The Weyl-BMS group and the asymptotic dynamics

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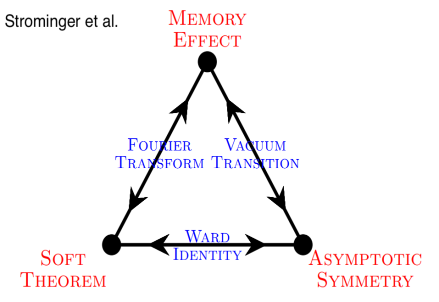
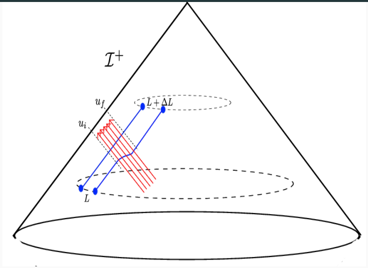
A quick overview: some highlights in the last 60 years!





[Ashtekar, 1409.1800],

[Strominger & Zhiboedov, 1411.5745]

1962 the BMS group = Lorentz x supertranslations:
asymptotic symmetry group for (asymptotically) flat spacetimes.

1965 Weinberg's graviton soft theorems:
relations among scattering amplitudes in the infrared regime.

1974 Gravitational memory/hereditary effects:
permanent shift in the relative position of two inertial detectors after GW passed.

Strominger's triangle arises new theoretical questions → new gravitational effects.

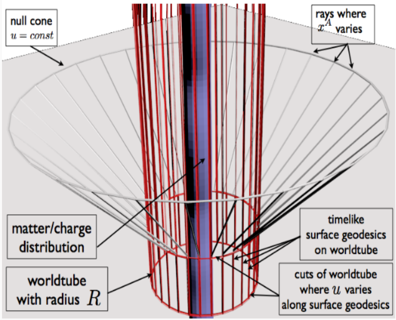
E.g., the larger the symmetry group → the more soft theorems/memory effects.

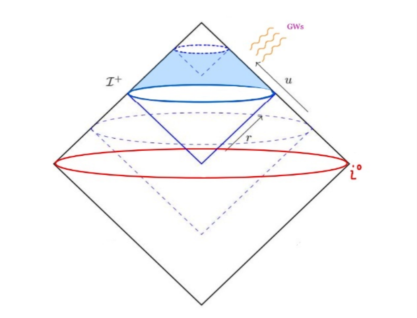
Main question today: What is the largest asymptotic symmetry group in gravity?

It serves as an organising principle.

Outline

The Weyl-BMS group





[Madler&Winicour, 1609.01731]

Bondi coordinates: $x^\mu = (u, r, x^A)$,
Bondi gauge: $g_{rr} = 0$, $g_{rA} = 0$, $\partial_r \det\left(g_{AB}/r^2\right) = 0$.

Bondi-Sachs metric:

$$ds^2 = -2e^{2\beta} du(F du + dr) + r^2 q_{AB}(dx^A - U^A du)(dx^B - U^B du),$$

where β, F, U^A, and q_{AB} are functions of (u, r, x^A).

What are the asymptotic boundary conditions for these quantities?

The boundary conditions

The ur and uA components obey the fall-off conditions:

$$g_{ur} = -1 + \mathcal{O}(r^{-2}), \quad g_{uA} = \mathcal{O}(1).$$

More "freedom" in the uu and AB components:

$$\begin{cases} g_{uu} = -1 + \mathcal{O}(r^{-1}), & q_{AB} = \mathring{q}_{AB} + \mathcal{O}(r^{-1}) & \text{(original BMS)} \\ g_{uu} = \mathcal{O}(r), & q_{AB} = e^{2\phi(u)}\mathring{q}_{AB} + \mathcal{O}(r^{-1}) & \text{(extended BMS)} \\ g_{uu} = \mathcal{O}(1), & q_{AB} = \bar{q}_{AB} + \mathcal{O}(r^{-1}) & \text{(generalized BMS)} \end{cases}$$

original BMS: $\mathring{q}_{AB}$ round metric on S^2 with Ricci scalar $\mathring{R} = 2$ [Bondi-Metzner-Sachs, 1962]

extended BMS: conformally related to $\mathring{q}_{AB}$ with u-dependence [Barnich-Troesseart, 2010]

generalized BMS: $\partial_u \bar{q}_{AB} = 0$ and $\delta\sqrt{\bar{q}} = 0$ [Campiglia-Laddha, 2014] [Compère et al., 2018]

Weyl-BMS: $\partial_u \bar{q}_{AB} = 0$ and $\delta\sqrt{\bar{q}} \neq 0$ [Freidel-RO-Pranzetti-Speziale, 2021]

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Remark 1: $\partial_u \bar{q}_{AB} = 0$ implies that $g_{uu} = \mathcal{O}(1)$.

