Topics in Multiple Time Scale Dynamics

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The BIRS five-day workshop brought together 33 in-person researchers (with an additional 3 virtual speakers and several virtual participants). The on-site participants included a mix of PhD students, postdocs, and senior researchers from North America, Europe, and Oceania.

1 Overview of the Field

Introduction

Several phenomena in nature occur at distinct time scales. For instance, certain regulatory processes in the human body are linked to the fact that a heartbeat takes approximately 0.6 seconds, while a circadian rhythm lasts 86400 seconds (24 hours). Another example occurs in social interactions where humans "tweet" in a time scale of seconds to minutes; chat, have conversations, meet, and take certain decisions in periods of hours to days; while they go to the polls in a time scale of years. The understanding of the subprocesses occurring at each time scale and how they interact often leads to a more accurate description of a system.

From a mathematical perspective, the study of dynamical systems with multiple time scales has been a long-standing topic of various research. A large part of this research focuses on systems with two time scales, also known as *slow–fast systems*, characterized by *singular perturbations*. In the deterministic context, in particular of ordinary differential equations (ODEs), classical results by Tikhonov [45] and Fenichel [17] say, roughly, that under a certain condition known as *normal hyperbolicity* one can isolate the slow and the fast dynamics for their respective analysis; and this allows for conclusions on critical objects of the original dynamics.

However, not all phenomena in nature can be modeled by slow-fast systems satisfying the normal hyperbolicity condition. In fact, emergent dynamical behaviour such as relaxation oscillations and mixed-mode oscillations are modeled by slow-fast systems for which normal hyperbolicity is lost and the coupling of the systems has to be accounted for. Real-life phenomena exhibiting such behaviour include neuron dynamics, population dynamics, complex networks, climate models and many others. A powerful tool to study systems with non-hyperbolic singularities, i.e. points where normal hyperbolicity is lost, is given by the *blow-up method* going back to works by Dumortier and Roussarie [10, 11]. This method, among others, has boosted strong theoretical progress in slow-fast systems which has had a considerable impact in applied sciences, most noticeably in neuroscience.

In addition to emergent phenomena in multiscale systems linked to the geometric subtleties of lost hyperbolicity, the emergence of stochastic behaviour has also been an intensively studied topic, most notably by the methods of *averaging* and *homogenization* [41]. While averaging is generally associated with the law of large numbers, homogenization is connected to central limit theorems. Recently Melbourne and coworkers [29, 30, 18, 37] have embarked on a programme to prove homogenization for a wide class of slow-fast systems where the fast system is chaotic but does not need to satisfy strong mixing properties, and, by that, have shown such results also for well-established models like the Lorenz equations.

The state of the art

Although the term *multiple time scale dynamics* involves several classes of dynamical systems such as those in terms of ordinary differential equations (ODEs), partial differential equations (PDEs), stochastic differential equations (SDEs), piecewise-smooth differential equations or also discrete time systems generated by maps, we mostly give an overview concerning ODEs. On the one hand, by doing so, we keep this section as brief as possible. On the other hand, the multiple time scale analysis beyond normal hyperbolicity and the theory of emergent stochastic behaviour by homogenization are most advanced in the context of ODEs.

Loss of normal hyperbolicity and desingularization

A slow-fast system (SFS) can be represented in any of the two following forms

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,z,\lambda,\varepsilon)\\ \varepsilon\frac{\mathrm{d}z}{\mathrm{d}t} &= g(x,z,\lambda,\varepsilon),\end{aligned} \qquad \text{or} \qquad \begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \varepsilon f(x,z,\lambda,\varepsilon)\\ \frac{\mathrm{d}z}{\mathrm{d}\tau} &= g(x,z,\lambda,\varepsilon).\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_s}$ are the slow variables, $z \in \mathbb{R}^{n_f}$ denotes the fast variables, $\lambda \in \mathbb{R}^m$ stands for parameters, t is the slow time variable, $\tau = t/\varepsilon$ is the fast time variable, the vector fields f and g are assumed to be smooth, and $0 < \varepsilon \ll 1$ is a small parameter accounting for the time scale separation between x and z. The *geometric singular perturbation theory* of dynamical systems with multiple time scales aims to understand the behavior of SFSs from considering the limit $\varepsilon = 0$ of (1) and then prove perturbation results for $\varepsilon > 0$ sufficiently small. When we set $\varepsilon = 0$ in (1), we obtain the so-called *constraint differential equation* (CDE), and *layer equation* respectively

$$\text{CDE:}\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t} &= f(x,z,\lambda,0),\\ 0 &= g(x,z,\lambda,0),\end{cases} \qquad \text{Layer:}\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}\tau} &= 0,\\ \dfrac{\mathrm{d}z}{\mathrm{d}\tau} &= g(x,z,\lambda,0).\end{cases}$$

Observe that the algebraic equation $g(x,z,\lambda,0) = 0$ defines the phase-space of solutions of the CDE, but also the set of equilibrium points of the layer equation, which leads us to define

$$S = \left\{(x,z,u) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f} \times \mathbb{R}^m \,|\, g(x,z,\lambda,0) = 0\right\}, \tag{2}$$

called *the critical manifold*. We say that S is *normally hyperbolic* when the matrix $\frac{\partial g}{\partial z}(x,z,\lambda,0)$ has no eigenvalues with zero real part. In such a scenario, Fenichel's Theory [17] says that S perturbs to a *slow normally hyperbolic invariant manifold* for $0 < \varepsilon \ll 1$ and that, in this situation, the dynamics of (1) are well approximated by the case $\varepsilon = 0$.

In contrast, a big challenge arises when S is not normally hyperbolic. To deal with SFSs with non-hyperbolic points (or singularities), a geometric desingularization technique, also referred to as the blow-up method, has been introduced in [10, 11, 31, 32], see also [23] for a comprehensive survey. Geometric desingularization allows for a more thorough and simpler analysis of the dynamics of SFSs near a non-hyperbolic point by lifting the dynamics onto an appropriate manifold. Once the technique is applied we gain (at least partial) hyperbolicity, so that other methods from dynamical systems theory (such as invariant and center manifold theory or Fenichel's perturbation theory) can be deployed.

