Fractals under quasiconformal maps

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Smooth Functions on Rough Spaces and Fractals with Connections to Curvature Functional Inequalities

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Definition

Let $F \subset \mathbb{R}^n$ bounded. For $r > 0$, denote by $N(F,r)$ the smallest number of sets of diameter at most r needed to cover F. The (upper) ***box-counting dimension*** *of F is*

$$\dim_B(F) = \limsup_{r \to 0} \frac{\log N(F,r)}{\log(1/r)} =$$

$$= \inf\{\alpha > 0 \,:\, \exists C > 0 \text{ s.t. }$$

$$N(F,r) \le Cr^{-\alpha} \text{ for all } 0 < r \le \operatorname{diam}(F)\}.$$

Example

Let $a > 0$ and $S_a := \{t^{-a}e^{it} \in \mathbb{C} : t > 1\}$. J. Fraser in 2019 proved

$$\dim_B(S_a) = \max\left\{\frac{2}{1+a}, 1\right\}.$$

Hence, for all $a < 1$ we have $\dim_H(S_a) = 1 < \dim_B(S_a) = \frac{2}{1+a}$.

Definition

*Let $F \subset \mathbb{R}^n$, the **Assouad dimension** of F is*

$$\dim_A(F) = \inf\left\{\alpha > 0 : \begin{array}{c} \exists\, C > 0 \text{ s.t. for all } 0 < r \leq R \text{ and } x \in F \\ N(B(x,R) \cap F, r) \leq C(R/r)^\alpha \end{array}\right\}.$$

Example

- $\dim_A(F_a) = 1$ *where* $F_a = \{1/n^a : n \in \mathbb{N}\}$,
- $\dim_A(S_a) = 2$

Not enough for even bi-Lipschitz classification of S_a.

Definition

For $0 < \theta < 1$ *and* $F \subset \mathbb{R}^n$, *define its (regularized* θ*-)* ***Assouad spectrum***

$$\dim_{A,reg}^{\theta}(F) = \inf\left\{\alpha > 0 \ : \ \begin{array}{c} \exists\, C > 0 \text{ s.t. } N(B(x,R)\cap F, r) \leq C(R/r)^{\alpha} \\ \forall\, 0 < r \leq R^{1/\theta} < R < 1, \forall\, x \in F \end{array}\right\}.$$

For all $x \in F$, $R > 0$ and $m \in \mathbb{N}$, denote by $N_d(B(x,R)\cap F, m)$ the number of m-dyadic cubes of $Q(x,R)$ needed to cover $B(x,R)\cap F$. We then have

$$\dim_{A,reg}^{\theta}(F) = \inf\left\{\alpha > 0 \ : \ \begin{array}{c} \exists\, C > 0 \text{ s.t. } N_d(B(x,R)\cap F, m) \leq C2^{m\alpha} \\ \forall\, 0 < 2^{-m}R \leq R^{1/\theta} < R < 1, \forall\, x \in F \end{array}\right\}.$$

Examples

- *For all* $0 < \theta < 1$,

$$\dim^{\theta}_{A,reg}(F_a) = \min\left\{\frac{1}{(1+a)(1-\theta)}, 1\right\} \in (\dim_B(F_a), \dim_A(F_a))$$

- *For all* $a > 0$ *and* $0 < \theta < 1$ *we have (by Fraser)*

$$\dim^{\theta}_{A,reg}(S_a) = \begin{cases} \min\left\{\frac{2}{(1+a)(1-\theta)}, 2\right\}, & \textit{if } 0 < a \le 1, \\ \min\left\{1 + \frac{\theta}{a(1-\theta)}, 2\right\}, & \textit{if } a \ge 1. \end{cases}$$

Hence, for all $a < 1$ *and* $\theta < \frac{a}{1+a}$ *we have*

$$\dim_B(S_a) = \frac{2}{1+a} < \dim^{\theta}_{A,reg}(S_a) = \frac{2}{(1+a)(1-\theta)} < \dim_A(S_a) = 2.$$