Enough to describe MPM spacetimes [Blanchet et al, 2021]

Remark 2: relaxing bcs $\rightarrow$ divergences $\rightarrow$ phase-space renormalization!

in AdS/CFT adding boundary action counter-terms [deHaro-Solodukhin-Skenderis, 2001], [Compère-Marolf, 2008]

in generalized BMS adding boundary Lagrangian (and associated symplectic potential) [Compère et al., 2018]

Additional investigation of this issue in [Freidel-Geiller-Pranzetti, 2020]

The asymptotic symmetry group: generators

We seek vector fields $\xi = \xi^u \partial_u + \xi^r \partial_r + \xi^A \partial_A$

a) preserving the Bondi gauge:

$$\xi^u = \tau, \quad \xi^r = -rW + \frac{r}{2}\left[D_A\left(I^{AB}\partial_B\tau\right) + U^A\partial_A\tau\right], \quad \xi^A = Y^A - I^{AB}\partial_B\tau$$

Here τ, W, and Y^A are functions of (u, x^A), and $I^{AB} = \int_r^{+\infty} dr' e^{2\beta} q^{AB}/r'^2$. Moreover, we allow the scale structure to vary:

$$\delta_\xi \sqrt{\bar{q}} = \left(D_A Y^A - 2W\right)\sqrt{\bar{q}}$$

b) preserving the boundary conditions:

$$\tau = T + uW, \quad \partial_u W = 0 = \partial_u T, \quad \partial_u Y^A = 0$$

Weyl-BMS generators at null infinity:

$$\boxed{\bar{\xi}_{(T,W,Y)} := T\partial_u + W\left(u\partial_u - r\partial_r\right) + Y^A\partial_A}$$

$T(x^A)$: super-translations; $W(x^A)$: Weyl rescaling of S^2; $Y^A(x^B)$: diffeos of S^2.

The asymptotic symmetry group: algebra

Weyl-BMS generators at null infinity:

$$\boxed{\bar{\xi}_{(T,W,Y)} := T\partial_u + W\left(u\partial_u - r\partial_r\right) + Y^A\partial_A}$$

Weyl-BMS Lie algebra:

$$\boxed{\left(\mathrm{diff}(S^2) \oplus \mathcal{W}_{S^2}\right) \oplus \mathcal{T}_{S^2}}$$

from the Lie commutators $[\bar{\xi}_{(T_1,W_1,Y_1)}, \bar{\xi}_{(T_2,W_2,Y_2)}] = \bar{\xi}_{(T_{12},W_{12},Y_{12})}$ with

$$T_{12} = Y_1[T_2] - W_1 T_2 - (1 \leftrightarrow 2), \quad W_{12} = Y_1[W_2] - Y_2[W_1], \quad Y_{12} = [Y_1, Y_2]$$

	background structure	restriction	parametrisation
Weyl-BMS	$\emptyset$	$\emptyset$	(T, W, Y)
generalized BMS	scale structure	$\delta\sqrt{q} = 0$	$(T, \frac{1}{2}D_A Y^A, Y)$
extended BMS	conformal structure	$\delta[q_{AB}] = 0$	$\left(e^{\phi}t, \frac{1}{2}(D_A Y^A - w), Y\right)$
original BMS	round sphere structure	$\delta q_{AB} = 0$	$(T, \frac{1}{2}D_A Y^A, Y)$

- Weyl-BMS group: $(\mathrm{Diff}(S^2) \ltimes \mathcal{W}_{S^2}) \ltimes \mathcal{T}_{S^2}$ [Freidel, RO, Pranzetti, Speziale, 2021]
- generalized BMS group: $\mathrm{Diff}(S^2) \ltimes \mathcal{T}_{S^2}$ [Campiglia-Laddha, 2014]-[Compère et al., 2018]
- extended BMS group: $(\mathrm{Vir} \times \mathrm{Vir}) \ltimes \mathcal{T}_{S^2}$ [Barnich-Troessaert, 2010]
- original BMS group: $SL(2, \mathbb{C}) \ltimes \mathcal{T}_{S^2}$

