Conformal Assouad dimension as the critical exponent for combinatorial modulus

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A potential theoretic notion of dimension

- The p-capacity ($p > 1$) between two sets $E, F \subset \mathbb{R}^n$ is given by

$$\mathrm{cap}_p(E, F) := \inf\{\int_{\mathbb{R}^n} |\nabla f|^p (x)\, dx : f \in C^1(\mathbb{R}^n), f\big|_E \equiv 1, f\big|_F \equiv 0\}.$$

  The capacity of the annulus $\mathrm{cap}_p(B(x_0, r), B(x_0, R)^c)$ is

$$\begin{cases} c_n \log(R/r)^{1-n}, & p = n, \\ c_{p,n} \left| R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)} \right|^{1-p}, & p \neq n. \end{cases}$$

- Question: Can we identify the dimension of a metric space as a critical value of p based on the behaviour of p-capacity of annuli?

Combinatorial modulus

- Let $G = (V, E)$ be a graph. Let Γ be a family of paths in G. Then the p-modulus of Γ is

  $$\mathrm{Mod}_p(\Gamma, G) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \sum_{v \in V} \rho(v)^p,$$

  where
  $\mathrm{Adm}(\Gamma) = \{\rho : V \to [0, \infty) : \sum_{v \in \gamma} \rho(v) \geq 1 \text{ for all } \gamma \in \Gamma\}$.
- One can use function ρ defined on edges (Duffin '62) instead of vertices (Cannon '94). This leads to a comparable quantity on bounded degree graphs (He and Schramm '95).
- If Γ is the family of paths that join A_1 and A_2, then the edge modulus of Γ is the (discrete) p-capacity between A_1 and A_2.
- We can understand modulus (or capacity) on metric spaces by approximating a metric space by a sequence of graphs at finer and finer scales.

Critical exponent for combinatorial modulus

- Let (X, d) be a compact metric space. Fix parameters $a, \lambda, L > 1$.
- For each $k \in \mathbb{N}$, let X_k be a maximal a^{-k}-separated subset of (X, d) with $X_k \subset X_{k+1}$ for all k.
- Define graphs G_n with vertex set X_n, where $x, y \in X_n$ are joined by an edge if $x \neq y$ and $B(x, \lambda a^{-n}) \cap B(y, \lambda a^{-n}) \neq \emptyset$.
- For $x \in V(G_n) = X_n$, let $\Gamma_{k,L}(x)$ denote all paths in G_{n+k} that begin at $B(x, a^{-n})$ and end at $B(x, La^{-n})^c$. Set

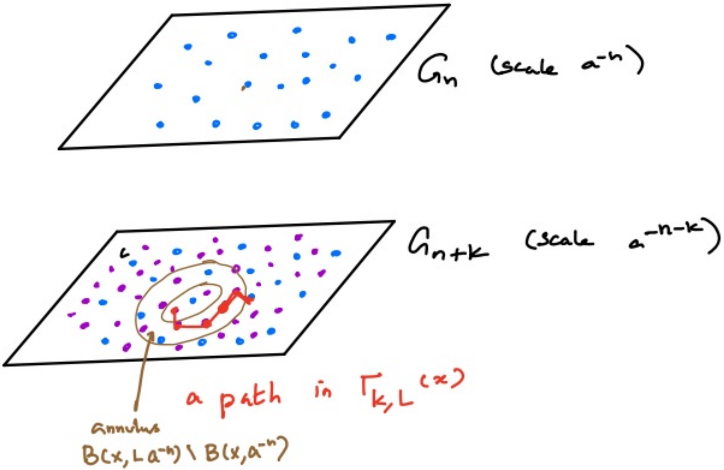
$$M_{p,k} = \sup\{\mathrm{Mod}_p(\Gamma_{k,L}(x) : x \in X_n, n \in \mathbb{N})\}$$

and the critical exponent $Q(X, d)$ is defined as

$$M_p = \liminf_{k\to\infty} M_{p,k}, \quad Q(X, d) = \inf\{p > 0 : M_p = 0\}.$$

- The above definition does not depend on choices of a, λ, L, X_n.
- This notion is due to Carrasco (2013) and Bourdon-Kleiner (2013).

Annulus viewed at a finer scale



Doubling metric space and Assouad dimension

- A metric space is doubling if there exists $N \in \mathbb{N}$ such that every ball of radius R can be covered by N balls of radii $R/2$.
- The Assouad dimension of a metric space (X, d) is the infimum of all $\alpha > 0$ such that there exists $C > 1$ so that every ball of radius R can be covered by $C(R/r)^{\alpha}$ balls of radii r for all $0 < r < R$.
- The Assouad dimension $d_A(X, d)$ is finite if and only if (X, d) is doubling.

