Reconfiguration of Regular Induced Subgraphs

◎ **Hiroshi Eto (Tohoku University)**

Takehiro Ito (Tohoku University)

Yasuaki Kobayashi (Hokkaido University)

Yota Otachi (Nagoya University)

Kunihiro Wasa (Hosei Uinversity)

Combinatorial Reconfiguration

Outline

- Reconfiguration Problems
- Regular Induced Subgraphs
- Our Problem
- Related work and Our Results

Outline

- Reconfiguration Problems
- Regular Induced Subgraphs
- Our Problem
- Related work and Our Results

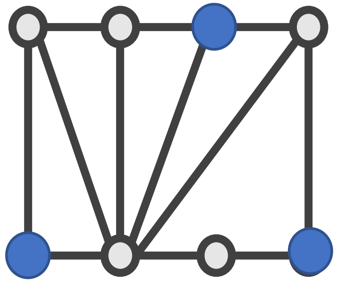
Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

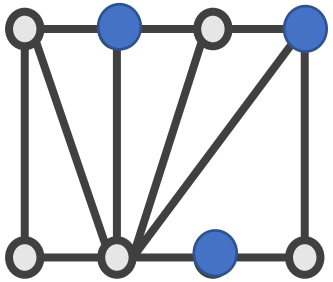
【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex



$\boldsymbol{S}$



$\boldsymbol{T}$

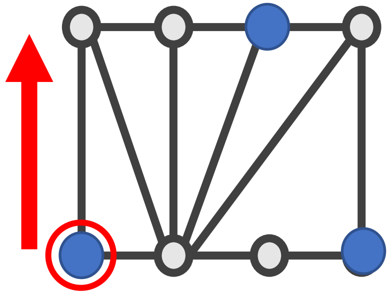
Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

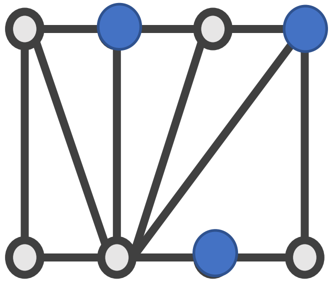
【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex



$\boldsymbol{S}$



$\boldsymbol{T}$

Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex

$\boldsymbol{S}$ $\boldsymbol{T}$

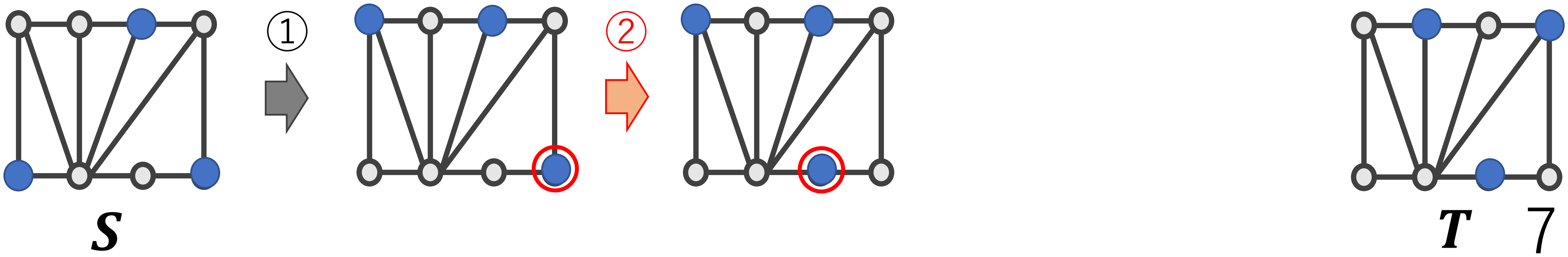
Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex



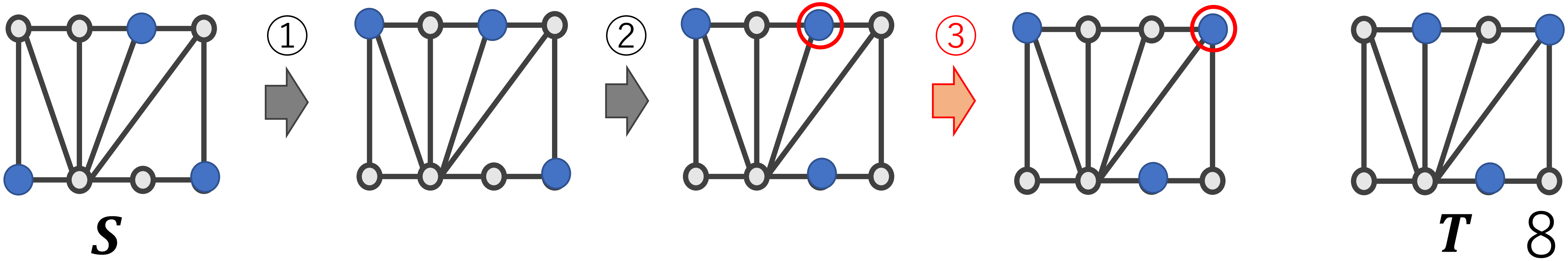
Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex



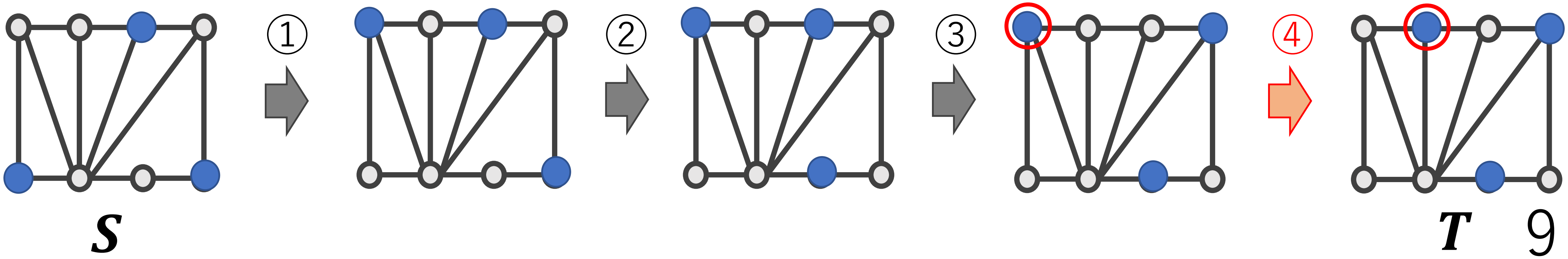
Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex



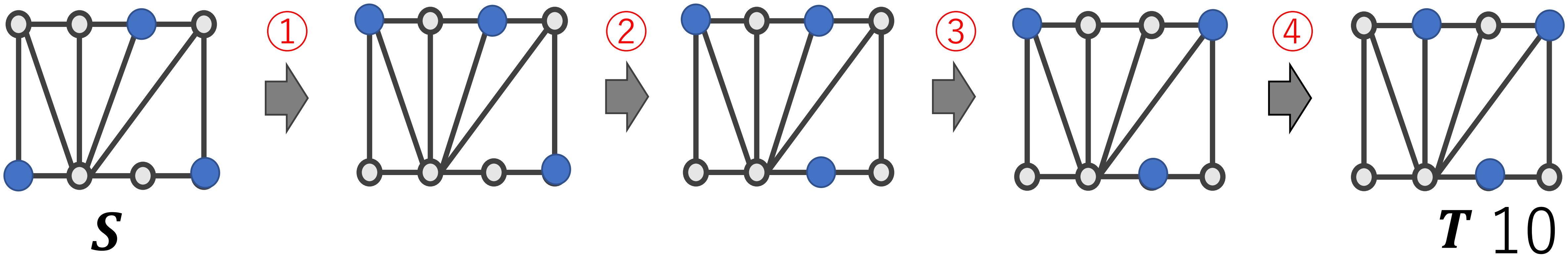
Reconfiguration Problems

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

【Independent Set Reconfiguration under Token Sliding rule】

Input:	A graph G and vertex sets S and T of G.
Question:	Is there a TS-sequence between S and T?

Token Sliding rule: A token can be moved to only an adjacent vertex



Outline

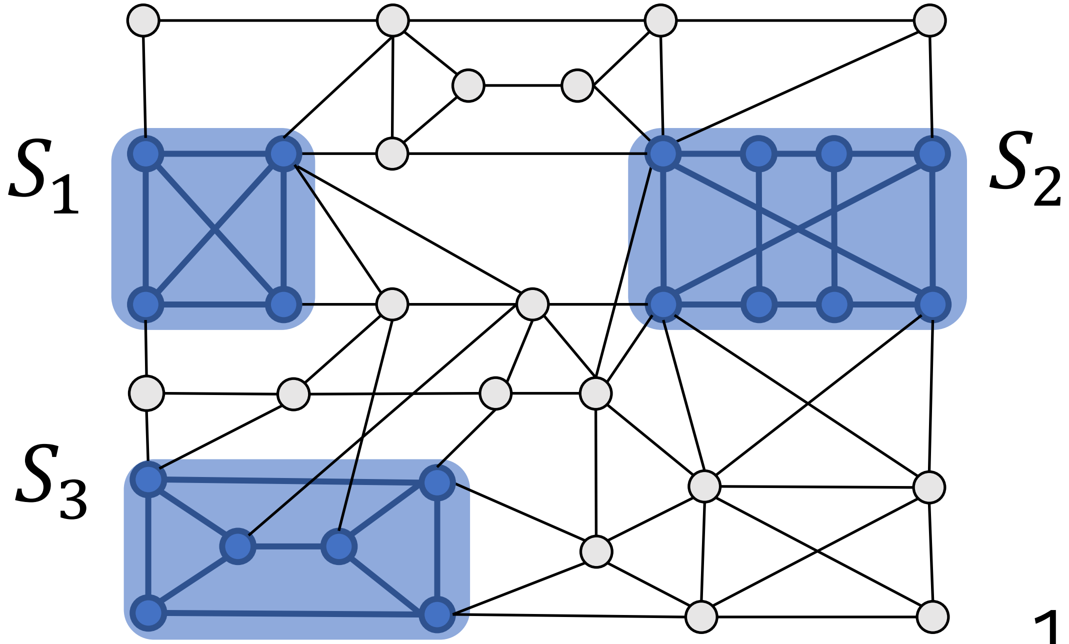
- Reconfiguration Problems
- Regular Induced Subgraphs
- Our Problem
- Related work and Our Results

Regular Induced Subgraphs

- We denote by $G[U]$ the subgraph of G induced by U.
- We say that a vertex subset U of a graph G is a d-regular set of G if $G[U]$ is $d\text{-}regular$.