The Weyl-BMS charge algebra and asymptotic Einstein's equations

Canonical analysis

Nomenclature:

$\{d, i_\xi, \mathcal{L}_\xi\}$, spacetime diff., contraction and Lie derivative: $\mathcal{L}_\xi = di_\xi + i_\xi d$

$\{\delta, I_\xi := I_{\mathcal{L}_\xi}, \delta_\xi := \delta_{\mathcal{L}_\xi}\}$, field-space diff., contraction and variation: $\delta_\xi = \delta I_\xi + I_\xi \delta$.

Given a Lagrangian L, $\delta L = d\theta_L - E$.

E stands for the e.o.m., and θ_L is the symplectic potential.

Noether's theorems say that

$$I_\xi E = dC_\xi, \quad j_\xi := I_\xi \theta_L - i_\xi L = C_\xi + dq_\xi \quad (dj_\xi \approx 0)$$

In gravity: $E = G_{\mu\nu}\delta g^{\mu\nu}\epsilon$, $C_\xi = \xi^\nu G_\nu{}^\mu \epsilon_\mu$, $\theta_L = 2g^{\rho[\sigma}\delta\Gamma^{\mu]}_{\rho\sigma}\epsilon_\mu$, where $\epsilon_\nu = i_{\partial_\nu}\epsilon$; and $q_\xi = \nabla^\mu \xi^\nu \epsilon_{\mu\nu}$.

The symplectic 2-form, the Noether charge and the flux read as

$$\Omega = \int_\Sigma \delta\theta_L, \quad Q_\xi = \int_{S^2} q_\xi, \quad \mathcal{F}_\xi = \int_{S^2} (i_\xi \theta_L + q_{\delta_\xi})$$

obey the fundamental canonical relation (see e.g., [Lee-Wald, 1990], [Iyer-Wald, 1994])

$$\boxed{-I_\xi \Omega \approx \delta Q_\xi - \mathcal{F}_\xi}$$

Contracting again with I_χ:

$$\boxed{\delta_\xi Q_\chi - I_\chi \mathcal{F}_\xi \approx -\left(\delta_\chi Q_\xi - I_\xi \mathcal{F}_\chi\right)}$$

Remark 1: invariant under the change of boundary Lagrangian $L \to L + dl$;

Remark 2: insensitive to phase-space renormalization: divergences cancel out!

Charge bracket

The antisymmetry of the symplectic form Ω suggests the charge bracket
(generalizes [Barnich-Troessaert, 2011], related work [Wieland, 2021])

$$\boxed{\{Q_\xi, Q_\chi\}_L := \delta_\xi Q_\chi - I_\chi \mathcal{F}_\xi + \int_{S^2} i_\xi i_\chi L}$$

Consider two (field-dependent) vector fields ξ and χ with modified Lie bracket

$$[\![\xi, \chi]\!] := [\xi, \chi]_{Lie} + \delta_\chi \xi - \delta_\xi \chi$$

s.t. the commutator of two field space variations is still a symmetry transformation

$$[\delta_\xi, \delta_\chi] = -\delta_{[\![\xi,\chi]\!]}$$

It can be proven that [technical step: $\Delta_\xi Q_\chi := (\delta_\xi - \mathcal{L}_\xi - I_{\delta_\xi})Q_\chi = Q_{\delta_\chi \xi} - Q_{[\![\xi,\chi]\!]}$]

$$\boxed{\{Q_\xi, Q_\chi\}_L = -Q_{[\![\xi,\chi]\!]} - \int_{S^2} i_\xi C_\chi \approx -Q_{[\![\xi,\chi]\!]}}$$

Property 1: it provides a representation of the vector field algebra on-shell.
Property 2: it is invariant under $L \to L + dl$.

This is the flux-balance relation, equivalent to the (asymptotic) Einstein's equations.