Proposition

(1) *For fixed* θ*, the set function* $F \mapsto \dim^{\theta}_{A,reg}(F)$ *is*

  (a) *monotonic,*

  (b) *finitely stable, i.e.,* $\dim^{\theta}_{A,reg}(E \cup F) = \max\{\dim^{\theta}_{A,reg}(E), \dim^{\theta}_{A,reg}(F)\}$*,*

  (c) *invariant under taking closures, and*

  (d) *invariant under bi-Lipschitz maps.*

(2) *For fixed* F*,* $\lim_{\theta\to 1^-} \dim^{\theta}_{A,reg}(F) := \dim_{qA}(F)$*.*

Moreover, if F *is bounded, then* $\lim_{\theta\to 0^+} \dim^{\theta}_{A,reg}(F) = \dim_B(F)$*,*

and

$$\dim_B(F) \le \dim^{\theta}_{A,reg}(F) \le \dim_{qA}(F) \le \dim_A(F).$$

Definition

A homeomorphism $f : \Omega \to \Omega'$ *between domains in* $\mathbb{R}^n$, $n \geq 2$, *is said to be* **K-quasiconformal** *(K-QC) if* f *lies in the local Sobolev space* $W^{1,n}_{loc}$ *and there is a* $K \geq 1$ *such that the inequality*

$$|Df|^n \leq K \det Df$$

holds a.e. in Ω*.*

The smallest value $K_O(f) \geq 1$ for which the above inequality holds a.e. in Ω is known as the **outer dilatation** of f.

QC maps are actually "more integrable" than initially expected.

Definition

For $n \geq 2$ and $K \geq 1$, define $p(n,K)$ to be the supremum of those values $p > 0$ so that there exists a constant $C > 0$ such that for every QC map $f : \Omega \to \Omega'$ in $\mathbb{R}^n$ with $K_O(f) \leq K$, the estimate

$$\left(⨍_Q |Df|^p \right)^{1/p} \leq C \left(⨍_{2Q} |Df|^n \right)^{1/n}$$

holds for every cube $Q \subset \Omega$ with $\operatorname{diam} Q < \operatorname{dist}(Q, \partial\Omega)$ and $\operatorname{diam} f(2Q) < \operatorname{dist}(f(2Q), \partial\Omega')$.

By Gehring ('73), $p(n,K) > n$ for each $n \geq 2$ and $K \geq 1$.
Astala ('94) showed that

$$p(2,K) = \frac{2K}{K-1}.$$

QC distortion of the Hausdorff and box dimensions:

Theorem (Gehring - Väisälä, '73)

Let E be a subset of $\Omega \subset \mathbb{R}^n$ with $\dim_H(E) = \alpha \in (0, n)$. Then

$$0 < \frac{(p(n, K^{n-1}) - n)\alpha}{p(n, K^{n-1}) - \alpha} \leq \dim_H f(E) \leq \frac{p(n, K)\alpha}{p(n, K) - n + \alpha} < n$$

for any K-QC map $f : \Omega \to \Omega' \subset \mathbb{R}^n$.

Theorem (Kaufman, 2000)

Let E be a bounded subset of $\Omega \subset \mathbb{R}^n$ with $\dim_B(E) = \alpha \in (0, n)$. Then

$$0 < \frac{(p(n, K^{n-1}) - n)\alpha}{p(n, K^{n-1}) - \alpha} \leq \dim_B f(E) \leq \frac{p(n, K)\alpha}{p(n, K) - n + \alpha} < n$$

for any K-QC map $f : \Omega \to \Omega' \subset \mathbb{R}^n$.

We proved similar estimates for $\dim_A$ and $\dim^{\theta}_{A,reg}$:

Theorem 1 (C.G. - Tyson, 2021)

Let $f : \Omega \to \Omega'$ be a K-QC map in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be a compact set with $\dim_A(E) = \alpha$. Then

$$\frac{(p(n, K^{n-1}) - n)\alpha}{p(n, K^{n-1}) - \alpha} \leq \dim_A f(E) \leq \frac{p(n, K)\alpha}{p(n, K) - n + \alpha}.$$

If $\Omega = \Omega' = \mathbb{R}^n$ then the conclusion holds for all $E \subset \mathbb{R}^n$.

