Remarks on the Black Hole Stability problem

Lars Andersson

Albert Einstein Institute

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joint work with Thomas Bäckdahl, Pieter Blue, and Siyuan Ma

arXiv:1903.03859, arXiv:2108.03148

- Einstein equations, 4D

  Action $S = \int R d\mu$

  $\delta S = 0 \quad \Rightarrow \quad R_{ab} = 0$

- Lorentz signature $+ - --$
  - hyperbolic (mod. gauge)
  - Cauchy problem: given Cauchy data, $\exists!$ maximal development

Introduction

Kerr

- Stationary, rotating, isolated, vacuum
- 2-parameter family: mass M, angular momentum per unit mass a
- Petrov D $\rightsquigarrow$ separability, integrability, decoupling, conservation laws

  subextreme $|a| < M \rightsquigarrow$ non-degenerate black hole

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Major conjectures $\sim$ 1970

- Strong/Weak C. C.
- BH Uniqueness/Stability
- P.I., End State Conjecture

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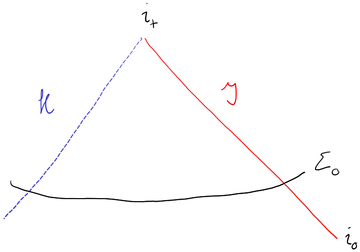
Open

Black Hole Stability Conjecture

The maximal development of Cauchy data close to Kerr data is asymptotic (to the future) to a member of the Kerr family.



Much work on the problem: Chandrasekhar, Carter, Teukolski, Whiting, Dafermos, Holzegel, Rodnianski, Klainerman, Szeftel, Giorgi, Schlapentokh-Rothman, Teixeira da Costa, Angelopoulos, Aretakis, Gajic, L.A., Blue, Bäckdahl, Ma, Hafner, Hintz, Vasy, ...

- TME, mode stability
- Morawetz estimate, decay for TME ($|a| \ll M$)
- linearized stability ($|a| \ll M$)
- steps towards nonlinear stability ($|a| \ll M$)
- Schwarzschild stability
- Kerr-dS stability ($|a| \ll M$)

Introduction

Kerr parameters

M mass, J angular momentum, $a = J/M$

- BH $\leftrightarrow$ spinning particle
- momentum in $\mathrm{Lie}(P)^*$ (10-dim)

  4-momentum p^a, angular momentum $\leftrightarrow J^a$, $J^a p_a = 0$.
- $M^2 = p^a p_a$, $W^2 = J^a J_a$ Casimirs — determine coadjoint orbit in $\mathrm{Lie}(P)^*$

Introduction

Gauge

- Kerr metric in B-L, E-F, etc. coordinates represents BH in rest frame, with aligned rotation axis

- Kerr metric in B-L, E-F, etc. coordinates represents BH in rest frame, with aligned rotation axis
- Only scalars M, a are Lorentz invariant, but there are more "global" gauge d.o.f.

$\rightsquigarrow$ need to consider both "local" and "global" gauge

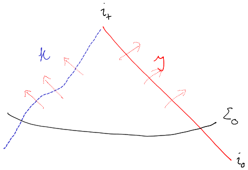
  - hyperbolicity — "local" gauge
  - alignment — "global" gauge

Introduction

Dynamical BH

- For linearized gravity, mass and angular momentum are quasi-locally conserved
- For nonlinear GR, mass and angular momentum radiate through $\mathcal{I}$ and $\mathcal{H}$

- final BH parameters can't be directly determined from the initial data.
- (apparent) horizon evolves



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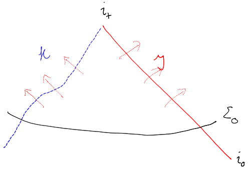
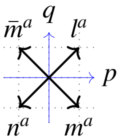


- trapping
- superradiance

GHP formalism

- null frame $l^a, n^a, m^a, \bar{m}^a$
- $l^a n_a = 1, m^a \bar{m}_a = -1$
- boost and spin rotations, properly weighted scalars

$$\eta \to \lambda^p \bar{\lambda}^q \eta, \quad \text{type } \{p, q\}$$



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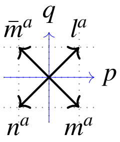

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$$\eta \to \lambda^p \bar{\lambda}^q \eta, \quad \text{type } \{p, q\}$$