To this date, the geometric desingularization method has been extremely useful to understand the dynamics near low codimension singularities such as folds [32], and to some extent to analyze some more degenerate scenarios [22, 34]. The blow-up method has also proven useful in the context of reaction-diffusion PDEs with different time scales, enabling the analysis of highly complicated behavior in the corresponding traveling wave ODEs [4, 5, 6]. In this context, the FitzHugh-Nagumo equation has been a paradigm model for describing complex dynamics in neuroscience where geometric desingularization has shown to be generally very successful (see e.g. also [7, 8]). Naturally, there has been active research from other viewpoints as is evidenced by the recent monograph [33] and references therein. In particular, we would like to highlight the work by Berglund and Gentz on geometric singular perturbation theory for SDEs, as summarized in their monograph [3].

Emergence of stochastic dynamics

For the context of homogenization, consider an ODE $\dot{y} = g(y)$ on $\mathbb{R}^m$ generating the flow $\phi_t : \mathbb{R}^m \to \mathbb{R}^m$ with invariant set Ω and ergodic probability measure μ on Ω. The slow-fast system, now with three different time scales, is given by

$$\dot{x}_\varepsilon = a(x_\varepsilon, y_\varepsilon) + \varepsilon^{-1} b(x_\varepsilon) v(y_\varepsilon), \quad x_\varepsilon(0) = \xi \in \mathbb{R}^d, \tag{3}$$

$$\dot{y}_\varepsilon = \varepsilon^{-2} g(y_\varepsilon), \quad y_\varepsilon(0) = \eta \in \Omega, \tag{4}$$

where $a : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^{d \times e}$ are C^3 and $v \in L^\infty(\Omega, \mathbb{R}^e)$ is an observable with $\int_\Omega v \, \mathrm{d}\mu = 0$. Define the family of random elements $W_\varepsilon \in C([0,1], \mathbb{R}^e)$ by

$$W_\varepsilon(t) = \varepsilon v_{t\varepsilon^2}, \quad v_t = \int_0^t v \circ \phi_s \, \mathrm{d}s.$$

It is then easy to observe that the slow equation (3) can be rewritten as

$$\dot{X}_\varepsilon = a(X_\varepsilon, y_\varepsilon) + b(X_\varepsilon) \dot{W}_\varepsilon, \quad X_\varepsilon(0) = \xi.$$

The question is if the solutions to this equation converge to the solution of an SDE with an appropriate stochastic integral, as $\varepsilon \to 0$. There are results for several different cases depending on the *weak invariance principle* $W_\varepsilon \to_w W$ in $C([0,1], \mathbb{R}^d)$, where $\to_w$ denotes weak convergence and W standard Brownian motion. Note that the weak invariance principle is equivalent to a central limit theorem for processes.

If, for example, $d = e$, $b = \mathrm{Id}$ and $W_\varepsilon \to_w W$ in $C([0,1], \mathbb{R}^d)$, then $X_\varepsilon \to_w X$ in $C([0,1], \mathbb{R}^d)$, where X is the solution to the Itô SDE [37]

$$\mathrm{d}X = (\int_\Omega a(X, y) \, \mathrm{d}\mu(y))\mathrm{d}t + \mathrm{d}W, \quad X(0) = \xi.$$

If $d = e = 1$ and $W_\varepsilon \to_w W$ in $C([0,1], \mathbb{R}^d)$, then $X_\varepsilon \to_w X$ in $C([0,1], \mathbb{R}^d)$, where X is the solution to the Stratonovich SDE [18]

$$\mathrm{d}X = (\int_\Omega a(X, y)\mathrm{d}\mu(y))\mathrm{d}t + b(X) \circ \mathrm{d}W, \quad X(0) = \xi.$$

This result has also been extended to multidimensional noise, where the theory of *rough paths* has been used, and the limiting SDE has turned out to be a Stratonovich equation with corrected drift term [29]. In all these works, the condition of the weak invariance principle $W_\varepsilon \to_w W$ in $C([0,1], \mathbb{R}^d)$ has been shown to hold for mild mixing conditions on the flow $\phi_t : \mathbb{R}^m \to \mathbb{R}^m$, not imposing uniform but only nonuniform hyperbolicity.

This programme for showing the emergence of stochastic behaviour in the slow variable derived from chaotic behaviour in the fast variable has proven to be very successful and entails important questions for generalizations, as indicated in the subsequent section. These theoretical results correspond with a far larger set of problems concerning emergent stochastic behaviour, as outlined also in the applied literature by Mackay [36].

Challenges

Most of our understanding of slow-fast systems with non-hyperbolic points (within the context of autonomous ODEs) is concerning singularities of low codimension, e.g. fold, cusp, Hopf, etc. While effort is being directed towards several generalizations, there are still many open questions; for instance, new challenges arise when one considers more than one fast variable or extra parameters leading to singular bifurcations of high codimension. Another exciting direction to be explored is the role of distinct time scales in the dynamics of and on networks. The interaction along the time scales may be coupled with the network's structure and can easily lead to emergent behaviour not present in the problems studied so far [24]. An interesting challenge arises as well when one considers non-autonomous dynamical systems with multiple time scales. In that front, the theory is far from its autonomous counterpart, yet motivations to carry out research in such a topic can be found in e.g. reservoir computing [19] and, in general, systems with delays [43]. From the perspective of applications, there is very active research that reflects the relevance of multiple time scales in a wide variety of scenarios. As an example, related to the present epidemic crisis, there has been some efforts to better understand epidemics that involve multiple time scales [25, 44].

