Applied problems of interest are drawn from classic combinatorics (lattice paths, permutations, integer partitions, combinatorics on words), graph theory, information theory (data compression), number theory, probability (random walks, branching processes such as trees), theoretical computer science (space and time complexity, sorting, searching, hashing), and applied areas, including biological sciences, information sciences, mathematical and statistical physics, and so on. The methods include analytic (complex-valued) approaches, such as analyzing the singularities of the relevant generating functions; symbolic computation (e.g., in Maple, Mathematica, or Sage); multivariate methods; mathematical transforms (Fourier, Laplace, Mellin); etc. One of the main goals of analytic combinatorics is the precise characterization of exact or asymptotic information about the enumeration of combinatorial objects, or about the mean, variance, distribution, etc., of randomly distributed objects. Since modern-day computing platforms allow researchers throughout the sciences to routinely study very large objects, the study of asymptotic properties of objects is more relevant today than ever before.

In probabilistic combinatorics, the objects of study often come from extremal combinatorics or graph theory, or computational complexity theory. The methods used can come from classic or modern probability theory, including the classic "Probabilistic Method" introduced by Paul Erdős in the 1930s. Existence proofs are a common feature in the work of this group, and so are constructive proofs and efficient algorithms.

There are other areas of mathematics related to both analytic and probabilistic combinatorics, like symbolic dynamics (for instance, a celebrated example of this relationship is Furstenberg's proof of Szemeredi's theorem that gave rise to the so called ergodic Ramsey theory). In symbolic dynamics, the objects of study are called *shift spaces*, they consist of sets of configurations of symbols that avoid a given set of finite configurations. It is relevant to consider probability measures defined on shift spaces, for example, they can model stochastic processes such as countable state Markov chains. In this context, analytic methods have been applied to study various classification problems of Markov shifts within the distinct probabilistic regimes (exponentially recurrent, positive recurrent, null recurrent and transient), through the asymptotic properties of the dynamic zeta functions that encode the periodic orbits. Furthermore, this type of classification problems have motivated questions in analytic combinatorics, some of which remain open as conjectures.

2 Open Problems and Recent Developments

Several open problems were presented and discussed, during both the open problem session and the open discussions throughout the workshop. We describe some of these.

2.1 Open Problem Session

2.1.1 Laura Eslava. *Random Recursive Trees.*

Random recursive trees are rooted labelled trees obtained by the following procedure: Let T_1 be a single vertex labeled 1. For $n > 1$ the tree T_n is obtained from the tree T_{n-1} by adding a directed edge from a new vertex labeled n to a vertex with label in $\{1, ..., n-1\}$, chosen uniformly at random and independent for each n; which is called the parent of n. The degree of a vertex v is the number of children of v, or equivalently, the number of edges directed towards v. We denote by $T_n(v)$ the subtree containing all descendants of v.

Enumerate the vertices of a random recursive tree T_n according to a decreasing order of their degrees; namely, let $(v^{(i)})_{i=1}^{n}$ be so that $\deg(v^{(1)}) \geq \cdots \geq \deg(v^{(n)})$.

- **Question.** Fix $i \in \mathbb{N}$. As $n \to \infty$, what can be said about the number of vertices in $T_n(v^{(i)})$? And how does it compare to the size of $T_n(v)$, if $v \in \{1, \ldots, n\}$ is chosen uniformly at random from the vertices in T_n?

2.1.2 Colin Defant. *Friends-and-Strangers Graphs.*

Suppose $X = (V(X), E(X))$ and $Y = (V(Y), E(Y))$ are two simple graphs, each of which has n vertices. Imagine that the vertices of X are chairs and the vertices of Y are people. Two people in Y are adjacent if and only if they are friends with each other. There are $n!$ different ways to arrange the people in the chairs (with every chair occupied by exactly 1 person). Starting with such an arrangement of people in chairs, we can perform a *friendly swap* by swapping the positions of two people who are sitting in adjacent chairs *and* who are friends with each other. The *friends-and-strangers graph* of X and Y, denoted $\mathsf{FS}(X, Y)$, is the graph whose vertices are the arrangements of people in chairs, where two arrangements are adjacent whenever one is obtained from the other by a friendly swap. These graphs were originally introduced in [2].