  - boost weight: $b = \frac{1}{2}(p+q)$
  - spin weight: $s = \frac{1}{2}(p-q)$
  - de-boosting $\rightsquigarrow$ spin-weighted scalars

$$\bar{\eta}\eta \quad \text{is true scalar}$$

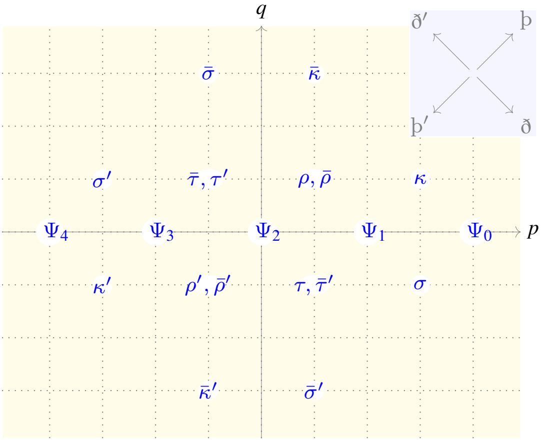
  - Necessary for estimates

GHP formalism

- operators þ, þ′, ð, ð′
- Connection components (spin coefficients):
- connection ⇝

  8 spin coefficients $\rho, \kappa, \sigma, \tau$ + ′ versions

  + 4 not properly weighted
- curvature scalars $\Psi_0, \ldots, \Psi_4$



Linearized gravity on Kerr

Special geometry

- Kerr is Petrov D $\Rightarrow \exists$ principal null frame
- only $\Psi_2, \rho, \tau, \rho', \tau' \neq 0$.

  $\Rightarrow$ Carter Killing tensor, symmetries, decoupling, integrability etc.

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Linearized gravity on Kerr

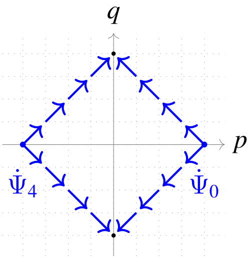
- $D\mathrm{Ric}.\dot{g}_{ab} = 0$
- gauge $\dot{g}_{ab} \to \dot{g}_{ab} + \nabla_{(a}\nu_{b)}$
- gauge invariants $\dot{\Psi}_0, \dot{\Psi}_4, \mathbb{I}_\xi, \mathbb{I}_\zeta$ Aksteiner et al., 2018, 2021
- $\mathbb{I}_\xi, \mathbb{I}_\zeta \leftrightarrow$ type D parameters $\dot{M}, \dot{a}, \dot{N}, \dot{c}$.

Linearized gravity on Kerr

- fields:
  - $\dot{\Psi}_0, \dot{\Psi}_4,$
  - linearized spin coeffcients & metric coefficients

Linearized gravity on Kerr

- fields:
  - $\dot{\Psi}_0, \dot{\Psi}_4$,
  - linearized spin coeffcients & metric coefficients
- $\dot{\Psi}_0, \dot{\Psi}_4$ solve TME, TSI:



- TME: spin, boost weighted wave eq.
- Carter symmetry operator $\rightsquigarrow$ separability
- TSI: 4th order differential relation $\dot{\Psi}_0 \leftrightarrow \dot{\Psi}_4$

Linearized gravity on Kerr

TME

- separated form of TME $\rightsquigarrow$ confluent Heun equation
- mode stability Whiting 1989, Shlapentokh-Rothman 2015, L.A. et al. 2017, Teixeira da Costa 2020
- Morawetz estimate (integrated local energy decay) Ma 2017, Dafermos & Holzegel & Rodnianski 2017 $|a| \ll M$
- Price law decay for $\dot{\Psi}_0, \dot{\Psi}_4$ Ma & Zhang 2021 $|a| \ll M$

Linearized gravity on Kerr

ORG Price & Shankar & Whiting 2006

Debye potential

- solution of $\mathrm{TME}^\dagger \psi = 0 \rightsquigarrow$ solution of $D\mathrm{Ric}.\dot{g}_{ab} = 0$
- $\dot{g}_{ab}$ in outgoing radiation gauge (ORG):

$$g^{ab}\dot{g}_{ab} = 0, \quad \dot{g}_{ab}n^b = 0$$

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- general solution of $D\mathrm{Ric}.\dot{g}_{ab} = 0$ on Kerr can be put in ORG
- 5 gauge conditions — requires special geometry
- $g^{ab}\dot{g}_{ab} = 0 \leftrightarrow$ residual gauge on Σ_0