Regarding the emergence of stochastic behavior in ODEs, there are several generalizations that are desirable for the theory outlined in the state of the art above. Firstly, one could dissolve the skew-product structure of the coupled systems (3) and (4) by replacing $g(y)$ with $g(x, y)$ such that the flow ϕ_t in the y-variable depends on the x-fibres. It is highly challenging to find the weakest conditions for homogenization results in such a scenario; recently some steps in this direction have been undertaken by Gkogkas, Engel and Kuehn [12]. Secondly, one could try and find systems where the convergence to the SDE is not only weak but pathwise and/or find rates of convergence. Generally, the whole research programme of Melbourne and his coworkers treats the problem of bounded noise converging to unbounded noise and by that touches a range of controversially discussed questions in the context of stochastic dynamical systems.

Concerning *discrete time systems*, the numerical study of a system of the form (1) entails time discretizations and, by that, leads to the question of an adequate discretization scheme and the behaviour of multiscale dynamics of maps in general. In the case of non-hyperbolic singularities, this gives rise to an analysis of non-hyperbolic fixed points and non-hyperbolic submanifolds of fixed points in maps with multiple time scales. Whereas the normally hyperbolic theory for discrete-time multiple time scale systems is already quite well developed in [21, 40], the geometric desingularization of non-hyperbolic objects for maps still needs several extensions. Engel and Kuehn [14] have, for example, studied the explicit Euler discretization of a *transcritical singularity* and, by using a blow-up method for maps, shown preservation of the qualitative behaviour by the scheme. A similar analysis was conducted by Arcidiacono, Engel and Kuehn [1] for the *pitchfork singularity*. However, also other schemes should be checked: it is well-known from the area of geometric integration and the general theory of structure-preserving discretizations [20] that only certain discrete-time schemes preserve relevant dynamical properties, e.g. adiabatic invariants for the Hamiltonian systems case [20] or certain asymptotic dynamics for the dissipative case [27]. For multiple time scale maps, Runge–Kutta methods have been studied from a geometric viewpoint [38, 39]. It remains to clarify more systematically, for which discretizations the geometric blow-up approach can be applied and how preservation properties of different schemes compare. In this context, an interesting problem are canard explosions in discrete-time, which have been treated by Engel et al. [16] with special integrable discretization schemes, and by Engel and Jardón-Kojakhmetov [13] with a detailed focus on quantifying the loss of stability at canard points for maps. Higher-dimensional problems are still to be explored.

A broad new challenge concerns the development of geometric techniques (á la blow up) to deal with singularities of slow-fast PDEs and SDEs (see [15] for a first analytical treatment with spectral methods and [28] for a more numerical approach) which would complement the current theory mainly elaborated for ODEs. In particular in the case of PDEs, systems with distinct spatial scales, as they naturally appear in climate dynamics [26], can exhibit singularities which are structurally similar to the case with distinct time scales. Additionally, there is the constant challenge of translating the theory to the benefit of applied sciences, and conversely of studying mathematical problems motivated by real-life phenomena with multiple time scales, with growing importance also in cell biology and chemical reaction networks [46]. Some other examples are related to population dynamics, epidemiology, tumor growth and tissue mechanics [2, 9, 35, 42].

2 Presentation Highlights

In this section, we list the abstracts of the workshop presentations.

Exploring Temporal Pulse Replication in the Fitzhugh-Nagumo Equation (Erik Bergland)

In 2018, Carter and Sandstede made use of geometric singular perturbation theory and blow-up analysis to determine the mechanism behind parametric pulse replication in the Fitzhugh-Nagumo equation: the presence of a canard point in the associated traveling-wave ODE leads to the existence of a one-parameter family of solutions the authors termed a homoclinic banana. As one travels along the banana, a transition occurs from single-pulse solutions to double pulses, with an intermediate phase of single pulses with oscillatory tails. As is typical with canard points, this transition takes place in an exponentially thin region of parameter space. Subsequently, in 2021 Carter et al. numerically observed the phenomenon of temporal pulse replication: starting with an initial condition close to a one-pulse on the banana causes the associated solution to mimic the parametric transition dynamically, despite the parameters being fixed. The authors speculated that the parametric transition may generate a nearby invariant manifold which guides the temporal transition. In this talk, we discuss the accompanying stability results for the parametric transition, why those results preclude a simple resolution of temporal pulse replication, and the numerical explorations carried out to address these challenges.

Stochastic resonance in stochastic PDEs (Nils Berglund)

Stochastic resonance can occur when a multi-stable system is subject to both periodic and random perturbations. For suitable parameter values, the system can respond to the perturbations in a way that is close to periodic. This phenomenon was initially proposed as an explanation for glacial cycles in the Earth's climate. While its role in that context remains controversial, stochastic resonance has since been observed in many physical and biological systems. This talk will focus on stochastic resonance in parabolic SPDEs, such as the Allen-Cahn equation, when they are driven by a periodic perturbation and by space- time white noise. We will discuss both the case of one spatial dimension, in which the equation is well-posed, and the case of two spatial dimensions, in which a renormalisation procedure is required. This talk is based on joint works with Barbara Gentz and Rita Nader.

References:

`https://dx.doi.org/10.1007/s40072-021-00230-w`

`https://arxiv.org/abs/2107.07292`

`https://arxiv.org/abs/2209.15357`

Nonlinear Laplacian Dynamics: Symmetries, Perturbations, and Consensus (Riccardo Bonetto)

Laplacian dynamics has been extensively used as a paradigmatic model for discrete linear diffusion processes. Nonlinear extensions of Laplacian dynamics display a richer behaviour that, for example, could model more complex diffusive phenomena on networked systems. We present a study of absolute Laplacian flows (ALFs) under small perturbations. The algebraic properties of such systems lead naturally to singular perturbations, acting as a drift in the (nonlinear) diffusion process. The main goal is to describe the near-consensus behaviour; in order to accomplish that we employ techniques from geometric singular perturbation theory, equivariant dynamical systems theory, and algebraic graph theory.