More formally, we can think of the vertices of $\mathsf{FS}(X, Y)$ as the bijections from $V(X)$ to $V(Y)$. Two such bijections σ and τ are adjacent if and only if there exist $a, b \in V(X)$ such that $\sigma(a) = \tau(b)$, $\sigma(b) = \tau(a)$, and $\sigma(c) = \tau(c)$ for all $c \in V(X) \setminus \{a, b\}$. This description makes it clear that $\mathsf{FS}(X, Y)$ and $\mathsf{FS}(Y, X)$ are isomorphic; the isomorphism is given by $\sigma \mapsto \sigma^{-1}$.

As an example, suppose $V(X) = V(Y) = [n]$. Then the vertex set of $\mathsf{FS}(X, Y)$ is the symmetric group S_n. Each edge $\{i, j\}$ in $E(X)$ corresponds to the transposition $(i\ j)$ in S_n. If Y is a complete graph, then $\mathsf{FS}(X, Y)$ is the Cayley graph of S_n generated by transpositions corresponding to the edges in X.

Theorem 2.1 (Alon–Defant–Kravitz [1]; generalized by Wang–Chen [6]) *Fix some small $\varepsilon > 0$. Let X and Y be independently-chosen Erdős–Rényi random graphs in $\mathcal{G}(n, p)$, where $p = p(n)$ depends on n. If*

$$p \leq \frac{2^{-1/2} - \varepsilon}{n^{1/2}},$$

then $\mathsf{FS}(X, Y)$ has an isolated vertex (and is therefore disconnected) with high probability. If

$$p \geq \frac{\exp(2(\log n)^{2/3})}{n^{1/2}},$$

then $\mathsf{FS}(X, Y)$ is connected with high probability.

Problem 2.2 *Understand $\mathsf{FS}(X, Y)$ when X and Y are independent Erdős–Rényi random graphs in $\mathcal{G}(n, p)$. If*

$$p \leq \frac{2^{-1/2} - \varepsilon}{n^{1/2}},$$

then how many connected components does $\mathsf{FS}(X,Y)$ *have? If*

$$p \geq \frac{\exp(2(\log n)^{2/3})}{n^{1/2}},$$

then what is the diameter of $\mathsf{FS}(X,Y)$*? What are the minimum and maximum degrees of* $\mathsf{FS}(X,Y)$*? Can we say "how connected"* $\mathsf{FS}(X,Y)$ *is?*

Problem 2.3 *Fix* X *and* Y*, and consider the Markov chain whose state space is the vertex set of* $\mathsf{FS}(X,Y)$ *where at each step, we choose two people (or maybe it is better to choose two friends) at random and swap them if they are allowed to swap (i.e., they are friends and are sitting in adjacent chairs). Here, maybe we would want to fix* Y *to be some specific graph like a path or a cycle (or the complement of a path or a cycle). Or maybe it is interesting to consider when* X *and* Y *are independent Erdős–Rényi random graphs in* $\mathcal{G}(n,p)$*.*

2.1.3 Sebastian Wild. *Range min-max entropy.*

Let $\mathcal{S}_n$ denote the set of permutations from $[n] = \{1,\ldots,n\}$ to $[n]$. Let $\mathcal{T}_n$ denote the set of binary trees on n vertices, i.e., trees where each node has a left and a right child; each of which could be the empty tree $\Lambda \in \mathcal{T}_0$.

Define $\mathrm{minTree}(x_1,\ldots,x_n)$ recursively as follows: $\mathrm{minTree}() = \Lambda$, and $\mathrm{minTree}(x_1,\ldots,x_n)$ for $n \geq 1$ is a new root with $\mathrm{minTree}(x_1,\ldots,x_{i-1})$ and $\mathrm{minTree}(x_{i+1},\ldots,x_n)$ as left (resp. right) subtrees, where $i = \arg\min_{1\leq j\leq n} x_j$. Define maxTree similarly, but using $i = \arg\max_{1\leq j\leq n} x_j$.