Theorem 2 (C.G. - Tyson, 2021)

Let $f : \Omega \to \Omega'$ be a K-QC map in $\mathbb{R}^n$, $n \geq 2$, and let $E \subset \Omega$ be a compact set with $\alpha_t := \dim^{\theta(t)}_{A,reg}(E)$, where $\theta(t) = 1/(t+1)$. Then

$$\frac{(p(n, K^{n-1}) - n)\alpha_{Kt}}{p(n, K^{n-1}) - \alpha_{Kt}} \leq \dim^{\theta(t)}_{A} f(E) \leq \frac{p(n, K)\alpha_{t/K}}{p(n, K) - n + \alpha_{t/K}},$$

for all $t > 0$.

Proof of Thm 2: We can assume w.l.o.g. the following:

- $E, f(E) \subset Q_0 := [-\frac{1}{5\sqrt{n}}, \frac{1}{5\sqrt{n}}]^n$
- $f(Q_0) \subset Q_0$
- there is a constant $C_{K,\eta} \in (0,1)$ such that

$$\operatorname{diam} Q_0 \leq C_{K,\eta} \min\{\operatorname{dist}(Q_0, \partial\Omega), \operatorname{dist}(Q_0, \partial\Omega')\}.$$

It suffices to prove

$$\dim_{A,reg}^{\theta(t)}(f(E)) \leq \beta_0 := \frac{p(n,K)\alpha_0}{p(n,K) - n + \alpha_0}, \qquad \alpha_0 = \dim_{A,reg}^{\theta(t/K)}(E).$$

Fix $p < p(n,K)$, $\alpha > \alpha_0$ and let $\beta := \frac{p\alpha}{p-n+\alpha}$.

Let $y \in f(E)$ and $0 < R' \leq 1$. By QCy of $g = f^{-1}$ we can find $B(x, R)$ "deep" in Q_0 with $x = g(y)$, $R < 1$ and

$$B(y, R') \cap f(E) \subset f(B(x, R) \cap E).$$

We consider cubes obtained via dyadic decomposition of $Q(x, R)$.

$$r_m := 2^{-m} R$$

$$r'_m := 2^{-m\alpha/\beta} R'$$

for **all** meaningful scales $m \geq m'_0$. Local Hölder continuity of f and choice of $\theta(t)$ ensure $m'_0 \geq m_0$.

Fix $m \geq m'_0$. For $j \geq m$, a j-dyadic cube Q^j is (m-)**minor** if $\operatorname{diam} f(Q^j) \leq r'_m$ and (m-)**major** otherwise.

Lemma 1

The total number of all m-major subcubes of $Q(x, R)$ is bounded above by $C_{K,n}2^{m\alpha}$.

Proof of Lemma 1: For $j \geq m$, let $M(j)$ be the number of m-major subcubes of $Q(x, R)$ of side length $2^{-j}R$. Denote Q_i^j such a cube, $1 \leq i \leq M(j)$.

By Morrey-Sobolev inequality on Q_i^j:

$$\operatorname{diam} f(Q_i^j) \leq C_2(\operatorname{diam}(Q_i^j))^{1-n/p} \left(\int_{Q_i^j} |Df|^p \right)^{1/p}$$

Sum over all m-major cubes of level j and then over all $j \geq m$:

$$\sum_{j=m}^{\infty} M(j) \leq C_4 \, 2^{\frac{m\alpha p}{\beta} - m(p-n)} \, (R')^{-p} R^{p-n} \int_{Q(x,R)} |Df|^p \, . \qquad (1)$$