Pattern-forming invasion fronts in the FitzHugh–Nagumo system (Paul Carter)

We consider the FitzHugh–Nagumo PDE in the so-called oscillatory regime in which one observes spatially oscillatory patterns that invade an unstable steady state. The resulting pattern is selected from a family of periodic traveling wave train solutions by an invasion front in the layer problem. Using geometric singular perturbation techniques, we construct pushed and pulled pattern-forming fronts as heteroclinic orbits between the unstable steady state and a periodic orbit representing the wave train in the wake. In the case of pushed fronts, the associated wave train necessarily passes near a pair of nonhyperbolic fold points on the critical manifold. We also briefly discuss implications for the stability of the wave trains and the challenges introduced by the fold points in the spectral stability problem.

This is joint work with Montie Avery, Björn de Rijk, and Arnd Scheel.

Optimal parameterizing manifolds and reduced systems for stochastic transitions (Mickael Chekroun)

A general, data-informed and theory-guided variational approach based on analytic parameterizations of unresolved scales/variables is presented to address the closure problem of stochastic systems. It relies on the Optimal Parameterizing Manifold (OPM) framework introduced in (Chekroun et al, J. Stat. Phys. 179, 2020) which allows, for deterministic turbulent systems away from the instability onset, to derive useful analytic formulas for such parameterizations. These are obtained as homotopic deformations of parameterizations near criticality such as e.g. arising in center manifold reduction, and whose homotopy parameters are optimized away from criticality using data from the full model. Contrarily to other nonlinear-parameterization approaches such as those based on invariant/inertial or slow manifolds, the superiority of the OPM approach lies in its ability to get rid of constraining spectral gap or timescale separation conditions.

In this work, this program is extended to stochastic partial differential equations (SPDEs) driven by additive noise, either white or of jump type. Analytic formulas of stochastic OPMs are derived. These parameterizations are optimized using a single solution path and are shown to represent efficiently the interactions between the noise and nonlinear terms in a given reduced state space, for the other solution paths. Path-dependent coefficients depending on the noise history are shown to play a key role in these parameterizations especially when the noise is acting along the ”orthogonal direction” of the reduced state space. Applications to stochastic transitions in SPDEs will be presented. This talk is based on a joint work with Honghu Liu and James C. McWilliams.

Estimating long-term behaviour of a flow with ergodic driving (Robin Chemnitz)

In this talk, we consider the long-term evolution of particle distributions in a time-inhomogeneous vector field with diffusion. The time dependence is governed by an ergodic driving system and particle distributions evolve in time through the Perron-Frobenius semigroup of the Fokker-Planck equation. We consider and connect two points of view:

- The Perron-Frobenius semigroup as a linear cocycle over the ergodic driving system and its Lyapunov-spectrum;
- The infinitesimal generator of the Perron-Frobenius semigroup on the augmented phase space and its spectrum.

Spectral Stability of Periodic Traveling Waves in Singularly Perturbed Systems (Björn de Rijk)

An issue in the stability analysis of periodic traveling waves in spatially extended systems is that the linearization about the wave possesses continuous spectrum, which is parameterized by the Floquet-Bloch variable and touches the origin due to translational invariance. Thus, the fine structure of the spectrum close to the origin is of importance, but often delicate to determine. In this talk we show how scale separation in singularly perturbed systems can be used to reduce complexity. We apply our techniques to study the spectral stability of periodic waves in the FitzHugh-Nagumo system, which are selected by invasion fronts. A challenge is that those waves are typically ‘nonhyperbolic’ in the sense that large and small spatial eigenvalues interact at so-called fold points. This is joint work with Montie Avery, Paul Carter and Arnd Scheel.

Function space methods for fast-slow neural field equations (Dirk Doorakkers)

Neural field equations (NFEs) have become an important tool in mathematical neuroscience to model dynamical behavior in the brain on macroscopic scales. They can be thought of as spatially continuous extensions of large-scale neuronal networks, and can therefore be interpreted as infinite-dimensional dynamical systems. A main difficulty in the analysis of NFEs is posed by their nonlocal nature, which is typically represented in such models by the convolution of a synaptic kernel with a nonlinear transformation of the neuronal activity variable.

NFEs often feature slowly-varying adaptation variables, which are highly relevant to the regulation of neuronal dynamics. For example, such variables can represent short-term synaptic plasticity. A rigorous fast-slow analysis of these models has so far not been explored. More generally, mathematical research on infinite-dimensional fast-slow systems of differential equations appears scarce in comparison to the vast literature that is available for ODE models.

I will present preliminary results on a functional analytic approach to the study of trajectories in the slow manifold of fast-slow NFEs. This approach relies on analyzing the behavior of bounded continuous solutions of the equation of perturbed motion at such trajectories. Such an approach has the advantage that it is closely related to methods based on exponential dichotomies for infinite-dimensional and non-autonomous dynamical systems. Therefore it could provide an integrated theoretical framework for multiple timescale dynamics in NFEs.

Rate-induced tipping in predator-prey systems (Ulrike Feudel)

Nowadays, populations are faced with unprecedented rates of global climate change, habitat fragmentation and destruction causing an accelerating conversion of their living conditions. Critical transitions in ecosystems often called regime shifts lead to sudden shifts in the dominance of species or even to species' extinction and decline of biodiversity. Many regime shifts are explained as transitions between alternative stable states caused (i) by certain bifurcations when certain parameters or external forcing cross critical thresholds, (ii) by fluctuations or (iii) by extreme events. We address a fourth mechanism which does not require alternative states but instead, the system performs a large excursion away from its usual behaviour when external conditions change too fast. During this excursion, it can embrace dangerously, unexpected states. We demonstrate that predator-prey systems can either exhibit a population collapse or an unexpected large peak in population density if the rate of environmental change crosses a certain critical rate. In reference to this critical rate of change which has to be surpassed, this transition is called rate-induced tipping (R-tipping). Whether a system will track its usual state or will tip with the consequence of either a possible extinction of a species or a large population peak like, e.g., an algal bloom depends crucially on the time scale relations between the ecological timescale and the time scale of environmental change. However, populations have the ability to respond to environmental change due to rapid evolution. We show how such kind of adaptation can prevent rate- induced tipping in predator-prey systems. This mechanism, called evolutionary rescue, introduces a third timescale which needs to be taken into account. Only a large genetic variation within a population would be able to successfully counteract an overcritically fast environmental change.