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ORG

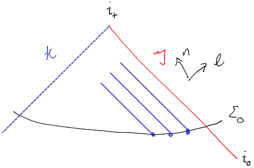
- kills 5 of 10 d.o.f. in $\dot{g}_{ab}$
- kills 3 of 12 spin coefficients

Linearized gravity on Kerr

L.A. & Bäckdahl & Blue & Ma 2019

$D\mathrm{Ric}.\dot{g}_{ab} = 0$ in ORG $\Rightarrow$

- TME for $\dot{\Psi}_0, \dot{\Psi}_4$
- hierarchy of transport equation
  $\text{þ}' \varphi = \psi$
- sourced by $\dot{\Psi}_4$



Linearized gravity on Kerr

L.A. & Bäckdahl & Blue & Ma 2019

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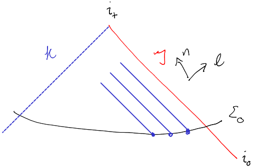

- $\dot{M}, \dot{a}$ decouple — represented by explicit solutions in ORG

Linearized gravity on Kerr

L.A. & Bäckdahl & Blue & Ma 2019

- Large PDE system
- use symbolic algebra to derive and manipulate the equations
- `xAct` for ™Mathematica xact.es
- packages for `xAct`:
  - `SpinFrames` Aksteiner&Bäckdahl 2015-2021,
  - `SymManipulator` Bäckdahl 2011-2021

