Inverse Coefficient Problems for Time-Fractional Diffusion Equations

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Inverse Problems for Anomalous Diffusion Processes
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Constant order (CO) time-fractional diffusion

Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 2$.

For $\alpha \in (0,1) \cup (1,2)$ and $T \in (0,+\infty)$, we consider the IBVP

$$\left\{ \begin{array}{rcll} \rho \partial_t^\alpha u - \nabla \cdot a \nabla u + qu & = & 0 & \text{in } Q := \Omega \times (0,T) \\ u & = & f & \text{on } \Sigma := \partial\Omega \times (0,T) \\ \partial_t^k u(\cdot,0) & = & 0 & \text{in } \Omega,\ k = 0,\ldots,[\alpha], \end{array} \right. \quad (1)$$

where ∂_t^α is the Caputo derivative of order α:

$$\partial_t^\alpha f(t) := \left\{ \begin{array}{ll} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & \text{if } \alpha \in (n-1,n),\ n \in \mathbb{N} := \{1,2,\ldots\} \\ \frac{d^n}{dt^n} f(t) & \text{if } \alpha = n \in \{0\} \cup \mathbb{N}. \end{array} \right.$$

Inverse problem 1

Let $\Gamma_{\rm in} \subset \partial\Omega$ and $\Gamma_{\rm out} \subset \partial\Omega$ be s.t.

$$\Gamma_{\rm in} \cap \Gamma_{\rm out} \neq \emptyset \text{ and } \Gamma_{\rm in} \cup \Gamma_{\rm out} = \partial\Omega.$$

We consider the *partial DN map at one fixed time* $T_0 \in (0, T)$:

$$\Lambda_{\rho,a,q} : \; f_{|\Gamma_{\rm in}} \mapsto a\partial_\nu u(\cdot, T_0)_{|\Gamma_{\rm out}}.$$

Inverse problem 1:

Does $\Lambda_{\rho,a,q}$ uniquely determine (ρ, a, q) ?

Obstruction to identifiability

- Liouville transform: $v := a^{1/2}u$ is a solution to

$$\begin{cases} a^{-1}\rho\partial_t^\alpha v - \Delta v + q_a v & = & 0 & \text{in } Q \\ v & = & a^{1/2}f & \text{on } \Sigma \\ \partial_t^k v(\cdot,0) & = & 0 & \text{in } \Omega,\ k=0,\ldots,[\alpha], \end{cases}$$

with

$$q_a := a^{-1/2}\Delta a^{1/2} + a^{-1/2}qa^{1/2}.$$

- DN maps relation:

$$\Lambda_{\rho,a,q} = a^{1/2}\Lambda_{a^{-1}\rho,1,q_a}a^{1/2} - a^{1/2}\partial_\nu a^{1/2}.$$

- Consequence: $(\rho, a, q) \mapsto \Lambda_{\rho,a,q}$ is not injective, as we have

$$a_{|\partial\Omega} = 1 \text{ and } \partial_\nu a_{|\partial\Omega} = 0 \Longrightarrow \Lambda_{\rho,a,q} = \Lambda_{a^{-1}\rho,1,q_a}.$$

The best we can hope is to recover two out of three coefficients.

Uniqueness result

Theorem (Kian, Oksanen, ÉS, Yamamoto '18)

Ω is smooth and connected.

Let $\rho_j \in C(\overline{\Omega})$, $a_j \in C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$, and $q_j \in L^\infty(\Omega)$, $j = 1, 2$, satisfy

$$\rho_j \geq c > 0,\ a_j(x) \geq c,\ q_j(x) \geq 0,\ x \in \Omega.$$

Assume that either of the three following conditions is fulfilled:

(i) $\rho_1 = \rho_2$ *and*

$$\nabla a_1(x) = \nabla a_2(x),\ x \in \partial\Omega. \quad (2)$$

(ii) $a_1 = a_2$ *and*

$$\exists C > 0,\ |\rho_1(x) - \rho_2(x)| \leq C \mathrm{dist}(x, \partial\Omega)^2,\ x \in \Omega. \quad (3)$$

(iii) $q_1 = q_2$ *and* (2)-(3).

Then, $\Lambda_{\rho_1, a_1, q_1} = \Lambda_{\rho_2, a_2, q_2}$ *yields* $(\rho_1, a_1, q_1) = (\rho_2, a_2, q_2)$.