Recall RHI for $p < p(n, K)$:

$$\frac{1}{\mathscr{L}_n(Q(x,R))^{1/p}} \left(\int_{Q(x,R)} |Df|^p \right)^{1/p} \leq \frac{1}{\mathscr{L}_n(Q(x,2R))^{1/n}} \left(\int_{Q(x,2R)} |Df|^n \right)^{1/n}$$

So the integral on the right of (1) is at most, up to some C_4':

$$\frac{R^n}{R^p} \mathscr{L}_n(f(Q(x,2R)))^{p/n} \leq R^{n-p} \mathscr{L}_n(Q(y,R'))^{p/n} \leq R^{n-p}(R')^p .$$

Hence, by the definition of $\beta = \frac{p\alpha}{p-n+\alpha}$, we obtain

$$\sum_{j=m}^{\infty} M(j) \leq C_5\, 2^{m\alpha}. \qquad \square_{Lemma 1}$$

Hence, we can use

$$C_\alpha \left(\frac{R}{r_m} \right)^{\alpha} + \sum_{j=m}^{\infty} 2^n M(j) \leq C_\alpha 2^{m\alpha} + C_5\, 2^{m\alpha+n} \leq C_6 2^{m\alpha} = \left(\frac{R'}{r_m'} \right)^{\beta}$$

images of minor cubes to cover $B(y, R') \cap f(E)$.

$\square$

An application of Theorem 2 is the following:

Theorem

For $a > b > 0$, there exists a QC map $f : \mathbb{C} \to \mathbb{C}$ with $f(S_a) = S_b$ if and only if $K_O(f) \geq \frac{a}{b}$.

Proof: $(\Leftarrow)$ $f(z) = |z|^{1/K-1}z$ is K-QC with $K = a/b$ and $f(S_a) = S_b$.
$(\Rightarrow)$ Suppose $K < \frac{a}{b}$ and there is a K-QC f with $f(S_a) = S_b$.

Using $p(2, K) = \frac{2K}{K-1}$ on the right of Theorem 2 gives

$$\frac{1}{K}\left(\frac{1}{\dim_{A,reg}^{\theta(t/K)}(S_a)} - \frac{1}{2}\right) \leq \frac{1}{\dim_{A,reg}^{\theta(t)}(S_b)} - \frac{1}{2}.$$

But for $t = 1/b$, $\theta(t) = b/(1+b)$ and $\theta(t/K) < a/(1+a)$, so the right is $= 0$ while the left > 0. $\square$

Remarks

- *We can actually improve Theorem 2 by replacing* $p(n, K^{n-1})$ *by the larger (or equal)* $p_I(n, K)$*, where* $p_I(n, K)$ *is defined as* $p(n, K)$ *but involving the inner dilatation* K_I*.*
- *Kaufman's Theorem only provides*

$$K \geq \frac{\min\{a, 1\}}{\min\{b, 1\}}$$

  for the spirals. Theorem 2 is necessary for the QC ***and*** *bi-Lipschitz classification of spirals.*
- *The theorems on* $\dim_H$ *and* $\dim_B$ *distortion can be stated for* $W^{1,p}(\Omega; \mathbb{R}^n)$ *(right hand side inequality), not necessarily quasiconformal. Our proofs rely on quasiconformality.*

Thank You!

Theorem

Let $f : \Omega \to \Omega'$ be a K-QC map in $\mathbb{R}^n$, $n \geq 2$, and let $E \subset \Omega$ be a compact set with $\alpha_t := \dim_{A,reg}^{\theta(t)}(E)$, where $\theta(t) = 1/(t+1)$. Then

$$\frac{(p(n,K^{n-1}) - n)\alpha_{Kt}}{p(n,K^{n-1}) - \alpha_{Kt}} \leq \dim_A^{\theta(t)} f(E) \leq \frac{p(n,K)\alpha_{t/K}}{p(n,K) - n + \alpha_{t/K}},$$

for all $t > 0$.

Theorem

For $a > b > 0$, there exists a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ with $f(S_a) = S_b$ if and only if $K_O \geq \frac{a}{b}$.