$$
\begin{aligned}
\text{þ}' G_{00'} &= -4\tilde\epsilon + 2\tilde\rho - 2\overline{\tilde\rho} - 2G_{10'}\tau - 2G_{01'}\bar\tau + G_{01'}\tau' + G_{10'}\bar\tau',\\
(\text{þ}' - \rho')G_{01'} &= -G_{02'}\bar\tau + 2\overline{\tilde\tau'},\\
(\text{þ}' - \rho')G_{02'} &= 2\overline{\tilde\sigma'},\\
(\text{þ}' - \bar\rho')G_{10'} &= -G_{20'}\tau + 2\tilde\tau',\\
(\text{þ}' - \bar\rho')G_{20'} &= 2\tilde\sigma',\\
(\text{þ}' - \rho' + \bar\rho')\tilde\tau' &= 2\tilde\beta'\rho' + (\eth - \tau + \bar\tau')\tilde\sigma',\\
(\text{þ}' - 2\rho' - \bar\rho')\tilde\beta' &= \rho'\tilde\tau' - \bar\rho'\tilde\tau' + (\eth - \tau)\tilde\sigma',\\
(\text{þ}' - \bar\rho')\tilde\sigma' &= \vartheta\Psi_4,\\
(\text{þ}' - \rho' - \bar\rho')\tilde\rho &= \tilde\epsilon\rho' + \bar{\tilde\epsilon}\rho' + \tfrac12 G_{00'}\rho'\bar\rho' + 2\tilde\beta'\tau + \tfrac12 G_{10'}\bar\rho'\tau - G_{01'}\bar\rho'\tau' - \tfrac12 G_{02'}\bar\tau\tau' + \tau'\overline{\tilde\tau'}\\
&\quad - (\eth - \bar\tau')\tilde\tau',\\
(\text{þ}' + \bar\rho')\tilde\kappa &= \tfrac54 G_{01'}\Psi_2 + \frac{G_{01'}\bar\Psi_2\overline{\kappa}_{1'}}{4\kappa_1} - 2\tilde\beta\rho - \tfrac32 G_{01'}\bar\rho\rho' - 2\tilde\epsilon\tau - \tfrac12 G_{00'}\bar\rho'\tau + \tfrac12 G_{02'}\rho\tau'\\
&\quad + G_{02'}\bar\rho\tau' + \tfrac12\tau'(\eth - \tau - \bar\tau')G_{01'} + (\eth - \tau + \bar\tau')\tilde\rho - (\eth' + \bar\tau - 2\tau')\tilde\sigma - \tfrac12\rho'\,\eth\, G_{00'},\\
(\text{þ}' - 2\rho')\tilde\epsilon &= \tilde\beta'\tau - \tilde\beta\bar\tau - \tilde\beta\tau' - \tfrac12 G_{01'}\rho'\tau' + \tfrac12 G_{02'}\tau'^2 - \tilde\beta'\bar\tau' - (\eth - \tau - \bar\tau')\tilde\tau',\\
(\text{þ}' - \rho')\tilde\beta &= -\tfrac12 G_{01'}\rho'^2 + \tfrac12 G_{02'}\rho'\tau',\\
(\text{þ}' - \rho')\tilde\sigma &= \tfrac34 G_{02'}\Psi_2 - \frac{G_{02'}\bar\Psi_2\overline{\kappa}_{1'}}{4\kappa_1} + \tfrac12 G_{02'}\rho\rho' - \tfrac12 G_{02'}\bar\rho\rho' - 2\tilde\beta\tau - \tfrac12\rho'(\eth + \tau)G_{01'}\\
&\quad + \tfrac12\tau'\,\eth\, G_{02'},\\
(\text{þ}' - 4\rho')\vartheta\Psi_3 &= (\eth - \tau)\vartheta\Psi_4,\\
(\text{þ}' - 3\rho')\vartheta\Psi_2 &= (\eth - 2\tau)\vartheta\Psi_3,\\
(\text{þ}' - 2\rho')\vartheta\Psi_1 &= (\eth - 3\tau)\vartheta\Psi_2,\\
(\text{þ}' - \rho')\vartheta\Psi_0 &= \tfrac32 G_{02'}\Psi_2\rho + 3\Psi_2\tilde\sigma - \tfrac32 G_{01'}\Psi_2\tau + (\eth - 4\tau)\vartheta\Psi_1\\
\tilde\beta &= -\tfrac12 G_{01'}\rho' + \tfrac12 G_{01'}\bar\rho' - \tfrac12\overline{\tilde\tau'} + \tfrac14(\eth' + \tau')G_{02'},\\
\tilde\beta' &= \tfrac12 G_{10'}\rho' + \tfrac12\tilde\tau' + \tfrac14(\eth - \bar\tau')G_{20'},\\
\tilde\kappa &= \tfrac12(\text{þ} - 2\bar\rho)G_{01'} - \tfrac12(\eth - \bar\tau')G_{00'},\\
\tilde\rho &= -\tfrac12 G_{00'}\rho' + \tfrac12 G_{01'}\tau' - \tfrac12(\eth - \bar\tau')G_{10'},\\
\tilde\sigma &= \tfrac12(\text{þ} - \bar\rho)G_{02'} - \tfrac12(\eth - 2\bar\tau')G_{01'},\\
\tilde\tau &= -\tfrac12 G_{01'}\rho' + \tfrac12 G_{02'}\tau',\\
(\text{þ} - \rho)\tilde\rho &= -\tfrac12 G_{00'}\Psi_2 + 2\tilde\epsilon\rho + (\eth' - \tau')\tilde\kappa,\\
(\text{þ} - 2\rho - \bar\rho)\tilde\beta &= -\tfrac12 G_{01'}(\Psi_2 + \rho\rho' - \bar\rho\rho') + \tilde\kappa\bar\rho' + \tfrac12 G_{02'}(\rho - \bar\rho)\tau' + (\eth - \bar\tau')\tilde\epsilon + (\eth' - \tau')\tilde\sigma - \eth\,\tilde\rho,\\
(\text{þ} - \rho)\tilde\beta' &= -G_{10'}\Psi_2 + \tilde\rho\tau' + \rho\tilde\tau' - (\eth' - \tau')\tilde\epsilon,\\
(\text{þ} - \rho)\tilde\sigma' &= \tfrac12 G_{20'}\Psi_2 - 2\tilde\beta'\tau' + (\eth' - \tau')\tilde\tau',\\
0 &= \tilde\epsilon(\rho' + \bar\rho') - \rho'\tilde\rho - (\eth - \bar\tau')\tilde\tau' + (\eth' - 2\tau')\tilde\beta + \eth\,\tilde\beta',\\
\vartheta\Psi_0 &= (\text{þ} - \bar\rho)\tilde\sigma - (\eth - \bar\tau')\tilde\kappa,\\
\vartheta\Psi_1 &= -\tilde\kappa\rho' + \tilde\sigma\tau' + (\text{þ} - \bar\rho)\tilde\beta - (\eth - \bar\tau')\tilde\epsilon,\\
\vartheta\Psi_2 &= -2\tilde\epsilon\rho' + 2\tilde\beta\tau' + (\eth - \bar\tau')\tilde\tau',\\
\vartheta\Psi_3 &= 2\tilde\beta'\rho' + (\rho' - \bar\rho')\tilde\tau' + \eth\,\tilde\sigma',\\
0 &= 3\Psi_2\tilde\kappa + \tfrac32 G_{01'}\Psi_2\rho - \tfrac32 G_{00'}\Psi_2\tau + (\text{þ} - 4\rho)\vartheta\Psi_1 - (\eth' - \tau')\vartheta\Psi_0,\\
0 &= -\tfrac32 G_{00'}\Psi_2\rho' - 3\Psi_2\tilde\rho + 3G_{10'}\Psi_2\tau + \tfrac32 G_{01'}\Psi_2\tau' + (\text{þ} - 3\rho)\vartheta\Psi_2 - (\eth' - 2\tau')\vartheta\Psi_1,\\
0 &= 3G_{10'}\Psi_2\rho' - \tfrac32 G_{20'}\Psi_2\tau + 3\Psi_2\tilde\tau' + (\text{þ} - 2\rho)\vartheta\Psi_3 - (\eth' - 3\tau')\vartheta\Psi_2,\\
0 &= -\tfrac32 G_{20'}\Psi_2\rho' - 3\Psi_2\tilde\sigma' + (\text{þ} - \rho)\vartheta\Psi_4 - (\eth' - 4\tau')\vartheta\Psi_3.
\end{aligned}
$$