Joint work with Anna Vanselow, Lukas Halekotte, and Sebastian Wieczorek.

Noise-induced synchronization in circulant networks of weakly coupled oscillators (Barbara Gentz)

Consider a finite-size system of coupled harmonic oscillators, and assume that the oscillators are commensurate and the coupling structure is circulant. We will present an averaging result for stochastic differential equations which will allow us to show that weak multiplicative-noise coupling can amplify some of the systems' eigenmodes and, hence, lead to asymptotic eigenmode synchronization.

Reference: PhD thesis of Christian Wiesel (formerly University of Bielefeld)

Lévy flights as an emergent phenomenon in a spatially extended system (Georg Gottwald)

Anomalous diffusion and Lévy flights, which are characterized by the occurrence of random discrete jumps of all scales, have been observed in a plethora of natural and engineered systems, ranging from the motion of molecules to climate signals. Mathematicians have recently unveiled mechanisms to generate anomalous diffusion, both stochastically and deterministically. However, there exists to the best of our knowledge no explicit example of a spatially extended system which exhibits anomalous diffusion without being explicitly driven by Le2vy noise. We provide the first explicit example of a stochastic partial differential equation which albeit only driven by normal Gaussian noise supports anomalously diffusive propagating front solutions. This is an entirely emergent phenomenon without explicitly built-in mechanisms for anomalous diffusion. This is joint work with Chunxi Jiao.

Stationary profiles of an area averaged PDE model for unidirectional pedestrian flows (Annalisa Iuorio)

In this talk, we investigate the stationary profiles of a convection-diffusion model for unidirectional pedestrian flows in domains with a single entrance and exit. The inflow and outflow conditions at both the entrance and exit as well as the shape of the domain have a strong influence on the structure of stationary profiles, in particular on the formation of boundary layers. We are able to relate the location and shape of these layers to the inflow and outflow conditions as well as the shape of the domain using geometric singular perturbation theory, both in the case of closing channels and for more intricate geometries such as bottlenecks. Furthermore, we confirm and interpret our analytical results by means of computational experiments in connection with real-life applications.

Geometric Blow-up for Pattern Forming Systems (Samuel Jelbart)

Many authors have demonstrated the utility of the geometric blow-up method as a tool for studying dynamic bifurcations in finite-dimensional slow-fast systems. We focus on the development and application of the geometric blow-up method for PDEs, applied in particular to the scalar Swift-Hohenberg equation with a slow parameter drift on an unbounded domain. In order to understand the dynamics near the dynamic Turing bifurcation which is responsible for the onset of patterned states, we show that the classical multiple scales approximation from modulation theory can be reformulated as a (geometric) blow-up transformation. This leads to an approximating set of non-autonomous Ginzburg-Landau equations, which can be analysed in the blown-up space. Analysing these equations and quantifying the magnitude of the approximation allows for a rigorous description of the solutions in a rich class of weighted Sobolev spaces. We also prove the existence of delayed stability loss phenomena.

Delayed Hopf bifurcations and space-time buffer curves in nonlinear PDEs (Tasso Kaper)

The talk will focus on Delayed Hopf Bifurcations in nonlinear reaction-diffusion equations, a phenomenon previously thought to occur mainly in analytic ODEs. In several recent works with Ryan Goh and Theo Vo, the speaker has shown that there can be long delays –past the time of the instantaneous Hopf bifurcation– before the onset of post-Hopf oscillations occurs. Space-time buffer curves are shown to be important for governing the spatially-dependent time of the delayed onset, as are homogeneous exit time curves. We also present formulas for these curves; quantify how they depend on the main system parameters, including the Hopf frequency, the magnitude of the diffusivity, the source terms, the initial data, and the duration of the approach to the instantaneous Hopf bifurcation; and, show that there is a competition between them.

The results presented in the talk are based on joint work with Ryan Goh and Theo Vo.

Sound-proof approximations for meteorological fluid dynamics as an asymptotic three-scale problem (Rupert Klein)

Air is a compressible medium. Yet, experience shows that sound waves play a negligible role in the vast majority of meteorologically relevant atmospheric processes. Nevertheless, the family of sound-proof flow models, which correspond to the incompressible or zero-Mach number approximations in engineering fluid mechanics, has met with severe scepticism from a large fraction of the meteorological community since they were first introduce many decades ago. In this lecture I will elucidate reasons for this scepticism, explain that a thorough analysis of nearly sound-free atmospheric flows involves a non-standard asymptotic three-scale problem, discuss formal estimates of the range of validity of available sound-proof models, and describe ongoing research aiming at an associated rigorous proof.

Collective variables in complex systems: from molecular dynamics to agent-based models and fluid dynamics (Peter Koltai)

The identification of persistent forecastable structures in complicated or high-dimensional dynamics is vital for a robust prediction (or manipulation) of such systems in a potentially sparse-data setting. Such structures can be intimately related to so-called collective variables known for instance from statistical physics. We have recently developed a first data-driven technique to find provably good collective variables in molecular systems. Here we will show that these generalize to other applications as well, such as fluid dynamics and social dynamics.

A surface of connecting orbits between two saddle slow manifolds in a return mechanism of mixed-mode oscillations (Bernd Krauskopf)

We employ a Lin's method set-up to compute a surface of heteroclinic connections between two saddle slow manifolds in the four-dimensional Olsen model for peroxidase-oxidase reaction. As will be shown, this surface organises the return mechanism of mixed-mode oscillations that also involve a slow passage through a Hopf bifurcation.