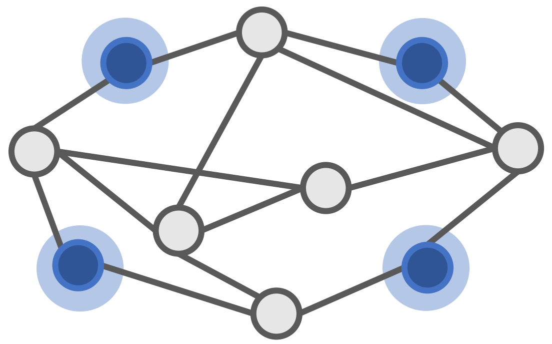
Vertex subsets $S_1, S_2, S_3 \subseteq U$

3-regular Induced subgraph $G[U]$

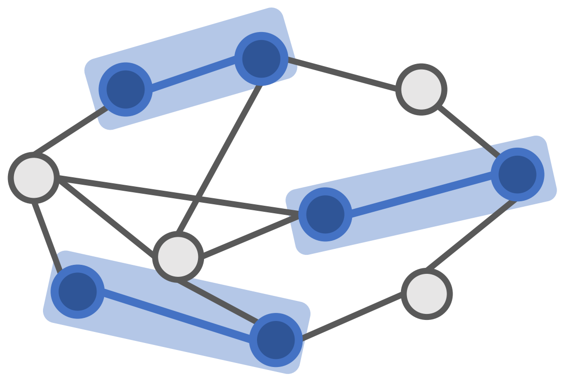


Regular induced subgraph

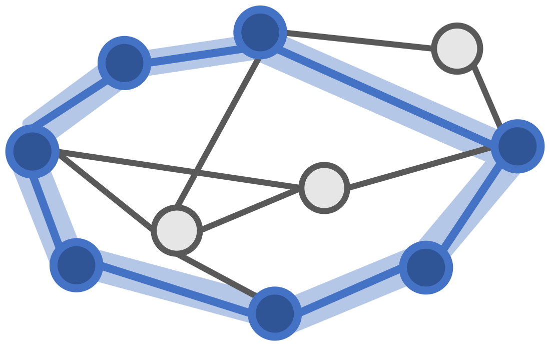
d =0
Independent set



d =1
Induced matching



d =2
Induced cycle



Outline

- Introduction
- Regular Induced Subgraphs
- Our Problem
- Related work and Our Results

Regular Induced Subgraph Reconfiguration

$d\text{-}Regular\ Induced\ Subgraph\ Reconfiguration\ under\ R\ (RISR_d)$	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of **G**.
Question:	Is there an R-sequence between $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$?

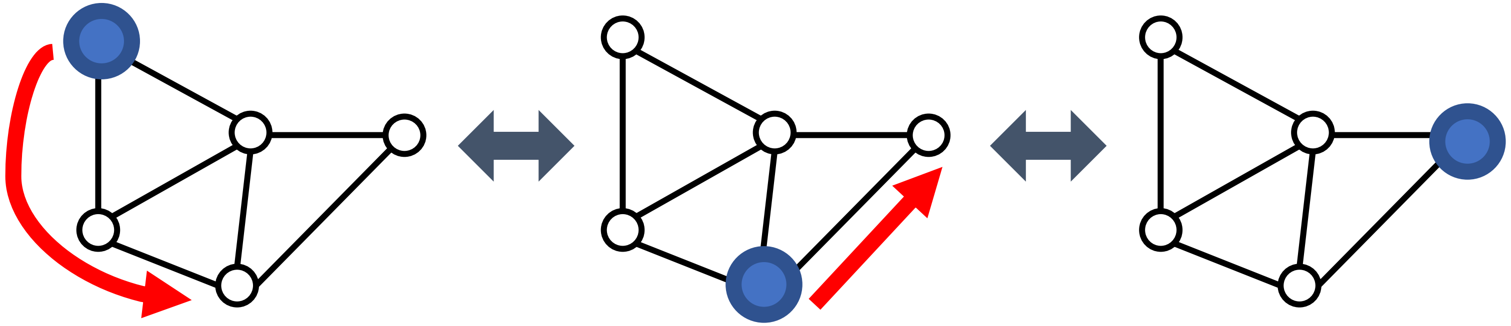
Reconfiguration rule (R ∈ TJ,TS)

- TJ:Token Jumping
- TS:Token Sliding

Reconfiguration Rule

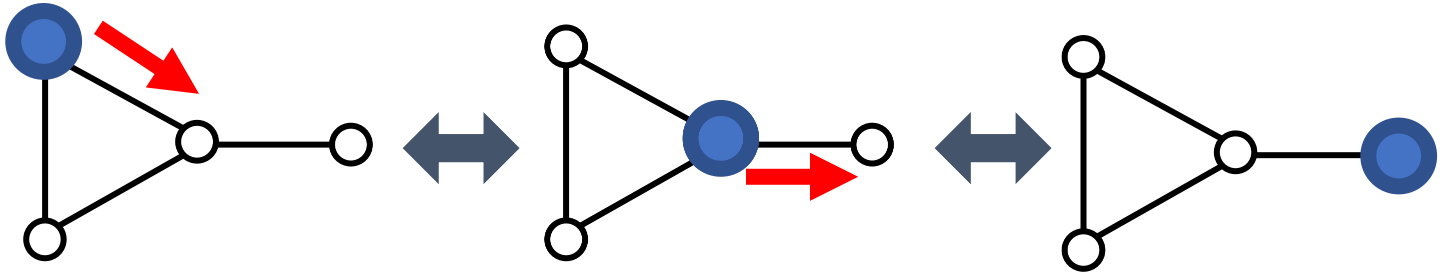
TJ(Token Jumping)

- $U_i \leftrightarrow U_{i+1}$ under **TJ** if $|U_i \setminus U_{i+1}| = |U_{i+1} \setminus U_i| = 1$



TS(Token Sliding)

- $U_i \leftrightarrow U_{i+1}$ under **TS** if $U_i \setminus U_{i+1} = \{v\},\ U_{i+1} \setminus U_i = \{w\}$, and $vw \in E(G)$



Regular Induced Subgraph Reconfiguration

d-$Regular\ Induced\ Subgraph\ Reconfiguration\ under\ R\ (RISR_d)$	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Reconfiguration rule (R ∊ TJ,TS)