Variable order (VO) time-fractional diffusion

$T \in (0, +\infty)$ and Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 2$.

$$\begin{cases} \left(\rho(x)\partial_t^{\alpha(x)} - \Delta + q(x)\right) u(x,t) = 0, & (x,t) \in Q \\ u(\sigma,t) = t^k g(\sigma), & (\sigma,t) \in \Sigma \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{4}$$

where $k \geq 2$, g is a suitable function, $q \in L^\infty(\Omega, \mathbb{R}_+)$ and

$$0 < \alpha_0 \leq \alpha(x) \leq \alpha_M < 1,\ 0 < \rho_0 \leq \rho(x) \leq \rho_M,\ x \in \Omega.$$

$\partial_t^{\alpha(x)}$ is the Caputo fractional derivative of order $\alpha(x)$:

$$\partial_t^{\alpha(x)} u(x,t) := \frac{1}{\Gamma(1-\alpha(x))} \int_0^t \frac{\partial_\tau u(x,\tau)}{(t-\tau)^{\alpha(x)}} d\tau.$$

Inverse problem 2

$\Gamma_{\rm in}$ and $\Gamma_{\rm out}$ are two suitable open subsets of $\partial\Omega$.
Let u_g be the solution to the IBVP (4) probed by the Dirichlet data

$$\Sigma \ni (\sigma, t) \mapsto t^k g(\sigma).$$

For $t \in (0, T)$, we introduce the partial DN map

$$\Lambda_{\alpha,\rho,q}(t) : \mathcal{H}_{\rm in} \ni g \mapsto \partial_\nu u_g(\cdot, t)_{|\Gamma_{\rm out}},$$

where $\mathcal{H}_{\rm in} := \{g \in H^{3/2}(\partial\Omega);\ \text{supp}\ g \subset \overline{\Gamma_{\rm in}}\}$.

Boundary data: $\{\Lambda_{\alpha,\rho,q}(t_n),\ n \in \mathbb{N}\}$ where $(t_n)_n \in (0, T)^{\mathbb{N}}$ has an accumulation point $\tau \in (0, T)$.

Inverse problem 2:

Does $\{\Lambda_{\alpha,\rho,q}(t_n),\ n \in \mathbb{N}\}$ uniquely determine (α, ρ, q) ?

Uniqueness result, $d \geq 3$

Theorem (Kian, ÉS, Yamamoto '20)

Assume that

a) Ω *is smooth and connected.*

b) $S_+ \subset \Gamma_{\mathrm{in}}$ *and* $S_- \subset \Gamma_{\mathrm{out}}$, *where*

$$S_\pm := \overline{\{x \in \partial\Omega;\ \pm(x - x_0)\cdot\nu \geq 0\}},$$

for some $x_0 \in \mathbb{R}^d$ *outside the convex hull of* $\overline{\Omega}$.

c) *The set of admissible unknown coefficients is defined by*

$$\mathcal{E} := \{(\alpha, \rho, q); \alpha, \rho \in L^\infty(\Omega),\ q \in L^\infty(\Omega; \mathbb{R}_+)\}.$$

Then, for all $(\alpha_j, \rho_j, q_j) \in \mathcal{E}$, $j = 1, 2$, *we have:*

$$\Lambda_{\alpha_1,\rho_1,q_1}(t_n) = \Lambda_{\alpha_2,\rho_2,q_2}(t_n),\ n \in \mathbb{N} \Longrightarrow (\alpha_1, \rho_1, q_1) = (\alpha_2, \rho_2, q_2).$$

Uniqueness result, $d = 2$

Theorem (Kian, ÉS, Yamamoto '20)

The same conclusion holds for $d = 2$, provided that:

a) Ω *is connected with*

$$\partial\Omega = \bigcup_{k=1}^{N} \gamma_k,\ \ \gamma_k = \textit{smooth closed contour},\ \ k = 1, \dots, N.$$

b) $\Gamma_{\mathrm{in}} = \Gamma_{\mathrm{out}} =$ *arbitrary non-empty open subset of* $\partial\Omega$.

c) *The set of admissible unknown coefficients is*

$$\mathcal{E}' := \left\{ (\alpha, \rho, q); \alpha, \rho \in L^{\infty}(\Omega), q \in W^{1,\kappa}(\Omega; \mathbb{R}_+), \kappa \in (2, +\infty) \right\}.$$