Doubling measures and Vol'berg-Konyagin theorem '87

- A non-zero measure μ is said to be doubling if there exists $C > 1$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X, r > 0$. Equivalently μ is q-homogeneous for some $q > 0$:

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \lesssim \left(\frac{R}{r}\right)^{q}, \quad \text{for all } x \in X, 0 < r < R.$$

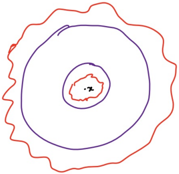
- (Vol'berg-Konyagin) The Assouad dimension of a compact metric space (X, d) is the infimum of all $q > 0$ such that there exists a q-homogeneous measure on (X, d).

Quasisymmetry and Conformal gauge

- Quasisymmetry (QS): A notion of ‘conformal maps’ on metric spaces (Ahlfors-Beurling ‘56, Tukia-Väisälä ‘80).
  $f : (X_1, d_1) \to (X_2, d_2)$ is a homeomorphism.
  $\eta : [0, \infty) \to [0, \infty)$ is a self-homeomorphism on $[0, \infty)$.
  Def. f is η-QS

  $$\frac{d_2(f(x), f(y))}{d_2(f(x), f(z))} \leq \eta\left(\frac{d_1(x, y)}{d_1(x, z)}\right) \quad \text{for all } x, y, z \in X_1,\ x \neq z.$$

  f is a QS (quasisymmetry) it is a quasisymmetry for some η.
  Def. Conformal gauge of a metric space (X, d)
  $\mathcal{J}(X, d) = \{\theta \text{ is a metric on } X | \mathrm{Id} : (X, d) \to (X, \theta) \text{ is a QS}\}$.



For all $x \in X$, $r > 0$, $A > 0$,
there exists $s > 0$, such that
$B_\theta(x, s) \subset B_d(x, r)$ and
$B_d(x, Ar) \subset B_\theta(x, \eta(A)s)$

Conformal dimensions

- The Ahlfors regular conformal dimension of a metric space (X, d) is

$$d_{\mathrm{ARC}} = \inf \left\{ Q \;\middle|\; \begin{array}{l} \text{there exists a measure } \mu \text{ and a metric} \\ \theta \in \mathcal{J}(X, d) \text{ such that } \mu(B_\theta(x, r)) \asymp r^Q \\ \text{for all } r < \operatorname{diam}(X, \theta). \end{array} \right\}.$$

- This is variant of Pansu's definition ('89) was introduced by Bonk-Kleiner ('05) and Bourdon-Pajot ('03).
- Possible values of $d_{\mathrm{ARC}} = \{0\} \cup [1, \infty]$ (Laakso'00, Kovalev'06).
- The conformal Assouad dimension $d_{\mathrm{CA}}(X, d)$ is

$$d_{\mathrm{CA}} = \inf\{d_A(X, \theta) : \theta \in \mathcal{J}(X, d)\}.$$

- **Questions**: Given a space, what is the value of d_{ARC} (or d_{CA})? Is the infimum attained? Both these questions are open for Sierpiński carpet.

Conformal dimensions: motivation and basic properties

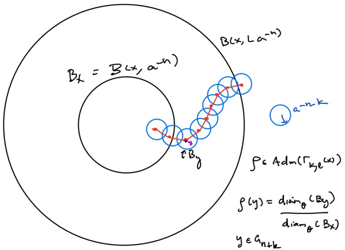
- In geometric group theory, the conformal dimension of the boundary of a hyperbolic group is a quasi-isometry invariant.
- In complex dynamics, the conformal dimension of the Julia set is invariant under Thurston equivalence.
- Quasisymmetry is a useful tool to understand Harnack inequalities (Kigami '08).
- $d_{\mathrm{CA}}(X, d) < \infty$ if and only if (X, d) is doubling
  $d_{\mathrm{ARC}}(X, d) < \infty$ if and only if (X, d) is doubling and uniformly perfect.
- (X, d) is uniformly perfect if there exists $C_P > 1$ such that for all balls $B(x, r) \neq X$ implies $B(x, r) \setminus B(x, r/C_P) \neq \emptyset$.