- TJ:Token Jumping
- TS:Token Sliding

Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$

Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$

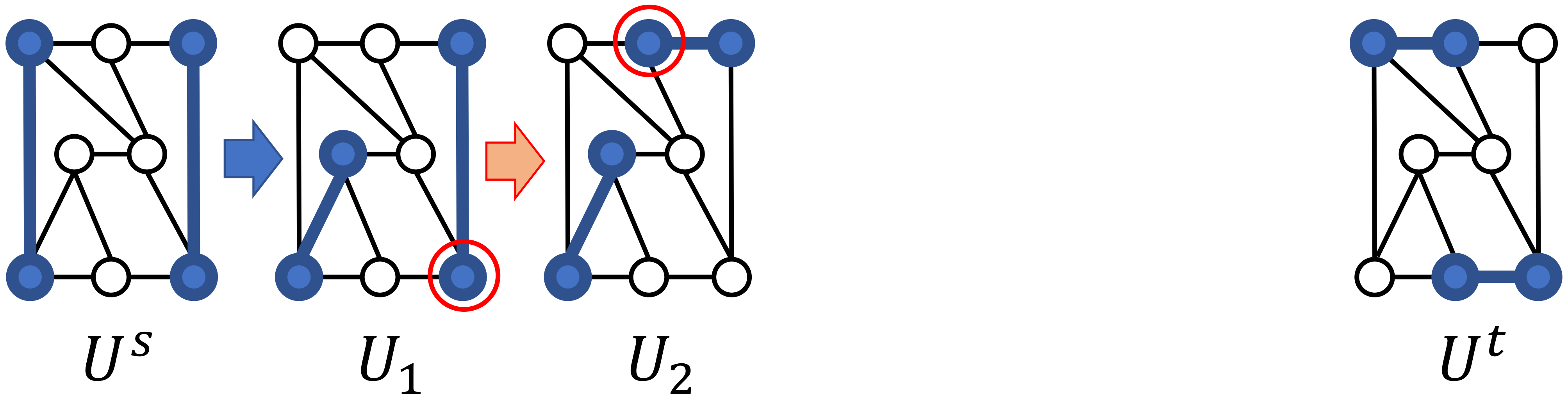
Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$



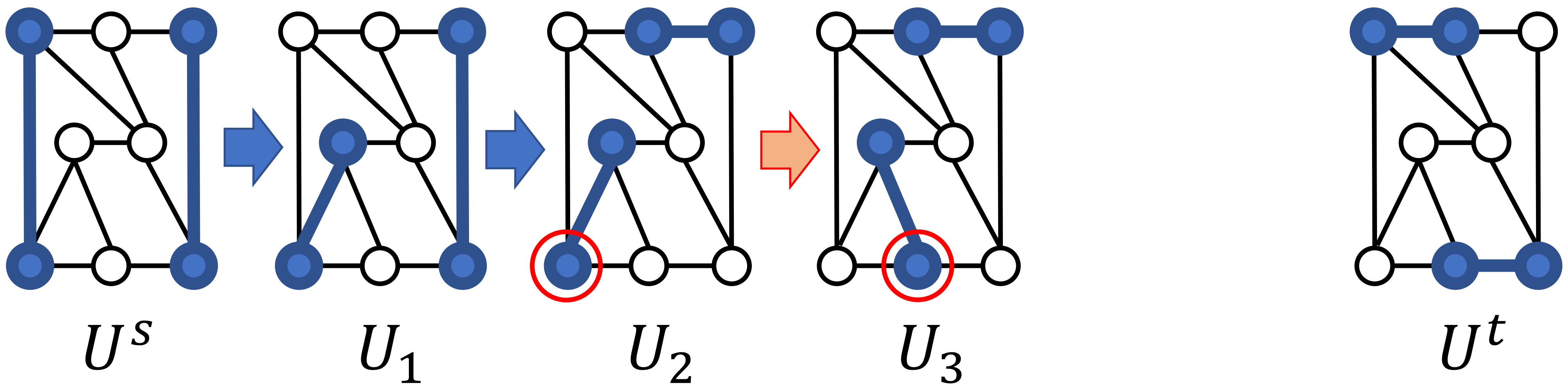
Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$



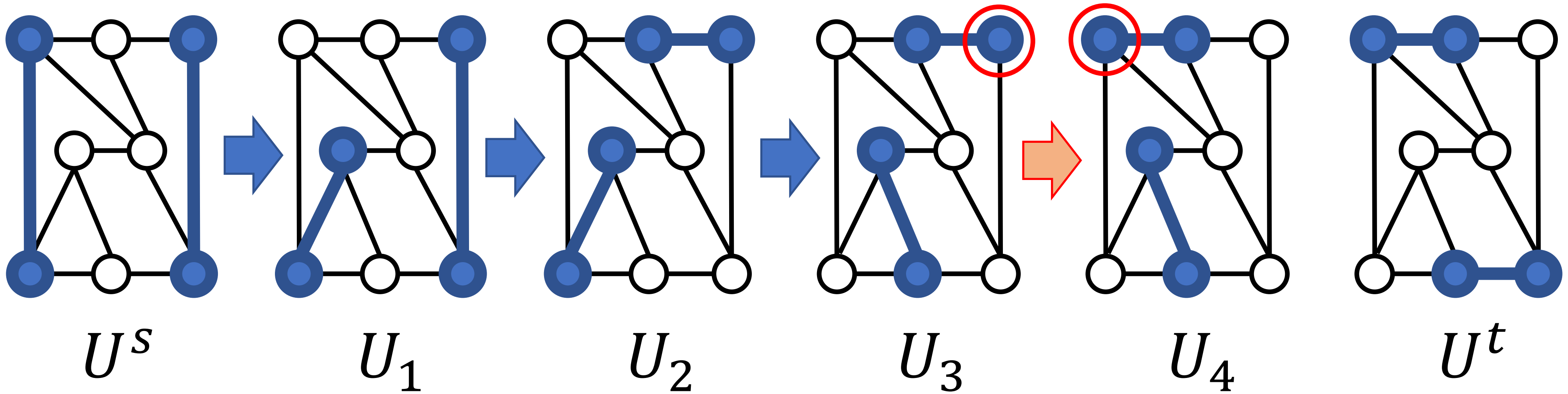
Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$