Comments and bibliography

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Outline

1. Inverse problem 1
   1. Inverse spectral result
   2. Linking the boundary data to the spectral data
   3. The case of Riemannian manifolds
2. Inverse problem 2
   1. DN maps identification
   2. Calderón problems

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Elliptic operator

$L^2_\rho(\Omega)$ is the Hilbert space $L^2(\Omega)$ endowed with the scalar product

$$\langle u, v\rangle := \int_\Omega \rho u v \mathrm{d}x,\ \ u, v \in L^2(\Omega).$$

- H is the operator generated in $L^2_\rho(\Omega)$ by the quadratic form

$$h[u] := \int_\Omega \left(a|\nabla u|^2 + qu^2 \right) \mathrm{d}x,\ \ u \in \mathrm{Dom}(h) := H^1_0(\Omega).$$

- H is self-adjoint in $L^2_\rho(\Omega)$ and acts on its domain as

$$Hu := \rho^{-1} \left(\nabla \cdot a\nabla u + qu \right),\ \ u \in \mathrm{Dom}(H) := H^1_0(\Omega) \cap H^2(\Omega).$$

Boundary spectral data

By compactness of $H_0^1(\Omega) \hookrightarrow L^2_\rho(\Omega)$, $\sigma(H)$ is purely discrete.

• $\{\lambda_n;\ n \in \mathbb{N}\}$ is the non-decreasing sequence of the eigenvalues (repeated according to multiplicities) of H.

• $\{\varphi_n;\ n \in \mathbb{N}\}$ is a family of eigenfuctions of H which form an orthonormal basis in $L^2_\rho(\Omega)$:

$$\left\{ \begin{array}{rcll} -\nabla \cdot a\nabla\varphi_n + q\varphi_n & = & \lambda_n \rho \varphi_n & \text{in } \Omega \\ \varphi_n & = & 0 & \text{on } \partial\Omega \\ \int_\Omega \rho \varphi_n^2 \mathrm{d}x & = & 1. & \end{array} \right.$$

• The boundary spectral data associated with (ρ, a, q):

$$\mathrm{BSD}(\rho, a, q) := \{(\lambda_n, \psi_n);\ n \in \mathbb{N}\},\ \text{where } \psi_n := a\partial_\nu \varphi_{n|\partial\Omega}.$$

Inverse spectral result

Theorem (Canuto, Kavian '04)

If either of the three assumptions (i), (ii) or (iii) is verified, then we have:

$$\mathrm{BSD}(\rho_1, a_1, q_1) = \mathrm{BSD}(\rho_2, a_2, q_2) \Longrightarrow (\rho_1, a_1, q_1) = (\rho_2, a_2, q_2).$$

- B. Canuto, O. Kavian, *Determining two coefficients in elliptic operators via boundary spectral data*: *a uniqueness result*, Bolletino Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8** (2004), no. 1, 207-230.

Thus, it is enough to prove that

$$\Lambda_{\rho_1, a_1, q_1} = \Lambda_{\rho_2, a_2, q_2} \Longrightarrow \mathrm{BSD}(\rho_1, a_1, q_1) = \mathrm{BSD}(\rho_2, a_2, q_2),$$

up to an appropriate choice of the eigenfunctions of the operator H associated with $(\rho, a, q) = (\rho_1, a_1, q_1)$.

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Direct problem: existence and uniqueness result

Let $\rho \in L^\infty(\Omega)$, $a \in C^1(\overline{\Omega})$, and $q \in L^\infty(\Omega)$ satisfy

$$\rho(x) \geq c > 0,\ a(x) \geq c,\ q(x) \geq 0,\ x \in \Omega.$$

Assume that $f \in C^{[\alpha]+1}([0, T], H^{3/2}(\partial\Omega))$ verifies the compatibility conditions

$$\partial_t^k f(\cdot, 0) = 0 \text{ in } \partial\Omega,\ k = 0, \ldots, [\alpha].$$

Then the IBVP (1) admits a unique solution

$$u \in C([0, T], L^2(\Omega)) \cap C((0, T], H^{2\gamma}(\Omega)),\ \gamma \in (0, 1).$$