Combinatorial modulus, $d_{\mathrm{CA}}(X, d)$ and $d_{\mathrm{ARC}}(X, d)$

- Issue with d_{ARC}: $Y \subset X$ need not imply $d_{\mathrm{ARC}}(Y, d) \leq d_{\mathrm{ARC}}(X, d)$. On the other hand, $Y \subset X$ implies $d_{\mathrm{CA}}(Y, d) \leq d_{\mathrm{CA}}(X, d)$ and $Q(Y, d) \leq Q(X, d)$.
- Doubling measures are preserved under quasisymmetry whereas Ahlfors regular measures are not.
- (Heinonen) If (X, d) is compact, doubling and uniformly perfect, then $d_{\mathrm{CA}}(X, d) = d_{\mathrm{ARC}}(X, d)$.
- (Carrasco'13, Keith-Kleiner) If (X, d) is compact, doubling and uniformly perfect, then $Q(X, d) = d_{\mathrm{ARC}}(X, d)$.
- (M. '22+) If (X, d) is compact and doubling, then $Q(X, d) = d_{\mathrm{CA}}(X, d)$.
- Carrasco's proof of $Q(X, d) \leq d_{\mathrm{ARC}}(X, d)$ and $Q(X, d) \geq d_{\mathrm{ARC}}(X, d)$ uses the uniform perfectness property.

Heuristics

- The construction of the metric $\theta \in \mathcal{J}(X, d)$ is such that the 'new' diameter of a ball is proportional to the optimizer for modulus of annuli at all locations and scales.
- Then the admissibility condition $\sum_{y \in \gamma} \rho(y) \geq 1, \gamma \in \Gamma_{k,L}(x)$ can be interpreted as a 'no shortcuts condition': $\sum_{y \in \gamma} \mathrm{diam}(B_y, \theta) \geq \mathrm{diam}(B_x, \theta)$.
- The smallness of p-modulus is similar to $\sum_{y \in G_{n+k}} \mathrm{diam}^p(B_y, \theta) \ll \sum_{x \in G_n} \mathrm{diam}^p(B_x, \theta)$ which could be interpreted as dimension bound $\dim(X, \theta) \leq p$.



Gromov hyperbolic spaces

- The proof uses Gromov hyperbolic spaces. In particular, it is helpful to view the given metric space as the boundary of a Gromov hyperbolic space .
- Let (Z, D) be a metric space. The Gromov product of x and y with respect to the base point w as

$$(x|y)_w = \frac{1}{2}(D(x,w) + D(y,w) - D(x,y)).$$

- (Z, D) is δ-hyperbolic, if for any four points $x, y, z, w \in Z$, we have

$$(x|z)_w \geq (x|y)_w \wedge (y|z)_w - \delta.$$

The boundary of a hyperbolic space

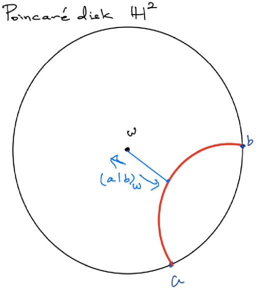
- A sequence of points $\{x_i\} \subset Z$ is said to converge at infinity, if $\lim_{i,j\to\infty}(x_i|x_j)_w = \infty$ (choice of w does not matter).
- Two sequences $\{x_i\}, \{y_i\}$ that converge at infinity are said to be equivalent, if $\lim_{i\to\infty}(x_i|y_i)_w = \infty$. This is an equivalence relation if (Z, D) is hyperbolic.
- The boundary of the hyperbolic space $\partial(Z, D) = \partial Z$ is the equivalence classes of sequences that converge at infinity.

Visual metric on the boundary

- The Gromov product on ∂Z with base point $w \in Z$ is

$$(a|b)_w = \sup \left\{ \liminf_{i \to \infty} (x_i|y_i)_w : \{x_i\} \in a, \{y_i\} \in b \right\}, \quad a, b \in \partial Z.$$

- A metric ρ on ∂Z is said to be a visual metric with visual parameter $\alpha \in (1, \infty)$ and base point w, if $\rho(a, b) \asymp \alpha^{-(a|b)_w}$.
- Visual metrics exist: for any δ-hyperbolic space (Z, d), there exists $\alpha_0 > 1$ (α_0 depends only on δ) such that if $\alpha \in (1, \alpha_0)$, then there exists a visual metric with parameter α.



Quasi-isometric stability of hyperbolicity

- A map $f : (X_1, d_1) \to (X_2, d_2)$ between two metric spaces is a quasi-isometry if there exist constants $A.B > 0$ such that

$$A^{-1} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B,$$

for all $x, y \in X_1$, and $\sup_{x_2 \in X_2} d(x_2, f(X_1)) \leq B$

- If (X_1, d_1) and (X_2, d_2) are almost geodesic spaces and $f : (X_1, d_1) \to (X_2, d_2)$ is a quasi-isometry, then (X_1, d_1) is hyperbolic if and only if (X_2, d_2) is hyperbolic.

