The Mazur pattern and concordance

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Definitions:

Satellite Operators

special functions $\{\text{knots}\} \to \{\text{knots}\}$

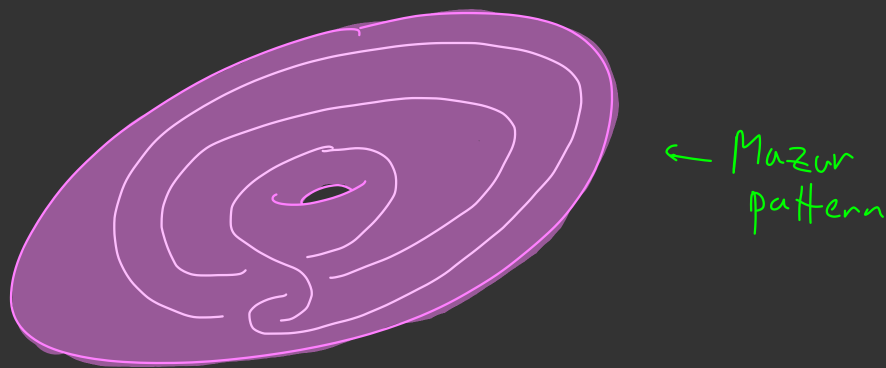
Concordance

equivalence rel'n on knots

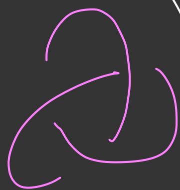
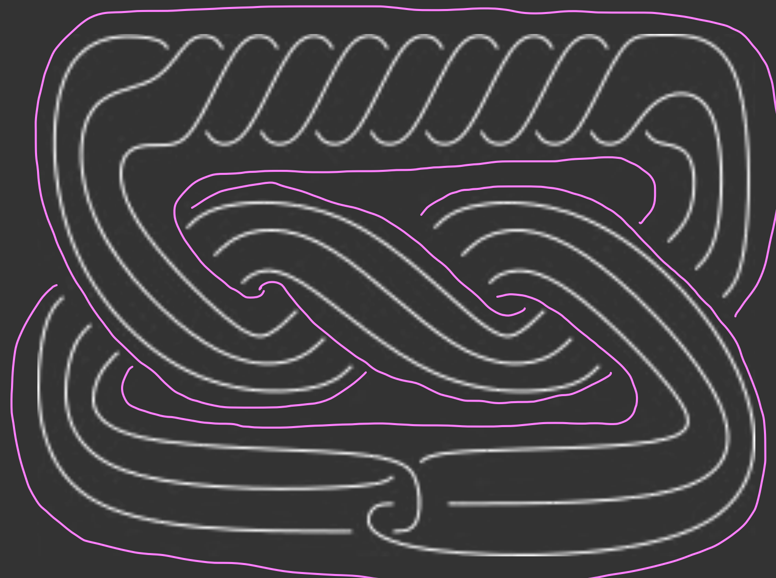
Question: When do satellite operators give the same induced maps $\{\text{knots}\}/\text{concordance} \to \{\text{knots}\}/\text{concordance}$?

Satellite Operators

The data of a satellite operator is given by a knot in a solid torus:



The action of a satellite operator on a knot J is given by tying up the solid torus into J:

$Q\Big($  $\Big) =$ 

white knot is the output

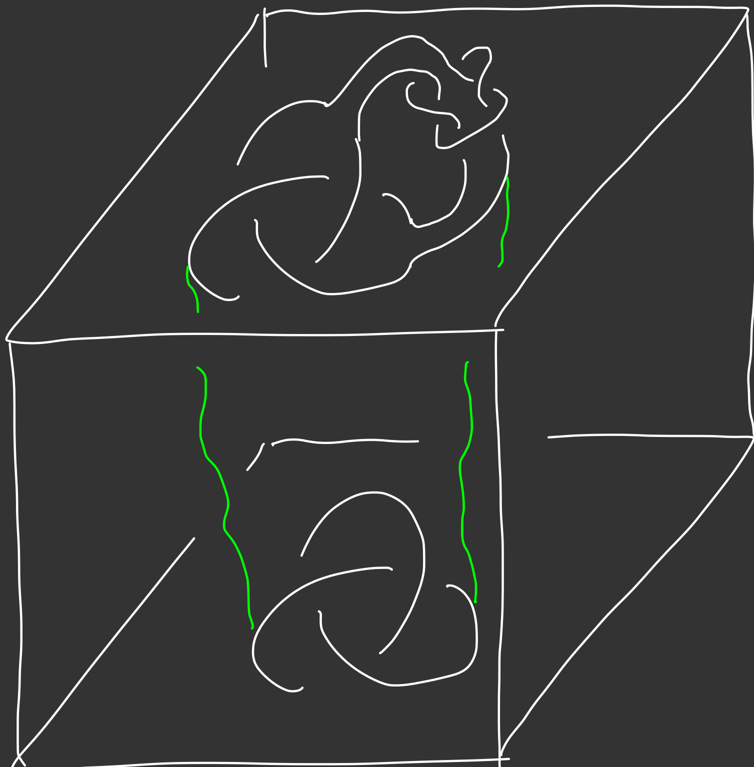
This is a fn

$\{\text{knots}\} \longrightarrow \{\text{knots}\}$

Concordance

Def Two (oriented) knots K & J are concordant if there is an embedded annulus in $S^3 \times I$ with boundary $(K \times \{0\}) \sqcup (-J \times \{1\})$

(mirror image w/ reversed orientation)



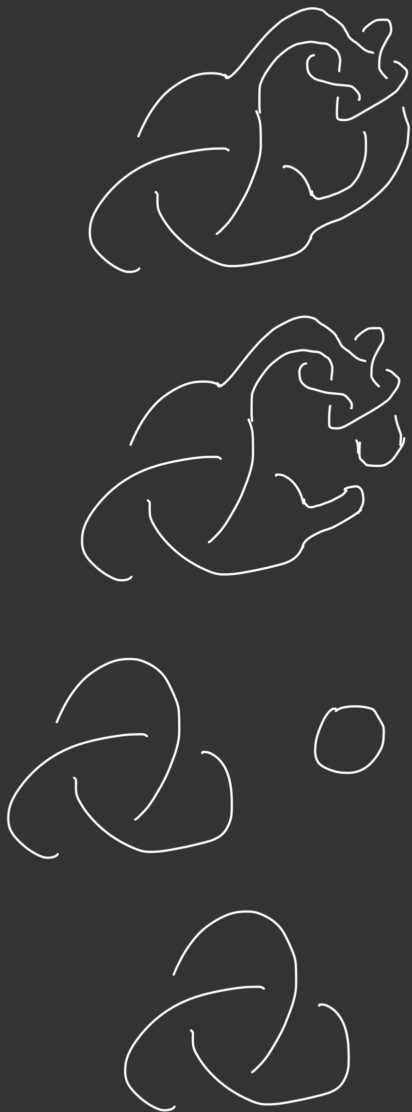
$S^3 \times I$

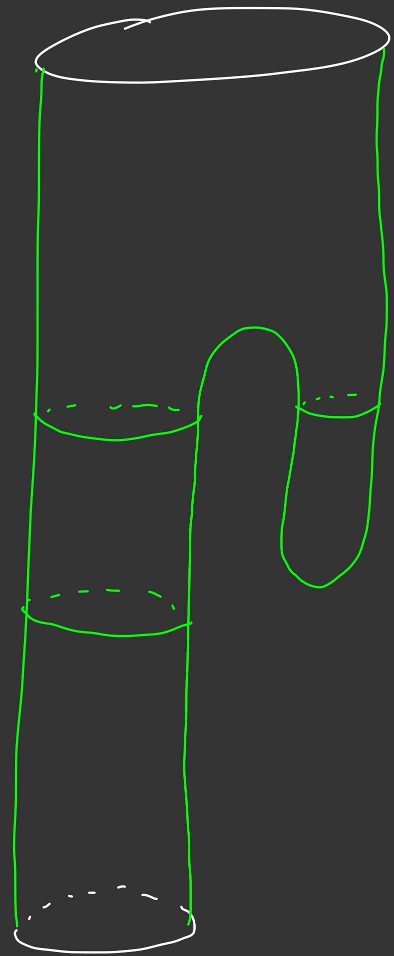
Rmk Satellite operators descend to functions