Obtaining the asymptotic Einstein's equations

Interplay among: geometric data – phase-space data – dynamics

$$\boxed{\{Q_\xi, Q_\chi\}_L + Q_{[\![\xi,\chi]\!]} \approx 0 \Longleftrightarrow \delta_\xi Q_\chi + Q_{[\![\xi,\chi]\!]} \approx I_\chi \mathcal{F}_\xi + \int_{S^2} i_\chi i_\xi L}$$

Weyl-BMS generators (ξ, χ)	$\{Q_\xi, Q_\chi\} + Q_{[\![\xi,\chi]\!]} = 0$	**Einstein's equations**
(∂_u, ξ_T)	$2\mathsf{E}_M - \frac{1}{4}\bar{\Delta}\mathsf{E}_{\bar{F}} = 0$	$\xi_T^\mu G_\mu{}^r = 0$
(ξ_T, ∂_u)	$2\mathsf{E}_M + \bar{D}^A\dot{\mathsf{E}}_{\bar{U}_A} + \frac{1}{4}\bar{\Delta}\mathsf{E}_{\bar{F}} = 0$	$\xi_T^u G_u{}^r - \xi_T^r G_u{}^u = 0$
(∂_u, ξ_W)	$\bar{D}^A\mathsf{E}_{\bar{U}_A} + u\left(2\mathsf{E}_M - \frac{1}{4}\Delta\mathsf{E}_{\bar{F}}\right) = 0$	$\xi_W^\mu G_\mu{}^r = 0$
(ξ_W, ∂_u)	$-\bar{D}^A\mathsf{E}_{\bar{U}_A} + u\left(2\mathsf{E}_M + \bar{D}^A\dot{\mathsf{E}}_{\bar{U}_A} + \frac{1}{4}\Delta\mathsf{E}_{\bar{F}}\right) = 0$	$\xi_W^u G_u{}^r - \xi_W^r G_u{}^u = 0$
(∂_u, ξ_Y)	$\mathsf{E}_{\bar{P}_A} + 2\bar{D}_A\dot{\mathsf{E}}_{\bar{\beta}} - 2\bar{F}\mathsf{E}_{\bar{U}_A} - \frac{1}{2}\bar{U}_A\mathsf{E}_{\bar{F}} = 0$	$\xi_Y^\mu G_\mu{}^r = 0$
(ξ_Y, ∂_u)	$0 = 0$	$0 = 0$

- original BMS: 1 flux-balance (energy);
- generalized BMS: 3 flux-balances (energy, angular mom) – importance of diff(S^2);
- Weyl-BMS: 5 flux-balances – importance of the Weyl rescalings.

Phase space renormalization

Rule-of-thumb: the weaker the boundary conditions, the more the divergences!

The divergent part of the symplectic potential reads as

$$\theta_{div} = d\vartheta_{div} - \frac{r}{2}\delta\left(\sqrt{\bar{q}}\ (\bar{R} - 4\bar{F})\right) du\ d^2\sigma$$

where (recall that $\delta\sqrt{\bar{q}} \neq 0$)

$$\vartheta_{div} = \left(\frac{r^2}{2}\delta\sqrt{\bar{q}} - \frac{r}{4}\sqrt{\bar{q}}C^{AB}\delta\bar{q}_{AB}\right) d^2\sigma + r\bar{\vartheta}^A\epsilon_{AB}d\sigma^B \wedge du, \quad \partial_A\bar{\vartheta}^A = \frac{1}{2}\delta(\sqrt{\bar{q}}\bar{R})$$

Strategy: make use of a boundary Lagrangian $L^R = L + d\ell$ to renormalize

$$\theta^R = \theta - d\vartheta + \delta\ell, \quad Q^R_\xi = Q_\xi + \int_{S^2}\left(i_\xi\ell - I_\xi\vartheta\right), \quad \mathcal{F}^R_\xi = \mathcal{F}_\xi + \int_{S^2}\left(\delta i_\xi\ell - \delta_\xi\vartheta\right)$$

Remark: we recover Barnich-Troessaert (and Wald-Zoupas) prescriptions for $\ell = \sqrt{q}\left(M - C_{AB}N^{AB}/8\right) dud^2\sigma$.