We write $\mathrm{minTree}(\pi)$ for $\mathrm{minTree}(\pi(1),\pi(2),\ldots,\pi(n))$ for $\pi \in \mathcal{S}_n$.

The connection between minTree and range-minimum queries is explained in detail here `https://www.wild-inter.net/publications/entropy-trees`.

Warmup: Range-min only

For $T \in \mathcal{T}_n$, we define

$$p(T) = \frac{|\{\pi \in \mathcal{S}_n : \mathrm{minTree}(\pi) = T\}|}{n!}.$$

Goal: What is the entropy of the distribution over $\mathcal{T}_n$

$$H_1(n) = \sum_{T\in\mathcal{T}_n} p(T)\log_2(1/p(T)) = \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n} \log_2(1/p(\mathrm{minTree}(\pi)))?$$

In this case, one can express $p(T)$ explicitly as the product of reciprocals of subtree sizes

$$p(T) = \prod_{v\in T} \frac{1}{\mathrm{nrDescendants(v)}};$$

(here we count v as one of its own descendants). inserting this into the sum above, one can obtain a recurrence relation for $H_1(n)$:

$$H_1(0) = H_1(1) = 0 \tag{1}$$

$$H_1(n) = \lg n + \frac{1}{n}\sum_{i=1}^{n}(H_1(i-1) + H_1(n-i)), \qquad (n \geq 2). \tag{2}$$

Kieffer, Yang and Szpankowski [5] resp. Hwang and Neininger [4] shows that this solves to

$$\begin{aligned} H_1(n) &= \lg(n) + 2(n+1)\sum_{i=2}^{n-1} \frac{\lg i}{(i+2)(i+1)} \\ &\sim 2n\sum_{i=2}^{\infty} \frac{\lg i}{(i+2)(i+1)} \\ &\approx 1.7363771n \end{aligned}$$

Open Problem: Range-min and max

Here, for $T_{\min}, T_{\max} \in \mathcal{T}_n$, define

$$p(T_{\min}, T_{\max}) = \frac{|\{\pi \in \mathcal{S}_n : \text{minTree}(\pi) = T_{\min} \text{ and } \text{maxTree}(\pi) = T_{\max}\}|}{n!}.$$

What is

$$H_2(n) = \frac{1}{n!} \sum_{T \in \mathcal{T}_n} \log_2(1/p(\text{minTree}(\pi), \text{maxTree}(\pi)))?$$

A great result would be a (somewhat) explicit form for $p(T_{\min}, T_{\max})$.

What is known. One can uniquely construct from $(T_{\min}, T_{\max})$ a Baxter permutation π so that $(T_{\min}, T_{\max}) = (\text{minTree}(\pi), \text{maxTree}(\pi))$ [3]. Hence $H_2(n) \le \lg |\text{Baxter}_n| \sim 3n$.

Empirically, we should have $H_2(n) \approx 2.64n$. (We are not sure if the accuracy of that estimate really is two decimal places; it is expensive to sample.)

2.1.4 Tewodros Amdeberhan. *Various problems.*

Problem 1. Let $\mathfrak{S}_n$ be the permutation group on $[n]$. Given the pattern $\sigma = k(k-1)\cdots 321$, let $I_n(\sigma)$ be the number of *involutions* in $\mathfrak{S}_n$ that **avoid the pattern** $\boldsymbol{\sigma}$. Amitai Regev proved the case $k = 4$:

$$I_n((4321)) = \sum_{k \ge 0} \binom{n}{2k} C_k$$

which are the Motzkin numbers. Here C_k are the Catalan numbers.

Given an integer partition λ, draw the Young diagram and fill in hook-lengths of each cell. If a number t is not among these hook-lengths then λ is called a $\boldsymbol{t}$**-core**. If it misses a, b, c then call it an $\boldsymbol{(a, b, c)}$**-core partition**.