$\mathcal{C} \longrightarrow \mathcal{C}$

{knots}/concordance

Mostly, we will work in the topological category.
But for the first part it makes no difference.

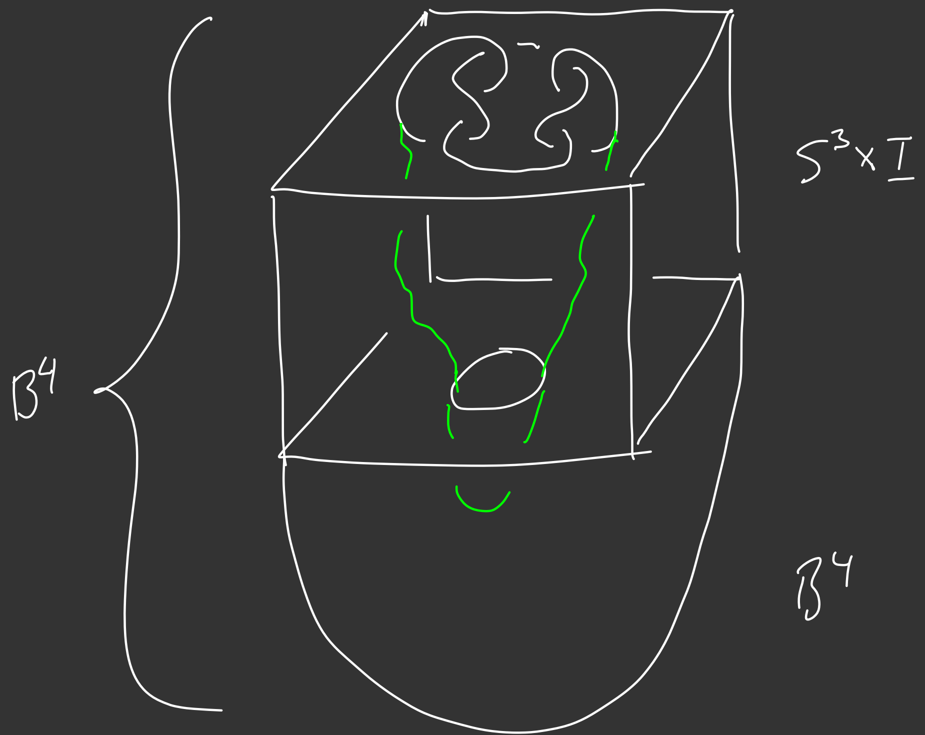




Def A knot is *slice* if it is concordant to the unknot.

Prop TFAE: 1) K is slice

2) $K \subset S^3 = \partial B^4$ bounds a properly embedded disk $D^2 \subset B^4$



After this slide, we always work in the topolgical category.

Thm (Cochran–Friedl–Teichner) P satellite operator. $P(\text{unknot})$ slice, η is nullhomotopic in $B^4 \setminus$ slice disk. Then $P: \mathcal{C}_{top} \to \mathcal{C}_{top}$ is the zero map, i.e. $P(J)$ is topologically slice for any J.