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Partial DN operator

Taking $\gamma \in (3/4, 1)$, we get that

$$(x, t) \mapsto a(x)\partial_\nu u(x, t) \in C((0, T], L^2(\partial\Omega)).$$

Thus, the partial DN map is well defined as a (bounded) operator

$$\Lambda_{\rho,a,q} : \mathcal{H}_{\rm in} \to L^2(\Gamma_{\rm out}),$$

where

$$\mathcal{H}_{\rm in} := \left\{ f \in C^{[\alpha]+1}([0, T], H^{3/2}(\partial\Omega));\ \text{supp } f \subset \Gamma_{\rm in} \times (0, T_0) \right\}.$$

OFDE

Let $f \in \mathcal{H}_{\rm in}$.

- For all $u \in C^0([0,T], L^2(\Omega))$, we have

$$u(\cdot,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n, \;\; u_n(t) := \langle u(\cdot,t), \varphi_n \rangle, \;\; t \in [0,T].$$

- Since u is solution to the IBVP (1), we have for each $n \in \mathbb{N}$,

$$\left\{ \begin{array}{rcll} \partial_t^\alpha u_n + \lambda_n u_n & = & -\int_{\partial\Omega} f \psi_n \mathrm{d}\sigma(x) & \text{in } (0,T) \\ \partial_t^k u_n(0) & = & 0, \;\; k = 0, \ldots, [\alpha], & \end{array} \right.$$

by the Green formula.

Representation formula

- Expression of the solution u to the IBVP (1), at $t = T_0$:

$$u(\cdot, T_0) = \int_0^{T_0} \left(\sum_{n=1}^{\infty} \gamma_n(s)\varphi_n \right) \mathrm{d}s \text{ in } H^{2\gamma}(\Omega),\ \gamma \in (0,1),$$

where

$$\gamma_n(s) = -s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^{\alpha}) \left(\int_{\partial\Omega} f(y, t-s)\psi_n(y)\mathrm{d}\sigma(y) \right)$$

and $E_{\alpha,\beta}$ is the two parameters ($\alpha > 0$, $\beta > 0$) Mittag-Leffler function

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ z \in \mathbb{C}.$$

- By the continuity of the trace operator,

$$a(x)\partial_{\nu} u(x, T_0) = \int_0^{T_0} \left(\sum_{n=1}^{\infty} \gamma_n(s)\psi_n(x) \right) \mathrm{d}s,\ x \in \partial\Omega.$$

Representation formula, continued

- Put

$$\Theta_n(x,y) := \sum_{p=1}^{m_n} \psi_{n,p}(x)\psi_{n,p}(y),\ (x,y) \in \partial\Omega,$$

where

- $m_n \in \mathbb{N}$ is the algebraic multiplicity of the eigenvalue λ_n,
- $\{\varphi_{n,p};\ p = 1,\dots,m_n\}$ is a family of eigenfunctions of H which form an orthonormal basis in $L^2_\rho(\Omega)$ of $\ker(H - \lambda_n)$ and $\psi_{n,p} := a\partial_\nu\varphi_{n,p}$.

- Representation formula of the partial DN map:

$$\begin{aligned} & \Lambda_{\rho,a,q} f \\ = \ & \int_0^{T_0} s^{\alpha-1} \left(\sum_{n=1}^{+\infty} E_{\alpha,\alpha}(-\lambda_n s^\alpha) \left(\int_{\partial\Omega} f(T_0 - s, y)\Theta_n(\cdot, y) \mathrm{d}\sigma(y) \right) \right) \mathrm{d}s. \end{aligned}$$

End of the proof

- $\Lambda_{\rho_1,a_1,q_1} = \Lambda_{\rho_2,a_2,q_2}$ entails for a.e. $x \in \Gamma_{\rm out}$ and all $f \in \mathcal{H}_{\rm in}$:

$$\int_0^{T_0} s^{\alpha-1} \left(\sum_{n=1}^{+\infty} \int_{\partial\Omega} \left(E_{\alpha,\alpha}(-\lambda_{1,n} s^{\alpha}) \Theta_{1,n}(x,y) - E_{\alpha,\alpha}(-\lambda_{2,n} s^{\alpha}) \Theta_{2,n}(x,y) \right) f(y, T_0 - s) {\rm d}\sigma(y) \right) {\rm d}s = 0.$$