Let $N(n, n+1, n+2)$ be the number of all partitions that are $(n, n+1, n+2)$-core partitions. Then, Amdeberhan-Leven proved

$$N(n, n+1, n+2) = \sum_{k \ge 0} \binom{n}{2k} C_k.$$

Question. Is there a direct bijection between the above (4321)-avoiding involutions in $\mathfrak{S}_n$ and $(n, n+1, n+2)$-core partitions?

Problem 2. Let $[n]_q = \frac{1-q^n}{1-q}$ and $[n]!_q = [1]_q \cdots [n]_q$. The MacMahon's $\boldsymbol{q}$**-Catalan polynomial** is

$$C_n(q) = \frac{1}{[n+1]_q} \binom{2n}{n}_q = \frac{[2n]!_q}{[n+1]!_q \, [n]!_q}.$$

William Chen (2015) conjectured that $C_n(q)$ **strictly convex functions**, i.e. $C_n''(q) > 0$ for $n \ge 2$.

We have an **almost proof** of this except the case $\boldsymbol{-1 < q < 0}$.

Question. For $0 < x < 1$ and $n \geq 2$, can you prove that

$$W_n(x) = \log\left(\frac{(1+x^{4n-1})(1+x^{2n})(1-x^{2n-1})}{(1+x^{2n+1})(1-x^{2n+2})}\right)$$

is a convex function of x?

Problem 3. The **hook-length** $h^\lambda(u) = \lambda_i + \lambda'_j - i - j + 1$ and **content** $c^\lambda(u) = j - i$ of a cell $u = (i, j)$ of a Young diagram of shape λ. The dimension of the irreducible representation of $GL(n, \mathbb{C})$ corresponding to λ with $\ell(\lambda) \leq n$ is given by

$$\dim_{GL}(\lambda, n) = \prod_{u\in\lambda} \frac{n + c^\lambda(u)}{h^\lambda(u)}.$$

Nekrasov-Okounkov's **hook-length formula**

$$\sum_{n\geq 0} q^n \sum_{\lambda\vdash n} \sum_{u\in\lambda} \frac{t + (h^\lambda(u))^2}{(h^\lambda(u))^2} = \prod_{k\geq 1} \frac{1}{(1-q^k)^{t+1}}.$$

R. Stanley's **hook-content identity**

$$\sum_{n\geq 0} q^n \sum_{\lambda\vdash n} \sum_{u\in\lambda} \frac{t + (c^\lambda(u))^2}{(h^\lambda(u))^2} = \frac{1}{(1-q)^t}.$$

For the irreducible representations of the **symplectic group** $\boldsymbol{Sp(2n)}$, the **symplectic content** of $u \in \lambda$ is

$$c_{sp}^\lambda(u) = \begin{cases} \lambda_i + \lambda_j - i - j + 2 & \text{if } i > j \\ i + j - \lambda'_i - \lambda'_j & \text{if } i \leq j. \end{cases}$$

Question. Can you prove this?

$$\sum_{n\geq 0} q^n \sum_{\lambda\vdash n} \sum_{u\in\lambda} \frac{t + (c_{sp}^\lambda(u))^2}{(h^\lambda(u))^2} = \prod_{k\geq 1} \frac{1}{(1-q^{4k-2})(1-q^k)^t}.$$

Remark. Cases $t = 0$ and $t = -1$ done (Amdeberhan-Andrews-Ballantine).