The boundary map

- If (X_1, d_1) and (X_2, d_2) are hyperbolic and almost geodesic, the quasi-isometry $f : (X_1, d_1) \to (X_2, d_2)$ extends to a well-defined map $\partial f : \partial X_1 \to \partial X_2$ on its boundary given by

$$\partial f(\{x_n\}) = \{f(x_n)\}.$$

- A sequence $\{x_n\}$ converges at infinity in (X_1, d_1) if and only if $\{f(x_n)\}$ converges at infinity in (X_2, d_2). Two sequences $\{x_n\}$ and $\{y_n\}$ that converge at infinity in (X_1, d_1) are equivalent if and only if $\{f(x_n)\}$ and $\{f(y_n)\}$ are equivalent in (X_2, d_2).
- (Bonk-Schramm '00) The boundary map is a bijection. If ρ_1, ρ_2 are visual metrics on $\partial X_1, \partial X_2$, then $\partial f : (\partial X_1, \rho_1) \to (\partial X_2, \rho_2)$ is a power quasisymmetry (quasisymmetry whose distortion function can be taken as $\eta(t) = C(t^\gamma \vee t^{1/\gamma})$for some $C \geq 1, \gamma > 0$).

Hyperbolic filling

- (Björn, Björn, Shanmugalingam '22) A compact metric space can be identified with the boundary of a hyperbolic graph (called hyperbolic filling) with visual metric.
- Similar earlier construction by Bourdon-Pajot '03 has hyperbolicity constant depend on the constant of uniform perfectness.
- The idea behind Carrasco's proof goes back to earlier work of Keith-Laakso '04.
- A bilipschitz change of the graph metric of hyperbolic filling is done using optimizers for modulus at various scales and locations.

Power quasisymmetry vs quasisymmetry

- Let $\mathcal{J}_p(X, d)$ denote the power quasisymmetric conformal gauge of (X, d).
- Possible issue with hyperbolic filling: The Bonk-Schramm theorem only produces metric in $\mathcal{J}_p(X, d)$ but not all quasi-symmetries are power quasisymmetries.
- (Tukia-Väisälä '84) If (X, d) is uniformly perfect, then $\mathcal{J}_p(X, d) = \mathcal{J}(X, d)$.
- In general, it is possible that $\mathcal{J}_p(X, d) \subsetneq \mathcal{J}(X, d)$ (Trostsenko-Vaisälä '99).
- (M. 22+) $d_{\mathrm{CA}}(X, d) = \inf\{d_{\mathrm{A}}(X, \theta) : \theta \in \mathcal{J}_p(X, d)\}$ for any compact doubling space.

Proof sketch

- To obtain $Q(X,d) \leq d_{\mathrm{CA}}(X,d)$, we construct the metric $\theta \in \mathcal{J}(X,d)$ by a bi-Lipshitz change of metric on the hyperbolic filling. The upper bound on $d_A(X,\theta)$ is obtained by constructing a doubling measure and using Vol'berg-Konyagin theorem (this requires a modification of the Vol'berg-Konyagin construction of doubling measures).
- To obtain $Q(X,d) \leq d_{\mathrm{CA}}(X,d)$, for $p > d_{\mathrm{CA}}(X,d)$, pick $\theta \in \mathcal{J}(X,d)$ and μ doubling measure that is q-homogeneous in (X,θ) for some $d_{\mathrm{CA}}(X,d) \leq q < p$ (using Vol'berg-Konyagin). A modification of the function

$$\rho(w) = \left(\frac{\mu(B_w)}{\mu(B_v)}\right)^{1/q}$$

where $v \in G_n$ is a 'parent' of $w \in G_{n+k}$ is admissible for combinatorial modulus and has small p-norm.

Question: monotonicty of exponents

- By a general sub-multiplicativity property of combinatorial modulus (Bourdon-Kleiner '13, Carrasco '13), $\beta_p = \lim_{k\to\infty} \frac{1}{k} \log M_{k,p}$ exists.
- It is easy to see that $p \mapsto \beta_p$ is non-increasing.
- Question: (Bonk) Is $p \mapsto \beta_p$ strictly decreasing in p?
- An affirmative answer would allow us characterize conformal dimension as the unique exponent p for which $\beta_p = 0$. We expect this to be true in most examples of interest: boundaries of hyperbolic groups, Sierpinski gasket/carpets, Julia sets.

Concluding remarks

- The conformal Assouad dimension is a better way to define Ahlfors regular conformal dimension.
- The construction of doubling measure by Vol'berg and Konyagin and the construction of metric by Carrasco is flexible enough to be adapated for different purposes.
- A modified version of Vol'berg-Konyagin construction played an important role in the proof of the stability of elliptic Harnack inequality (Barlow, M., 2018).
- Similarly, a modification of Carrasco's construction helped us to understand a new relationship between elliptic and parabolic Harnack inequalities (Kajino, M., 2022).

Thank you for your attention

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