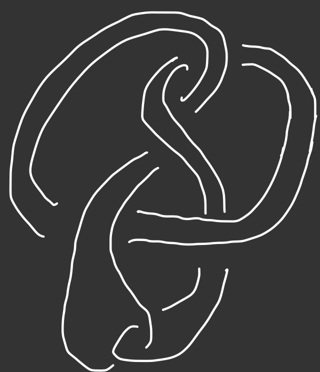
Whitehead doubling operator

Wh(unknot) =

η is the image of the meridian of the solid torus

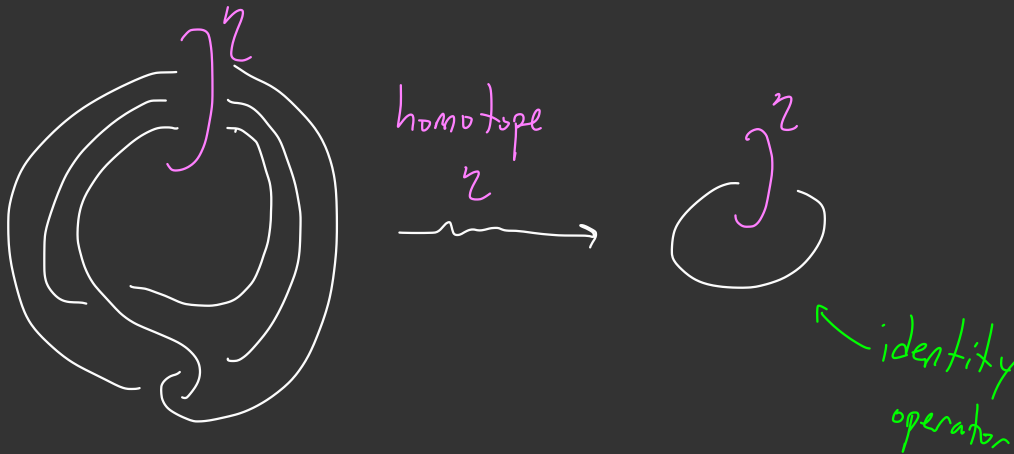
Cor Whitehead doubles are topologically slice.

First proved by Freedman (proved that Alexander polynomial $=1 \Rightarrow$ topologically slice)



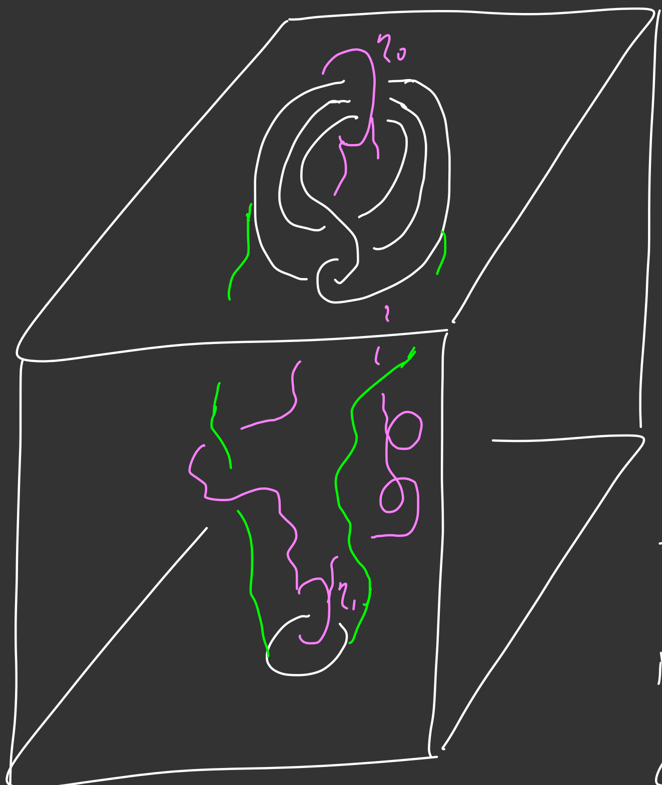
topologically slice!

Mazur pattern — we have a situation that "feels related":



Question: Are $Q(J)$ & J topologically concordant?

Question: P_0, P_1 satellite operators. $P_0(\text{unknot})$, $P_1(\text{unknot})$ concordant. η_0, η_1 freely homotopic through $(S^3 \times I) \setminus$ annulus.