Linearized gravity on Kerr

L.A. & Bäckdahl & Blue & Ma 2019

- Weighted Hardy estimates for transport equations:

  decay estimate for TME $\rightsquigarrow$ decay for fields

- due to scaling in transport equations, leading order terms at $\mathcal{I}$ need to be treated separately

- Get decay at i_+ using TSI at $\mathcal{I}$

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- Proof is modular: Morawetz estimate for TME $\rightsquigarrow$ stability for linearized gravity

  $\rightsquigarrow$ get linearized stability for $|a| < M$ given Morawetz estimate for TME for $|a| < M$

- Harmonic gauge $g^{ab}\dot{\Gamma}^{c}_{ab} = 0 \rightsquigarrow$

$$\nabla^c \nabla_c \dot{g}_{ab} + 2R_{acbd}\dot{g}^{cd} = 0$$

- harmonic gauge + "robust Fredholm" Hafner & Hintz & Vasy 2019 $|a| \ll M$
- Chandrasekhar transform, Regge-Wheeler type equation Dafermos et al., Johnson, Hung & Keller & Wang, ... $|a| \ll M$
- double null coordinates Dafermos et al., ...

Nonlinear Radiation Gauge

L.A. & Bäckdahl & Blue & Ma 2021

Spacetimes $(M, g_{ab}$ near Kerr $(M, \mathring{g}_{ab})$

- $\mathring{g}_{ab}$ Kerr background
- n^a ingoing principal null direction (background)
- $\delta g_{ab} = g_{ab} - \mathring{g}_{ab}$

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- $\mathring{g}_{ab}$ Kerr background
- n^a ingoing principal null direction (background)
- $\delta g_{ab} = g_{ab} - \mathring{g}_{ab}$
- Diffeomorphism gauge

$$
\begin{aligned}
n^a \delta g_{ab} &= 0 \\
g^{ab} \delta g_{ab} &= O(\epsilon^2)
\end{aligned}
$$

Nonlinear Radiation Gauge

L.A. & Bäckdahl & Blue & Ma 2021

NLORG

- Bianchi $\rightsquigarrow$ FOSH system for Ψ_i
- difference variables for metric, spin coefficients, curvature
- introduce frame gauge

$\rightsquigarrow$ Bianchi + transport system $\text{Þ}' \varphi = \psi$

$\rightsquigarrow$ FOSH system for EFE

Nonlinear Radiation Gauge

L.A. & Bäckdahl & Blue & Ma 2021

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  $\rightsquigarrow$ Bianchi + transport system $\text{þ}' \varphi = \psi$

  $\rightsquigarrow$ FOSH system for EFE

- Bianchi $\rightsquigarrow$ nonlinear TME for Ψ_4
- quadratic and higher order couplings

  $\rightsquigarrow$ coupled wave-transport system for EFE

Concluding remarks

Challenges in the black hole stability problem

- Dynamical background
- Coupled system $\rightsquigarrow$ loss of regularity in the estimates

  $\rightsquigarrow$ use tame (Nash-Moser) type estimates
- Fixed point problem: Kerr-dS stability Hinz & Vasy 2016
- Bootstrap: double null + TME + TSI + GCM sphere condition Klainerman et al

Concluding remarks

NLORG

- NLORG: TME + transport system
- TME is the origin of decay — use transport system to construct solution
- The reduced field equation depends on the unknown final Kerr parameters

  $\rightsquigarrow$ coupled system — unknowns include final Kerr parameters
- Fixed point problem