- Taking $f(x,t) = \psi(t) h(x)$, for two arbitrary

$$\psi \in C_0^{\infty}(0, T_0) \text{ and } h \in H^{3/2}(\partial\Omega),\ \text{supp } h \subset \Gamma_{\rm in},$$

yields

$$\boxed{F_{1,h}(x, s^{\alpha}) = F_{2,h}(x, s^{\alpha}),\ s \in (0, T_0),\ x \in \Gamma_{\rm out}} \quad (*)$$

where

$$F_{j,h}(z,x) := \sum_{n=1}^{+\infty} E_{\alpha,\alpha}(-\lambda_{j,n} z) \left(\int_{\Gamma_{\rm in}} \Theta_{j,n}(x,y) h(y) {\rm d}\sigma(y) \right).$$

- $(*) \overset{\text{analytic continuation}}{\Longrightarrow} \text{BSD}(\rho_1, a_1, q_1) = \text{BSD}(\rho_2, a_2, q_2)$.

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FDE on Riemannian manifolds

Let (M, g) be a compact connected Riemannian manifold of dimension $d \geq 2$, with boundary ∂M.

- Weighted Laplace Beltrami operator

$$\begin{aligned} \Delta_{g,\mu} u &:= \mu^{-1} \nabla \cdot_g \mu \nabla_g u \\ &= \sum_{i,j=1}^{d} \mu^{-1} |g|^{-1/2} \partial_{x_i} (\mu |g|^{1/2} g^{ij} \partial_{x_j} u), \ u \in C^\infty(M), \end{aligned}$$

where $g^{-1} := (g^{ij})_{1 \leq i,j \leq d}$ and $|g| := \det g$.

- For $\alpha \in (0,1) \cup (1,2)$ and $q \in C^\infty(M)$, we consider the IBVP

$$\begin{cases} \partial_t^\alpha u - \Delta_{g,\mu} u + qu = 0 & \text{in } M \times (0,T) \\ u = f & \text{on } \partial M \times (0,T) \\ \partial_t^k u(\cdot, 0) = 0 & \text{in } M, \ k = 0, ..., [\alpha]. \end{cases}$$

Inverse problem

Let $T_0 \in (0, T)$ and let $\Gamma_{\rm in}$ and $\Gamma_{\rm out}$ be two non empty open subsets of ∂M.

- Dirichlet data :

$$\mathcal{H}_{\rm in} := \{f \in C^{[\alpha]+1}([0, T], H^{\frac{3}{2}}(\partial M));\ \mathrm{supp} f \subset \Gamma_{\rm in} \times (0, T_0)\}.$$

- The partial DN map

$$\Lambda_{M,g,\mu,q} :\ \mathcal{H}_{\rm in} \ni f \mapsto \partial_\nu u(\cdot, T_0)_{|\Gamma_{\rm out}} := \sum_{i,j=1}^{d} g^{ij} \nu_i \partial_{x_j} u(\cdot, T_0)_{|\Gamma_{\rm out}},$$

is linear bounded from $\mathcal{H}_{\rm in}$ into $L^2(\Gamma_{\rm out})$.

Inverse problem:

Does $\Lambda_{M,g,\mu,q}$ determine (M, g), and μ and q, uniquely ?

Two obstructions to uniqueness

- (μ_1, q_1) and (μ_2, q_2) are *gauge equivalent* if there exists a positive function $\kappa \in C^\infty(M)$ obeying

$$\kappa(x) = 1,\ \partial_\nu \kappa(x) = 0 \text{ on } \partial M,$$

such that

$$\mu_2 = \kappa^{-2}\mu_1,\ q_2 = q_1 - \kappa \Delta_{g,\mu_1} \kappa^{-1}.$$

In this case we have $\Lambda_{M,g,\mu_1,q_1} = \Lambda_{M,g,\mu_2,q_2}$.

- If $\Phi : M \to M$ is a diffeomorphism fixing the boundary ∂M, then

$$\Lambda_{M,g,\mu,q} = \Lambda_{M,\Phi^* g,\mu,q},$$

where $\Phi^* g$ is the pull back of g by Φ.