Do P_0, P_1 induce the same function $\mathcal{C}_{top} \to \mathcal{C}_{top}$?

Thm (M) Under same conditions, induced maps are the same $\mathcal{C}_{top}/\mathcal{F}_1 \to \mathcal{C}_{top}/\mathcal{F}_1$

In the smooth category:

__Thm (Akbulut)__ Wh() not smoothly slice. (Gauge Theory)

__Open__ Is Wh() smoothly slice?

Many knots J are not smoothy concordant to $Q(J)$.

(Cochran-Franklin-Hedden-Horn, Collins, using Heegaard-Floer)

(h)-solvability

Equivalent characterization of topological sliceness:

3-mfld associated with K

K is topologically slice $\Leftrightarrow$ it's 0-surgery M_K bounds a 4-mfld W s.t.:

1) $\pi_1(W)$ normally generated by meridian μ of K in M_K
2) $H_1(W) = \mathbb{Z}$ generated by μ
3) $H_2(W) = 0$

Eg $W = B^4 \setminus$ slice disk

Think of as Poincaré conjecture for slice disk complements

Hard to build 4-mfld w/ required properties

Equivalent characterization of topological sliceness:

3-mfld associated with K

K is topologically slice $\Leftrightarrow$ it's 0-surgery M_K bounds a 4-mfld W s.t.:

Def of (n)-solvability

K is (n)-solvable if M_K bounds a 4-mfld W s.t.:

1) $\pi_1(W)$ normally generated by meridian μ of K in M_K
2) $H_1(W) = \mathbb{Z}$ generated by μ

same for both

3) $H_2(W) = 0$

weaker

3) $H_2(W) = \mathbb{Z}^{2k}$ generated by embedded surfaces w/ trivial normal bundles

$\pi_1(S_i), \pi_1(T_i) \to \pi_1(W)^{(n)}$

K (n.5)-solvable if $\pi_1(T_i) \to \pi_1(W)^{(n+1)}$

(n)-solvability gives filtration on set of concordance classes

$$\begin{bmatrix}\text{top}\\ \text{slice}\\ \text{knots}\end{bmatrix} \subseteq \cdots \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_{1.5} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}_{top}$$

$\mathcal{F}_{1.5}$: $\Rightarrow$ Casson-Gordon sliceness obstruction vanishes

(Cochran-Orr-Teichner)

$\mathcal{F}_{0.5}$: iff K algebraically slice

$\mathcal{F}_0$: iff $\mathrm{Arf}(K)=0$

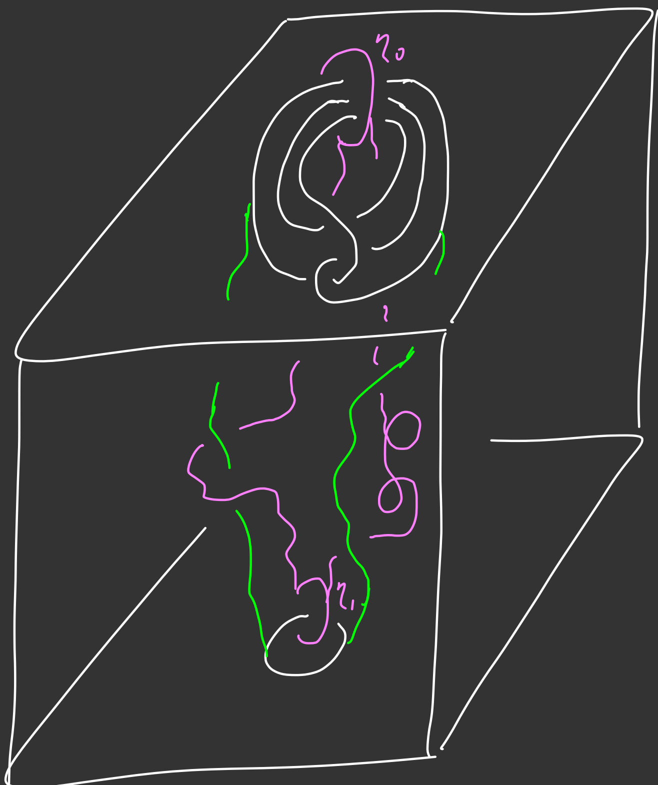
<u>Thm</u> (Cochran-Harvey-Leidy) $\mathbb{Z}^\infty \subseteq \mathcal{F}_{(n)}/\mathcal{F}_{(n.5)}$

<u>Open</u> $\mathcal{F}_{(n.5)}/\mathcal{F}_{(n+1)} = 0$?