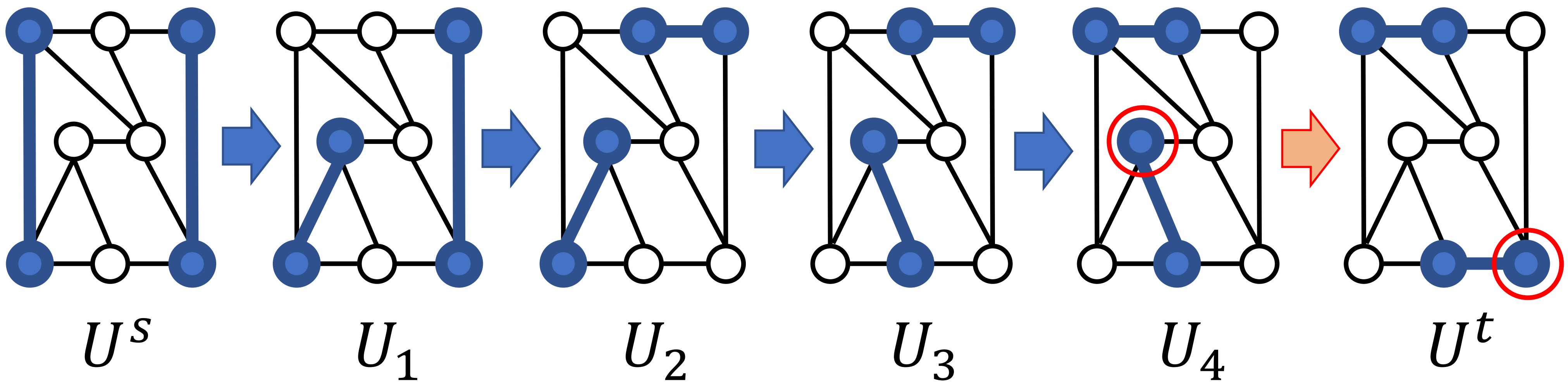
Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$



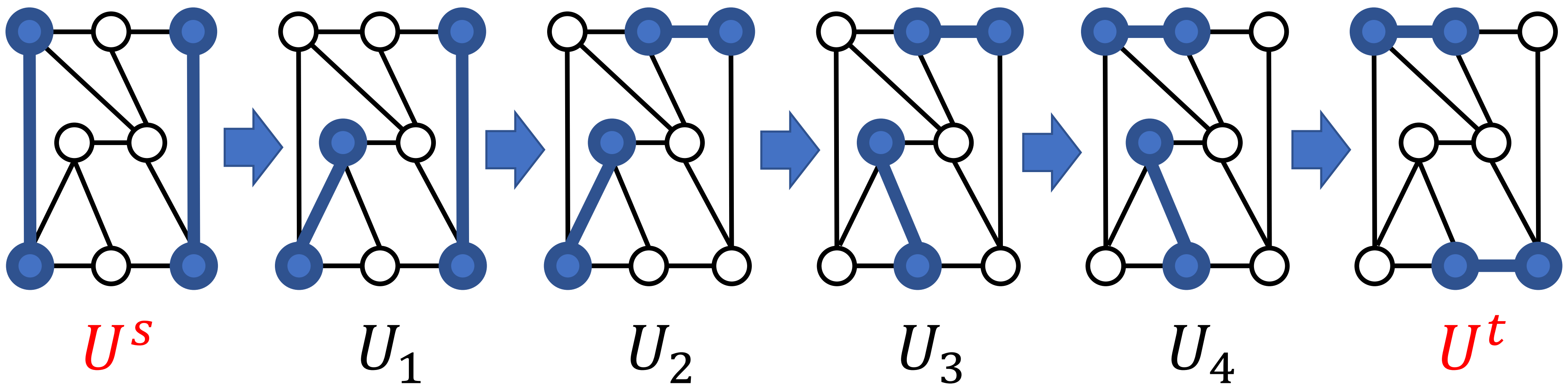
Regular Induced Subgraph Reconfiguration

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TJ (Token Jumping)

d=1, $\boldsymbol{RISR_1}$



$\boldsymbol{RISR_d}$

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Reconfiguration rule (R ∊ TJ,TS)

- TJ:Token Jumping
- TS:Token Sliding

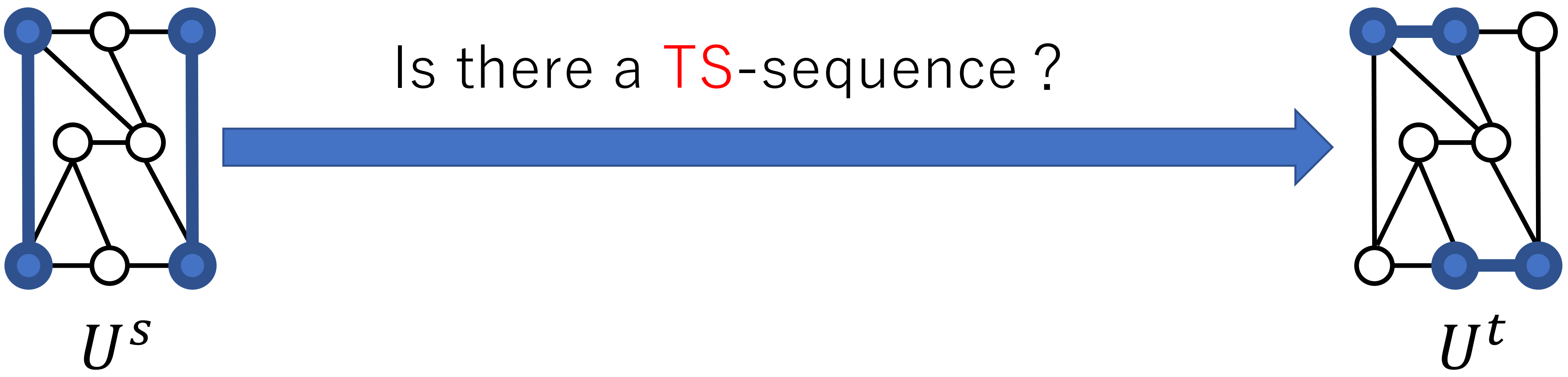
$\boldsymbol{RISR_d}$

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TS (token Sliding)