Joint work with Elle Musoke and Hinke M. Osinga

Towards Geometric Singular Perturbation Theory for PDEs (Christian Kuehn)

Systems with multiple time scales appear in a wide variety of applications. Yet, their mathematical analysis is challenging already in the context of ODEs, where many decades were needed to develop a more comprehensive theory based upon invariant manifolds, desingularization, variational equations, and many other techniques. Yet, for PDEs the progress has been extremely slow due to many obstacles in generalizing several ODE methods. In my talk, I shall report on several recent advances for fast-slow PDEs, namely the extension of slow manifold theory for unbounded operators driving the slow variables, and the design of blow-up methods for PDEs to tackle normal hyperbolicity.

Critical scales for noise-driven tipping in nearly non-smooth Stommel-type models (Rachel Kuske)

We overview a combination of deterministic and stochastic methods for studying dynamic bifurcations in canonical climate-related models. Our focus is on dominant factors in different scenarios of tipping, that is, where the transition related to the dynamic bifurcation may be advanced or delayed. Previous work has contrasted non-smooth and smooth dynamic bifurcations in the deterministic setting, indicating how noise and "nearly" non-smooth behavior can play a larger role in more realistic tipping models. The presence of high and low frequency forcing must also be considered, resulting in a competition between different important contributions, including stochastic forcing, high and low frequency components, the "non-smoothness" of the underlying bifurcations, bi-stability, and the slow variability of critical physical and environmental process. The analysis points to some fundamental differences in the smooth and non-smooth cases, which lead to a wider variety of tipping mechanisms in non-smooth-like settings.

Joint work with Chris Budd.

Using coupling method to detect underlying dynamics (Yao Li)

In this talk I will present our recent result about how to use a numerical coupling method to classify dynamics at different time scales. I will first discuss how to use numerical coupling technique to estimate the speed of convergence of a stochastic differential equation towards its steady state. Then I will show how to connect the property of deterministic dynamics and coupling time distributions by running the coupled process with different magnitudes of noise. Applications of studying loss surfaces of deep neural networks will be demonstrated.

Bispectral Density of Squared Gaussian Processes (Adam Monahan)

In a manner analogous to the partitioning by the spectral density of the variance of a stochastic process among frequency components, the bispectral density partitions the third statistical moment into contributions from interacting frequency pairs. This quantity allows the asymmetry of fluctuations of a stochastic process to be related to timescale interactions. Various quantities of geophysical interest (e.g. ocean surface wave power density, wind speed) can be related to the square of processes which well approximated as Gaussian. In this talk, I show how the bispectral density of a squared Gaussian process x_t can be expressed as a convolution-type integral involving the spectral density of x_t. This integral can be evaluated analytically for several classical stochastic processes (e.g. Ornstein–Uhlenbeck process, damped oscillator). The relevance of these results to the physical characterization of non-Gaussian variability in atmosphere/ocean fields will be discussed.

Bifurcation theory for stochastic partial differential equations (Alexandra Neamtu)

Detecting bifurcation points for stochastic partial differential equations is a subtle task because even for finite-dimensional stochastic systems the question of how to describe a bifurcation is not fully answered. There are several concepts of bifurcations, which can lead to different results. For instance, using order-preserving random dynamical systems, the famous result by Crauel and Flandoli indicates that additive noise destroys a pitchfork bifurcation. However, we show that even in the presence of additive noise a phenomenological bifurcation still occurs. This can be explained by a different qualitative behavior of the equilibrium before and after the bifurcation and it can be quantified by finite-time Lyapunov exponents.

This talk is based on a joint work with Alex Blumenthal and Maximilian Engel and on an ongoing work with Dirk Blömker.

Finite-Time Dynamics in a Stochastic Brusselator (Guillermo Olicón Méndez)

In this talk, we present some qualitative behaviour in finite-time windows in a stochastic Brusselator, where one of the parameters in the classical model is assumed to be random. In particular, we focus our attention on the so-called Finite-Time Lyapunov Exponents (FTLE), which are an indicator of exponential expansion/contraction of nearby orbits in short time scales. On the other hand, we make use of the Covariant Lyapunov Vectors (CLV) in order to analyse the fast-slow structure when the parameters of the system change. Specifically, we study the relation between their alignment within the fast regime.

Phase resetting as a two-point boundary value problem (Hinke M. Osinga)

Phase resetting is a common experimental approach to investigating the behaviour of oscillating neurons. Assuming repeated spiking or bursting, a phase reset amounts to a brief perturbation that causes a shift in the phase of this periodic motion. The observed effects not only depend on the strength of the perturbation, but also on the phase at which it is applied. The relationship between the change in phase after the perturbation and the unperturbed old phase, the so-called phase resetting curve, provides information about the type of neuronal behaviour, although not all effects of the nature of the perturbation are well understood. Mathematically, resetting is closely related to the concept of isochrons of an attracting periodic orbit, which are the submanifolds in its basin of attraction of all points that converge to the periodic orbit with a specific phase. A phase reset maps each isochron in the family of isochrons to another isochron in this family. Recently, we developed a numerical method that computes phase resetting curves in this precise context of mapping one isochron to another. The method is based on the continuation of a multi-segment boundary value problem and can be applied to systems of arbitrary dimension. In this talk, we show how this new approach can be used to study parameter-dependent deformations of phase resetting curves; we give a detailed overview of its properties, and investigate how the resetting behaviour is affected by phase sensitivity in the system.

Joint work with Bernd Krauskopf and Peter Langfield.