Problem 4. Let

$$F(t, x, z) := \prod_{j=0}^{\infty} \frac{1}{(1-tx^j)^{z-1}}.$$

(a) If $z = 2$ then on the one hand we get Euler's

$$F(t, x, 2) = \sum_{n\geq 0} \frac{(-1)^n x^{\binom{n}{2}}}{(x;x)_n} t^n,$$

on the other we get Pólya's formula (the "cycle index decomposition")

$$F(t, x, 2) = \sum_{n\geq 0} Z(S_n, (1-x)^{-1}, \ldots, (1-x^n)^{-1}) t^n.$$

(b) If $t = x$ then we get Nekrasov-Okounkov's

$$F(x, x, z) = \sum_{n\geq 0} x^n \sum_{\lambda\vdash n} \prod_{\square\in\lambda} \left(1 - \frac{z}{h_\square^2}\right).$$

where $h_\square$ is the hook-length of a cell. **Question.** Is there a unifying combinatorial right-hand side in

$$\prod_{j\geq 0}\frac{1}{(1-tx^j)^{z-1}}=?$$

Problem 5. Given a sequence of **positive numbers** $(a_k)_{k\geq 0}$, define the operator $\mathcal{L}a_k=a_k^2-a_{k-1}a_{k+1}$. We say $(a_k)_k$ is **log-concave** provided that $\mathcal{L}a_k\geq 0$ for all $k\geq 0$.

If, after a repeated action of the operator $\mathcal{L}$, we find $\mathcal{L}^i a_k\geq 0$ for $1\leq i\leq m$ and for all k, then $(a_k)_k$ is named $\boldsymbol{m}$**-fold log-concave**. The sequence is called **infinitely log-concave** if $\mathcal{L}^i a_k\geq 0$ for all $i\geq 1$.

Given a graph G and x distinct colors, denote the number of proper colorings by $\kappa_G(x)$, referred as the **chromatic polynomial of $\boldsymbol{G}$**.

Theorem (June Huh). The absolute values of coefficients in $\kappa_G(x)$, of any graph G, are log-concave.

Question. Are the (absolute values) coefficients of any chromatic polynomial infinitely log-concave?

Problem 6. The "quantum" version **qTSPP** of the number of *totally symmetric plane partitions*, contained in the cube $[0,n]^3$, is enumerated by

$$f_n(q):=\prod_{j=1}^{n}\prod_{k=1}^{j}\prod_{\ell=1}^{k}\frac{1-q^{j+k+\ell-1}}{1-q^{j+k+\ell-2}}.$$

L'Hôpital $f_n(1)=\lim_{q\to 1}f_n(q)$ restores the classical version $\prod_{1\leq\ell\leq k\leq j\leq n}\frac{j+k+\ell-1}{j+k+\ell-2}$.

Although $f_n(-1)=0$ trivially, when n is odd, we observe the case n even is decidedly striking; namely that,

$$f_{2n}(-1)=\lim_{q\to -1}f_{2n}(q)=\prod_{k=0}^{n-1}\frac{(3k+1)!}{(n+k)!},$$

the number A_n of $n\times n$ *Alternating Sign Matrices* **ASMs**.

Question. Is there a non-analytic (more conceptual) reason for this connection between **qTSPP** and **ASMs**?

Problem 7. Consider the rational functions (in fact, polynomials)

$$F_n(q)=\frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}(-q)^k\frac{2k+1}{n+k+1}\binom{2n}{n-k}\prod_{j=0,\,j\neq k}^{n}\frac{1+q^{2j+1}}{1+q}.$$

The numbers $\frac{2k+1}{n+k+1}\binom{2n}{n-k}$ belong to a family of **Catalan triangle** of which the special case $k=0$ yields the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$.

Of further interest is $F_n(1)=E_{2n}$ the **Euler numbers**.

Question. Is it true that $F_n(q)$ has non-negative coefficients?

Problem 8. Given a Laurent polynomial g, let $CT(g)$ denote its **constant term**.

Consider the specific Laurent polynomial

$$f_n(x_1,\dots,x_r) = \left(1 + \prod_{j=1}^{r}(1+x_j) + \prod_{j=1}^{r}\left(1+\frac{1}{x_j}\right)\right)^n.$$

Question. Is there a *Combinatorial Nullstellensatz type* (of Alon Noga) proof of this fact:

$$CT(f_n) = \sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}\binom{m}{k}^{r+1}.$$

Problem 9 Let $\lambda_n = (n, n-1, \dots, 2, 1)$ be the **staircase partition** and its **Young diagram** Y_n.