<u>Open</u> If K is (n)-solvable for all $n \in \frac{1}{2}\mathbb{Z}$, is K then topologically slice?

Thm (M) Given two satellite operators P_0 & P_1, suppose P_0(unknot) & P_1(unknot) are topologically concordant, and η_0 & η_1 are freely homotopic through the complement of the concordance. Then P_0 & P_1 induce the same function

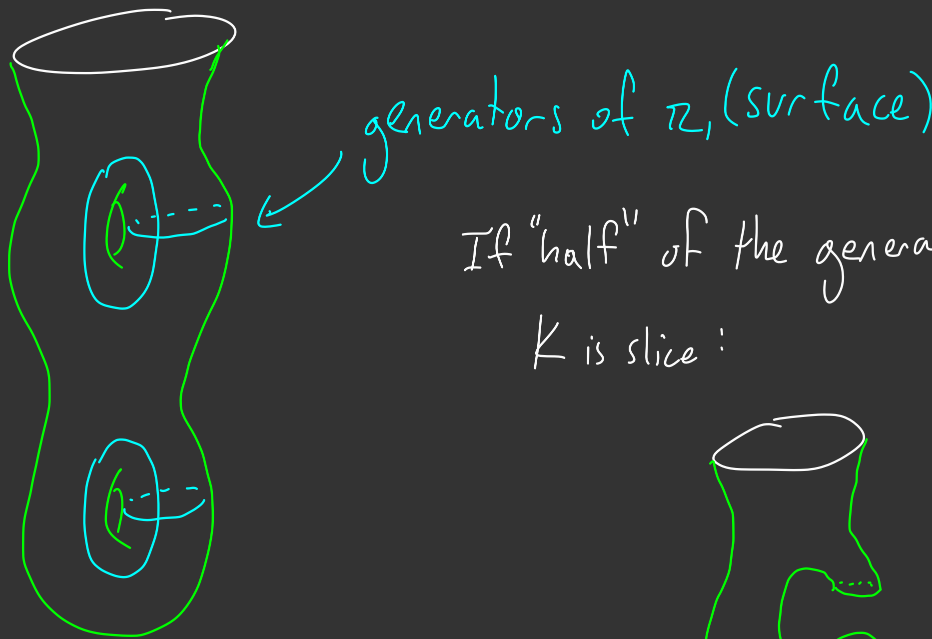
$$\mathcal{C}_{top}/\mathcal{F}_1 \to \mathcal{C}_{top}/\mathcal{F}_1$$



(Moreover, the Casson-Gordon obstruction to sliceness vanishes for $P_0(J) \# -P_1(J)$, as does the metabelian ρ-invariant obstruction.)

(both are (1.5)-solvability obstructions)

Every K bounds a surface.



If "half" of the generators bound disks,

K is slice:

Can ask that generators also bound surfaces:

and so on...

called a surface tower
of height n

Can also have a surface tower of height $n.5$
by only attaching surfaces to "half" of the curves.

Note: requires generators to be nullhomologous in $B^4 \setminus$ surface
— not always true!

Algebraic loosening of bounding a surface: curve $\in$ commutator subgp of π_1

(ie. nullhomologous since $H_1 = \pi_1^{ab}$)

" " surface tower:

curve deep in derived series of π_1

$$\left(\begin{array}{l} G^{(0)} := G \\ G^{(n)} := [G^{(n-1)}, G^{(n-1)}] \end{array} \right)$$