d=1, $\boldsymbol{RISR_1}$



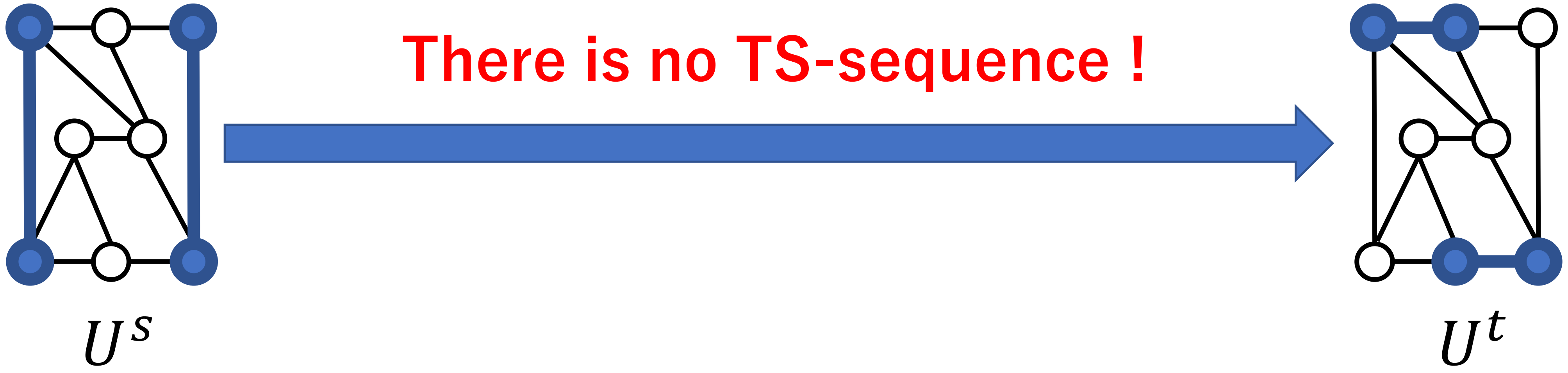
$\boldsymbol{RISR_d}$

$d\text{-}Regular\ Induced\ Subgraph\ Reconfiguration\ under\ R\ (RISR_d)$	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Example

Reconfiguration rule TS (token Sliding)

d=1, $\boldsymbol{RISR_1}$



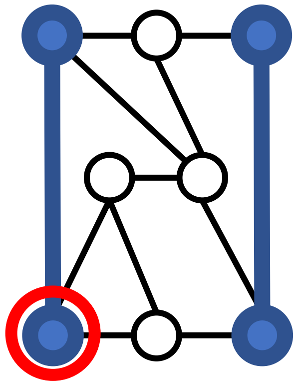
$\boldsymbol{RISR_d}$

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	I… ween U^s and U^t?

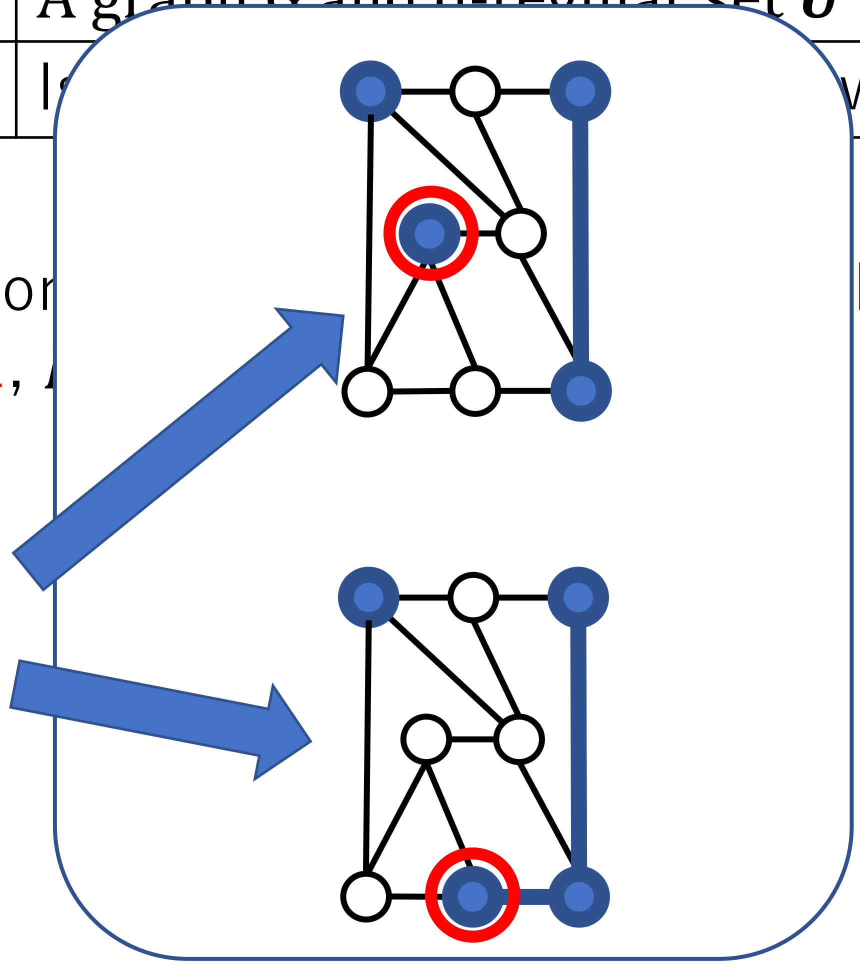
Example

Recon… liding)