Question. In how many different ways a_n can one tile Y_n using monomers (1×1 squares) and dimers (1×2 or 2×1 rectangles)? Is there a determinant (Pfaffian) formulation of this enumeration, in **Kasteleyn's style**?

Problem 10. If $0 \leq k \leq< b$ are integers, prove the **coefficient-wise inequality**

$$\binom{a}{k}_q\binom{a+b}{b-k}_q \geq \binom{b}{k}_q\binom{a+b}{a-k}_q$$

or equivalently

$$\binom{a}{k}_q\binom{b}{k}_q\binom{a+b}{b}_q\left[\frac{1}{\binom{a+k}{k}_q} - \frac{1}{\binom{b+k}{k}_q}\right] \geq 0.$$

For related and different problems that we have conjectures for, follow the link

`http://math.tulane.edu/~tamdeberhan/conjectures.html`

POSTSCRIPT. Problem 2 has received interest from a number of the workshop participants. More concretely, with Stephan Wagner, we are completing a joint paper on this problem under the general umbrella of *Convexity of q-Catalan polynomials*.

2.1.5 Robin Pemantle. *Counting paths in oriented percolation.*

We compare two exponential growth rates. One is for the expected number of open paths to $(n(1-b), nb)$. Taking logs and dividing by n, we get

$$f(p,b) := \lim_n (1/n)\log EZ(n(1-b), nb) = h(b) + \log p$$

where $h(b)$ is the entropy

$$b\log(1/b) + (1-b)\log(1/(1-b)).$$

Let

$$g(p,b) := \lim \text{ in probability of} (1/n)\log Z(n(1-b), nb),$$

conditioned on $Z(n(1-b), nb) > 0$, if such a limit exists.

Problems.

1. Prove the limit exists (maybe can be done with subadditive ergodic theorem?).
2. Is $g(p,b)$ ever equal to $f(p,b)$ or is it always strictly less?
3. For fixed $p < 1$, there is a critical b_c going to zero as p goes to 1, such that open paths in directions with slope less that $b_c/(1-b_c)$ are exponentially unlikely. Does $g(p,b)$ go to zero as $b \to b_c$ from above?

2.2 Open Discussions

During the workshop, several collaborators had the chance to be together to continue their ongoing projects. Gómez and Ward continued their work on certain generalizations of partition functions. The idea is that the classical integer partition function $P(z) = \prod_{n\geq 1}(1-z^n)^{-1}$ admits a plethora of generalizations, and they consider a very general, countable family of partition functions. They have the form

$$\prod(1 - z^{n_1\cdots n_i d_1\cdots d_j e_1\cdots e_k})^{-n_1\cdots n_i/d_1\cdots d_j}$$

with the $(i+j+k)$-fold product is taken over all positive integers n's, d's and e's. The main problems are (1) to give a unified combinatorial description, and (2) to solve the "partition problem", which is to determine the asymptotic growth of the coefficients of the partition functions. During the workshop they had the chance to mostly conclude the writing of a manuscript where they present solutions to (1) and (2). For (1), certain colorings of Young tableaux arise, with colorings controlled by some arithmetic functions defined by divisor functions. The main difficulty is for (2): although the techniques are standard (namely Mellin transformation, then residue analysis, and finally saddle point method), to perform the previous analysis in such generality can be technically challenging and requires care. Gmez and Ward solve the partition problem in its generality only for the logarithm of the coefficients, and they find the actual asymptotic growth of coefficients only in a few particular cases when certain saddle point equations can be solved.

2.3 Presentation Highlights

Each of the first four days of the workshop started by a one-hour talk, providing a substantial insight into the current research in the field.

Robert Sedgewick presented a new algorithm for the cardinality estimation problem. The algorithm was demonstrated to be effective with an approximate analysis and showed much potential for further research. It can be implemented in a dozen lines of code and can accurately estimate the number of distinct elements in an input stream, using a very small amount of memory.