The result on manifolds

Theorem (Kian, Oksanen, ÉS, Yamamoto '18)

Let (M_j, g_j), $J = 1, 2$, *be two compact and smooth connected Riemannian manifolds of dim.* $d \geq 2$ *with the same boundary, and let* $\mu_j \in C^\infty(M_j)$ *and* $q_j \in C^\infty(M_j)$ *satisfy*

$$\mu_j(x) \geq c > 0, \ q_j(x) \geq 0, \ x \in M_j.$$

Let $\Gamma_{\rm in}, \Gamma_{\rm out} \subset \partial M_1$ *be relatively open and suppose that*

$$\Gamma_{\rm in} \cap \Gamma_{\rm out} \neq \emptyset.$$

Suppose, moreover, that

$$g_1 = g_2, \ \mu_1 = \mu_2 = 1 \text{ and } \partial_\nu \mu_1 = \partial_\nu \mu_2 = 0 \text{ on } \partial M_1.$$

Then, $\Lambda_{M_1, g_1, \mu_1, q_1} = \Lambda_{M_2, g_2, \mu_2, q_2}$ *yields that*

- (M_1, g_1) *and* (M_2, g_2) *are isometric*
- (μ_1, q_1) *and* (μ_2, q_2) *are gauge equivalent.*

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DN maps identification

- $\Lambda_j := \Lambda_{\alpha_j,\rho_j,q_j}$, $j = 1, 2$, and recall that for all $n \in \mathbb{N}$:

$$\Lambda_1(t_n) = \Lambda_2(t_n) \Longleftrightarrow \partial_\nu u_1(\cdot, t_n)_{|\Gamma_{\text{out}}} = \partial_\nu u_2(\cdot, t_n)_{|\Gamma_{\text{out}}}.$$

- $g \in H^{3/2}(\partial\Omega) \Rightarrow t \mapsto u_g(\cdot, t) \in \mathcal{A}(0, +\infty; H^2(\Omega))$:

$$\left. \begin{array}{c} h := (\partial_\nu u_2 - \partial_\nu u_1)_{|\Gamma_{\text{out}}} \in \mathcal{A}(0, +\infty; L^2(\Gamma_{\text{out}})) \\ h(t_n) = 0 \text{ for } n \in \mathbb{N} \\ (t_n)_n \text{ has an acc. point in } (0, T) \end{array} \right\} \Rightarrow h = 0,$$

$$\Rightarrow \boxed{\partial_\nu u_1(\cdot, t)_{|\Gamma_{\text{out}}} = \partial_\nu u_2(\cdot, t)_{|\Gamma_{\text{out}}}, \ \ t \in (0, +\infty)} \qquad (\star)$$

$$\Rightarrow \Lambda_1(t) = \Lambda_2(t), \ \ t \in (0, +\infty).$$

Calderón problems

- Since $t \mapsto e^{-st}u_j(\cdot,t) \in L^1(0,+\infty; H^2(\Omega))$, $j=1,2$, $s \in (0,+\infty)$, the Laplace transform $U_j(s) = \int_0^{+\infty} e^{-st}u_j(\cdot,t)dt$ satisfies:

$$(\star) \Rightarrow \boxed{\partial_\nu U_1(\cdot,s)_{|\Gamma_{\rm out}} = \partial_\nu U_2(\cdot,s)_{|\Gamma_{\rm out}},\ s \in (0,+\infty)}\ (\star\star)$$

- For all $s \in (0,+\infty)$, $\tilde{U}_j(s) := \frac{s^{k+1}}{k!} U_j(s)$ solves

$$\begin{cases} (-\Delta + q_j(x,s))w(x) = 0, & x \in \Omega \\ w(x) = g(x), & x \in \partial\Omega, \end{cases}$$

with $q_j(\cdot,s) := q_j + \rho_j s^{\alpha_j}$.

- Putting $\Lambda_{V_j(\cdot,s)} g := \partial_\nu \tilde{U}_j(s)_{|\Gamma_{\rm out}}$, we see that :

$$\begin{aligned} (\star\star) \quad &\Rightarrow \quad \Lambda_{V_1(\cdot,s)} = \Lambda_{V_2(\cdot,s)},\ s \in (0,+\infty) \\ &\Rightarrow \quad V_1(\cdot,s) = V_2(\cdot,s),\ s \in (0,+\infty). \end{aligned}$$

- $s \to 0 \Rightarrow q_1 = q_2$, $s = 1 \Rightarrow \rho_1 = \rho_2$, $s = e \Rightarrow \alpha_1 = \alpha_2$.

Thank you for your attention.