d=1, …



U^s

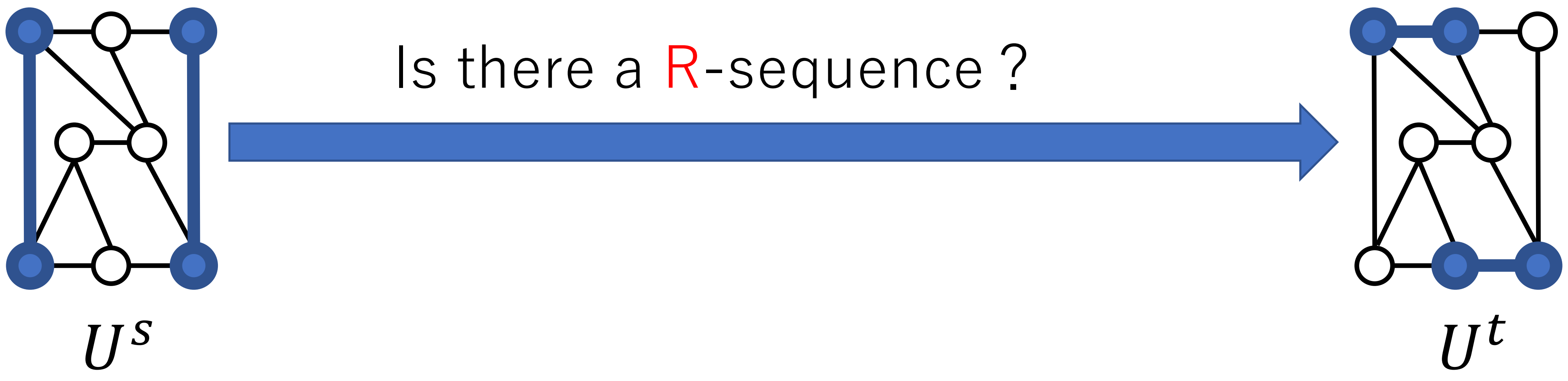


$\boldsymbol{RISR_d}$

d-Regular Induced Subgraph Reconfiguration under R ($RISR_d$)	
Input:	A graph G and d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Reconfiguration rule (R ∊ TJ,TS)

- TJ:Token Jumping
- TS:Token Sliding

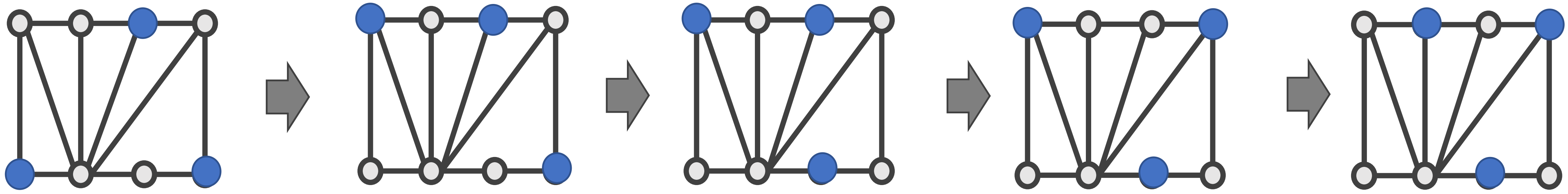


Outline

- Introduction
- Regular Induced Subgraphs
- Our Problem
- Related work and Our Results