James Allen Fill discussed the celebrated algorithm QuickQuant and proved that the limiting distribution of the scale-normalized number of key comparisons used by QuickQuant to find the tth quantile has a Lipschitz continuous density function that is bounded above by 10. Furthermore, this density has superexponential decay in the right tail. Fill also pointed out that the presented results also enable perfect simulation from the limiting distribution.

Robin Pemantle discussed generating functions such as those for restricted random walks, trees by size and type of node, and statistical mechanical models, with an emphasis on the computational analysis of these functions. These generating functions are often implicitly defined by algebraic or transcendental equations.

Laura Eslava focused on random recursive trees and weighted random recursive trees. These trees are rooted and give a label to each vertex which is connected with a previous vertex according to a probability correlated with the associated weight. The weight remains constant throughout the process. Unlike linear preferential attachment trees, the high-degree vertices in the random recursive and weighted trees keep changing during the process. Eslava proposed a structure to describe the order of the degree and the height of Eslava provided a description of both the order of the degree and the height of such high-degree vertices for (a wide class of weighted) random recursive trees and presented some applications and open problems.

2.4 Software Session

Our workshop ended with a software session. We had the following presentations.

2.4.1 Stephen Melczer.

Stephen Melczer presented a new Sage package to derive the asymptotics of multivariate generating functions using the theory of analytic combinatorics in several variables (ACSV). This is the first package that rigorously checks assumptions that must hold for the results of ACSV to apply and is currently available to users through the pip package manager.

2.4.2 Benjamin Hackl.

The subject of this talk was the module for computations with asymptotic expansions in SageMath.

2.4.3 Ricardo Gomez.

A prototype of a software application to produce subsets of musical scales was presented. The main mathematical ideas were explained, starting from the fact that musical scales are combinatorially isomorphic to integer compositions. The GUI gives the user very quick access to all the musical scales. The search engine is based on the Ising model and the parameters in the Hamiltonian energy functions that determine the corresponding Gibbs measures can be manipulated to calibrate specific configurations with precise combinatorial properties. The simulation of generic configurations is carried out through Monte Carlo Markov chain models like Glauber dynamics. It was announced that the software soon will be accessible to the public, in particular to musicians that could make use of it for multiple purposes.

3 New Projects and Scientific Progress

The wide variety of specialties present at the workshop led to numerous new collaborations, even new perspectives to tackle old open problems. For instance:

Luc Devroye and Stephen Melczer told Miklós Bóna about their work on *hipster trees*, that is, rooted trees in which no two siblings have subtrees of identical size. Miklós Bóna and Boris Pittel started a new collaboration on this topic.

Robin Pemantle has started working on a problem presented by Robert Sedgewick, the problem of estimating to within a factor of 1.05 (with high probability) the number of distinct entries in a stream, using a one-pass algorithm with as little space and time as possible.

Following a suggestion by Robin Pemantle, Yin Mei started investigating the expansion of the probability distribution for a random parking function around the uniform distribution by powers of n. Pemantle conjectured based on the total variance distance given in Mei's talk. This looks like a very promising research direction.

Sebastian Wild suggested that Yin Mei try another way of doing hashing in computer science. Rather than linear probing hashing as discussed in Mei's talk, Wild suggested Robin Hood hashing. The hope is that interesting new statistics can be derived.

4 Summary of the Meeting

The workshop brought together two main groups of researchers working in analytic and probabilistic combinatorics. Many of the presentations added an algorithmic flavor to the workshop. The variety of subjects was enriching and motivating, which gave rise to several new collaborations and projects. In addition to creating synergy to promote teamwork and to facilitate learning different tools, general diversity was achieved. In particular, even though international travel still poses numerous challenges, we can report the following.

- In-person participants came from the US, Canada, México, Chile, the United Kingdom, Sweden, Austria and France.
- In addition to this, virtual participants attended from The Netherlands, South Africa, Taiwan, Uruguay, and Argentina.
- One of the organizers, eight of the in-person participants, and 11 of the virtual participants were women.
- Five of the talks were given by women.
- The participants ranged from consolidated and internationally recognized experts to young researchers and some students.

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