Renormalized expression for the symplectic 2-form at null infinity:

$$\Omega^R = \int_{\mathcal{I}}\Bigg[+\frac{1}{4}\delta N_{AB} \wedge \delta(\sqrt{\bar{q}}C^{AB}) \qquad (\delta\sqrt{\bar{q}} = 0 = \delta\bar{q}_{AB}\text{, [Ashtekar \& Streubel, 1981]})$$

$$-\frac{1}{4}\delta\left(\frac{\bar{R}}{2}C_{AB} - D_{\langle A}D^C C_{B\rangle C}\right) \wedge \delta(\sqrt{\bar{q}}\bar{q}^{AB}) \qquad (\delta\sqrt{\bar{q}} = 0 \neq \delta\bar{q}_{AB}\text{, [Compère et al., 2018]})$$

$$+\delta\left(M + \frac{1}{4}D_A D_B C^{AB}\right) \wedge \delta\sqrt{\bar{q}}\Bigg] du\ d^2\sigma$$

Rediscovering the Weyl-BMS group: pushing extended corner symmetry to scri

Extended corner symmetry

Corner symmetry group: surface diffeomorphisms "plus" surface boosts

[Donnelly-Freidel, 2016], [Donnelly-Freidel-Moosavian-Speranza, 2020]

$$\mathfrak{g}_{S^2} = \mathrm{diff}(S^2) \oplus \mathfrak{sl}(2,\mathbb{R})$$

Extended corner symmetry includes surface translations (see also [Ciambelli-Leigh, 2021])

$$\boxed{\mathfrak{g}^{ext}_{S^2} = \big(\mathrm{diff}(S^2) \oplus \mathfrak{sl}(2,\mathbb{R})\big) \oplus \mathbb{R}^2}$$

To prove this, consider the following metric around the corner S^2:

$$ds^2 = h_{ab}dx^a dx^b + \gamma_{AB}(d\sigma^A - U^A_a dx^a)(d\sigma^B - U^B_b dx^b)$$

One defines $Y^A = \xi^A|_{x^a=0}$, ${W_a}^b = \partial_a \xi^b|_{x^a=0}$, $T^a = \xi^a|_{x^a=0}$ and the associated charges

$$P_A = \frac{1}{2}\gamma_{AB}\epsilon^{ab}(\partial_a + U^A_a\partial_A)U^B_b, \quad {N_b}^a = \frac{1}{2}h_{bc}\epsilon^{ca}$$

$$Q_a = \frac{1}{2}\epsilon^{cb}(\partial_b + U^A_b\partial_A)h_{ac} - U^B_a P_B - D_C({N_a}^b U^C_b),$$

Pushing these charges to scri, one gets (after renormalization) the Weyl-BMS algebra

$$\boxed{\mathfrak{g}^{ext}_{S^2} = \big(\mathrm{diff}(S^2) \oplus \mathfrak{sl}(2,\mathbb{R})\big) \oplus \mathbb{R}^2 \quad \xrightarrow{\mathcal{I}} \quad \mathfrak{bmsw} = \big(\mathrm{diff}(S^2) \oplus \mathcal{W}_{S^2}\big) \oplus \mathcal{T}_{S^2}}$$

The factor $\mathfrak{sl}(2,\mathbb{R})$ is typical of GR; it might change in modified theories of gravity.
Deformation/extension of $\mathrm{diff}(S^2)$? [Rojo-Prochazka-Sachs, 2021]

Conclusions and future directions

Conclusions

Recap:

- new asymptotic symmetries in GR: the Weyl-BMS group;
- derivation of (asymptotic) Einstein's equations from first principles;
- phase-space renormalization.

Follow-ups:

- relax $\partial_u \bar{q}_{AB} = 0$ and extend the Weyl-BMS group;
- explore the consequences of Weyl-BMS for memory effects and soft theorems.

Other interesting directions:

- make advantage of asymptotic symmetries to improve gravitational waveforms;
  e.g., [Ashtekar et al., 2019, 2020], [Mitman et al, 2020, 2021a,b]
- coupling QNM and BMS modes [Gasperin-Jaramillo, 2021]
- asymptotic symmetries in dS; [Fernández-Álvarez & Senovilla, 2020-2021], [Compère et al., 2020]
  asymptotically and spatially flat FLRW; [Bonga-Prabhu, 2020], [Rojo-Heckelbacher-RO, 2022]
- investigate the "triangle" in the cosmological setting;
- explore the role of asymptotic symmetries in modified theories of gravity.

THANK YOU FOR YOUR ATTENTION!