Front propagation in two-component reaction-diffusion systems with a cut-off (Nikola Popovic)

The Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation with a cut-off was popularised by Brunet and Derrida in the 1990s as a model for many-particle systems in which concentrations below a given threshold are not attainable. While travelling wave solutions in cut-off scalar reaction-diffusion equations have since been studied extensively, the impacts of a cut-off on systems of such equations are less well understood. As a first step towards a broader understanding, we consider various coupled two-component reaction-diffusion equations with a cut-off in the reaction kinetics, such as an FKPP-type population model of invasion with dispersive variability due to Cook, a FitzHugh-Nagumo-style model with piecewise linear Tonnelier-Gerstner kinetics and, finally, a more general FKKP-type system with a cut-off in both components that is motivated by models for the spatial spread of hitchhiking traits. Throughout, our focus is on the existence, structure, and stability of travelling fronts, as well as on their dependence on model parameters; in particular, we determine the correction to the front propagation speed that is due to the cut-off. Our analysis is, for the most part, based on a combination of geometric singular perturbation theory and the desingularisation technique known as “blow-up”.

Joint work with Zhouqian Miao, Panagiotis Kaklamanos, and Tasso Kaper.

Noise-induced transient dynamics (Weiwei Qi)

Many complex processes exhibit transient dynamics - intriguing or important dynamical behaviors over a relatively long but finite time period. A fundamental issue is to understand transient dynamics of different mechanisms. In this talk, we focus on a class of randomly perturbed processes arising in chemical reactions and population dynamics where species only persist over finite time periods and go to extinction in the long run. To capture such transient persistent dynamics, we use quasi-stationary distributions (QSDs) and study their noise-vanishing asymptotic. Special attention will be paid to essential differences between models with and without environmental noises. The talk ends up with some discussions.

Snaking of Contact Defects in the Brusselator (Timothy Roberts)

The Brusselator is one of the oldest systems studied in spatial dynamics, first conceived as a result of Turing's landmark work on the formation of stripe patterns in the 1960's. Despite the decades and myriad studies since then, it remains a system of interest due to its ability to display a zoo of different complex behaviors. In this work we look at a newly discovered behavior, snaking of contact defects. Numerical studies by Tzou et al. (2013), found that an interaction between two distinct types of stable oscillations allows for the production of contact defects: a temporally constant core region sitting in a temporally oscillating background. Their results suggest that these patterns form through a process called snaking. In this work we look to extend their numerical studies with the aim of producing a proof for the existence and stability of these patterns and the snaking bifurcation that produces them.

Self-organized sacrifice: the death of vegetation patches during dry spells (Arnd Scheel)

In ecosystems that cannot sustain dense uniform vegetation, patterns of concentrated vegetation emerge to leverage benefits of dense growth and economize overall resources. As environmental conditions change, such patterns adapt by varying the spacing between and number of vegetated regions. In a prototypical model for pattern formation, the complex Ginzburg-Landau equation, we study the dynamics of wavenumbers in spatially periodic solutions. As parameters slowly pass through an Eckhaus instability, patterns with higher wavenumber become unstable. We predict the resulting drop in wavenumber and the time delay of this transition based on spatio-temporal resonances.

Joint work with Anna Asch, Montie Avery, and Anthony Cortez.

Chaotic dynamics in slow-fast neural systems (Andrey Shilnikov)

Several basic mechanisms of chaotic dynamics in phenomenological and biologically plausible models of individual neurons are discussed. We show that chaos occurs at transitions between generic activity types in neurons such as tonic spiking, bursting, and quiescence, where the system can also become bi-stable. Bifurcations underlying these transitions give rise to period-doubling cascades, various homoclinic and saddle phenomena, torus breakdown, and chaotic mixed-mode oscillations in diverse model of individual neurons.

Vegetation Pattern Formation in Drylands: a Multi-Time-Scale Approach (Mary Silber)

A beautiful example of spontaneous pattern formation occurs in certain dryland environments around the globe. Stripes of vegetation alternate with stripes of bare soil, with striking regularity and on a scale readily monitored via satellites. Though the vegetation is a showstopping spectacle, water, which is the limiting resource for these ecosystems, is the unseen player behind the scenes. Water concentrates into the vegetated zones, essentially reinforcing vegetation patterning, via positive feedbacks, and its dynamics play out on the short timescales of the rare storms. In contrast, the vegetation may change very little over decades. I will describe a "stochastic pulsed precipitation" model framework that allows us to capture the impacts of variability in storm characteristics, such as storm intensity and duration, as well as seasonality. We identify an intrinsic length scale associated with these storm characteristics that sets the vegetation pattern scale in the model. This work is motivated by the question of how these vulnerable ecosystems might respond to climate change which may lead to increased variability in storm intensity.

The work highlighted in this talk was done in collaboration with Punit Gandhi.

Dynamics of chemical reaction systems with several slow manifolds (Peter Szmolyan)

Dimension reduction in chemical reaction systems is often based on quasi-steady-state approximations. The mathematical justification of these approximations can be based on the concept of a slow manifold in the framework of geometric singular perturbation theory (GSPT). More recently it was observed, that the dynamics of specific biochemical models with switch-like or oscillatory behavior is organized by several slow manifolds, corresponding to different scaling regimes of the variables. Matching of these different scaling regimes is carried out by the blow-up method which is needed to analyse slow manifolds in situations where normal hyperbolicity breaks down. In this talk, I will survey these developments and highlight some ongoing activities.

Rigorous derivation of the Michaelis–Menten kinetic in the presence of diffusion for enzyme reactions (Bao Quoc Tang)

Michaelis–Menten kinetic (MM) is one of the most used when modelling enzyme- (or more generally catalytic-) reactions. In the case of homogeneous medium, i.e. the (bio-)chemical concentrations depend solely on time, both formal and rigorous derivations of MM from mass action kinetics have been studied extensively and thoroughly in the last decades. For heterogeneous medium, the modelling should take into account the diffusion of substances, which leads to a system of partial differential equations. In this case, interestingly, only formal derivation of MM from mass action kinetic has been investigated. In this talk, we present, up to our knowledge, the first rigorous derivation of MM in the presence of diffusion. The proof utilises an improved duality technique and a modified energy method.

This is based on a joint work with Bao-Ngoc Tran (University of Graz).

Spatially periodic solutions of a singularly perturbed three-component reaction-diffusion system (Peter van Heister)

Following pattern formation away from onset is still a challenge as no general theory is available. Here, we consider a singularly perturbed three-component reaction-diffusion system and show how the singular perturbed structure can be used to study pattern formation away from onset. We show analytically how a near-equilibrium periodic pattern emerges through a Hamiltonian- Hopf bifurcation and, upon continuing in a system parameter, evolves to various far-from equilibrium periodic patterns that can be described rigorously by geometrical singular perturbation techniques.

This is joint work with Christopher Brown, Gianne Derks, and David J.B. Lloyd from the University of Surrey in the UK.

Geometric analysis of fast-slow PDEs with fold singularities (Thomas Zacharis)

We study a singularly perturbed fast-slow system of two partial differential equations (PDEs) of reaction-diffusion type on a bounded domain. We assume that the reaction terms in the fast variable contain a fold singularity, whereas the slow variable assumes the role of a dynamic bifurcation parameter, thus extending the classical analysis of a fast-slow dynamic fold bifurcation to an infinite-dimensional setting. Our approach combines a spectral Galerkin discretisation with techniques from Geometric Singular Perturbation Theory (GSPT) which are applied to the resulting high- dimensional systems of ordinary differential equations (ODEs). In particular, we show the existence of invariant manifolds away from the fold singularity, while the dynamics in a neighbourhood of the singularity is described by geometric desingularisation, via the blowup technique. Finally, we relate the Galerkin manifolds that are obtained after the discretisation to the invariant manifolds which exist in the phase space of the original system of PDEs.

3 Recent Developments and Open Problems

The workshop ended with a round table where various insights from the presentations and potential new directions were discussed. One particular topic of debate was the role of transient dynamics, i.e. phenomena that happen on finite time scales and cannot be observed by purely analyzing asymptotic behavior. Naturally, this is highly relevant for applications as time is always finite in the real world. Mathematically, such phenomena are well-captured in the deterministic fast-slow analysis (e.g. relaxation-oscillations, mixed-mode oscillations, etc.), with new challenges arising particularly in high-dimensional systems. In stochastic dynamics, this constitutes a vibrant area of research connecting physics-driven and mathematical approaches including topics like escape rates from bounded domains, large deviations, finite-time Lyapunov exponents, covariant Lyapunov vectors and many other concepts. Still, a unified theory, especially when stochastic bifurcations are involved, is missing and object of future research.

In terms of applications, the role of mathematical neuroscience was emphasized as key to new models and their mathematical analysis and challenges. More generally, the meeting led to a reflection and debate about the role of mathematical modelling in the context of large data sets and advanced computational tools. There was some agreement that the role of mathematicians will still be to thoroughly analyze and understand the structure within such models but also be open to an advanced integration of data-based methodology. Additionally, it was emphasized that the field of multiscale and PDE dynamics, including traveling waves and pattern formation, should look out for new benchmark models beyond the celebrated FitzHugh–Nagumo equations, from an application-oriented as well as a mathematical perspective. This is clearly a crucial new pathway for the next years.

Conclusively, two more open directions were mentioned, with already some progress, though still involving considerable challenges. For the stochastic part, this concerns the homogenization of fast-slow ODEs that are fully coupled. While the skew-product case with chaotic driving is well-understood in terms of stochastic limit equations (see the talk by Georg Gottwald), the fully coupled case has so far only been handled in very restricted situations with strong assumptions. New techniques seem to be required here, making use of innovations in rough path theory and stochastic regularization. On the deterministic side, an emerging field of interest concerns non-smooth bifurcations and slow passages through such singularities (see the talk by Rachel Kuske). Such non-smooth points or even whole manifolds will require new geometric insight and inspire highly intriguing research that can be understood well by having the smooth case as comparable benchmark.

Overall, the final discussion on recent and new developments and open problems was very lively and provided a suitable and worthy conclusion of this inspiring workshop.

4 Outcome of the Meeting

The BIRS workshop brought together 31 on-site researchers from North America, Europe, and Oceania. Thanks to the hybrid format, 3 virtual speakers were able to contribute remotely and several virtual participants could attend the event. Together with the videos of the talks, which are freely available on the BIRS website, this successfully contributed to reaching a wider public, not restricted to the invited participants, and to connecting different communities and different career stages even more.

In particular, one of the organisers' goals was in fact to bring different communities together:

- early-career and more-experienced researchers: it was very important for PhD students to attend and to present their work during the workshop. This allowed them to get to know and integrate into the community, to expand their network and to contribute with a fresh point of view to the new trends.
- researchers from GSPT, stochastic, more theoretical and more applied communities: thanks to the time allocated to the discussion, new connections were formed and cross-fertilisation of different communities ignited. As first short-term outcome, several new invitations to seminars and workshops among the participant of the workshop were made, hopefully leading to new collaborations.

Regarding the gender balance, 8 speakers were great female scientists (out of 34), either early career or consolidated researchers. Among the organisers, the ratio was 1/4.

During the event, participants contributed in creating a friendly and very productive atmosphere, posing a lot of questions during the sessions and actively participating in the discussion. As evident from feedback following the meeting, they also expressed great appreciation for the event, schedule and selection of topics.

Finally, the contributions of the speakers and the organisers are going to be collected in a volume of AMS Contemporary Mathematics dedicated to the proceedings of the workshop, with due acknowledgement to the Banff International Research Station. It is the intention of the organizers to prepare a review paper covering the content of lectures and discussions that took place during the workshop as part of this volume, to reach out to the wider international community.

The organisers are very grateful to the BIRS Staff, in particular to Linda and the IT Technicians.

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