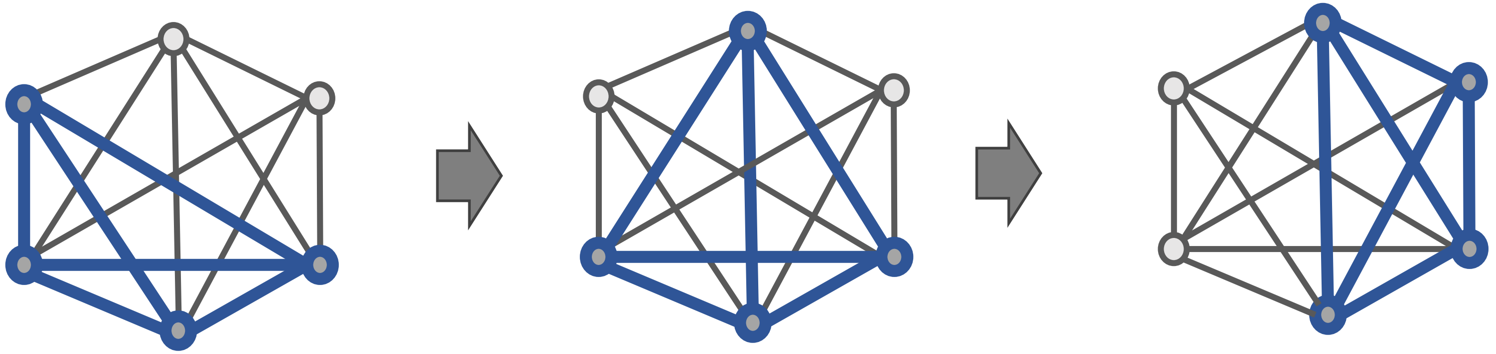
Related work

Independent set reconfiguration $= \boldsymbol{RISR_0}$

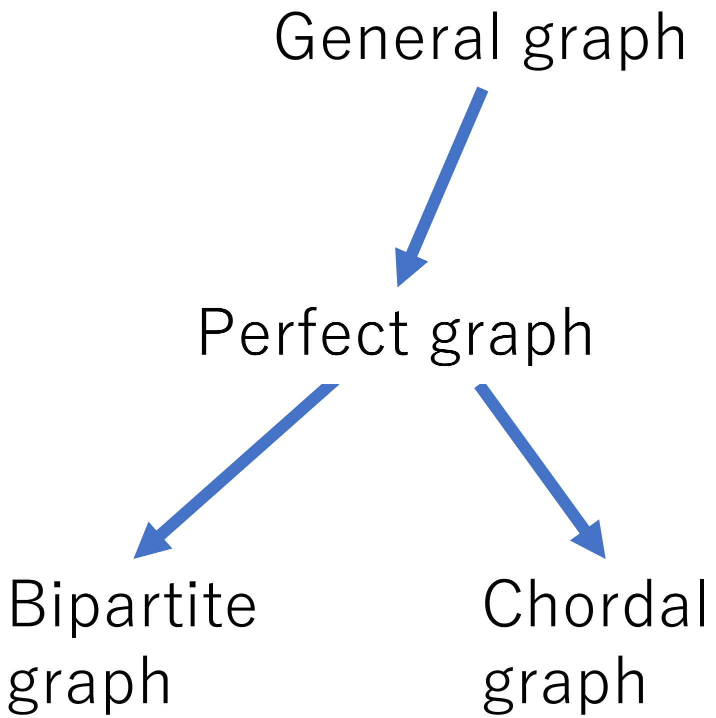


k-clique reconfiguration is a k-1 regular induced subgraph reconfiguration

- $k = 4$
- $\boldsymbol{CRISR_3}$

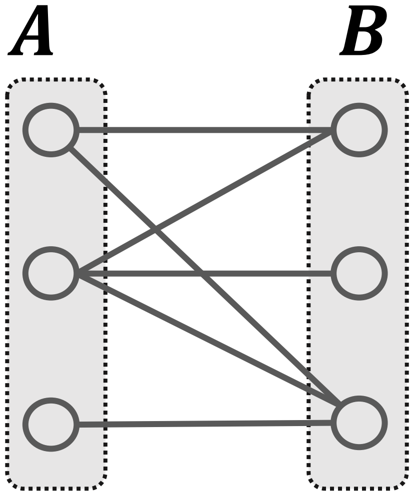


Graph class



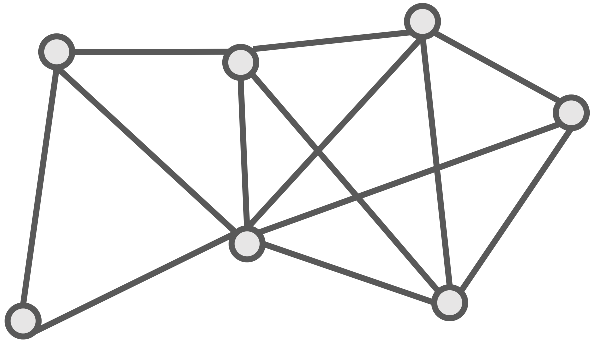
Bipartite graph

A graph is bipartite if its vertex set can be partitioned into 2 independent sets.

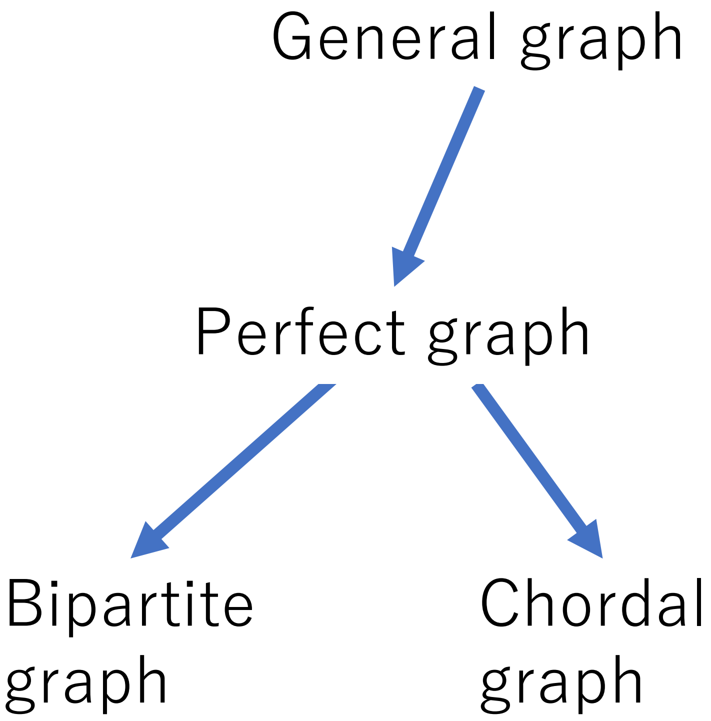


Chordal graph

A graph is chordal if every cycle of length at least 4 has a chord.



Related work and Our Results



	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: **PSPACE-c**	[Kamiński et al., 2012] $d = 0$:**P**
Bipartite graphs	[Lokshtanov et al., 2019] $d = 0$: **PSPACE-c**	[Lokshtanov et al., 2019] $d = 0$: **NP-c**

Related work and Our Results

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**
Bipartite graphs	[Lokshtanov et al., 2019] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:P**	[Lokshtanov et al., 2019] $d = 0$: NP-c [Our Results] **$d \geq 1$:PSPACE-c**

	$RISR_d$	
	TS	TJ
Bipartite grapgh	[Lokshtanov et al., 2019] $d = 0$: PSPACE-c [Our Results] $\boldsymbol{d \geq 1}$**:P**	[Lokshtanov et al., 2019] $d = 0$: NP-c [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**

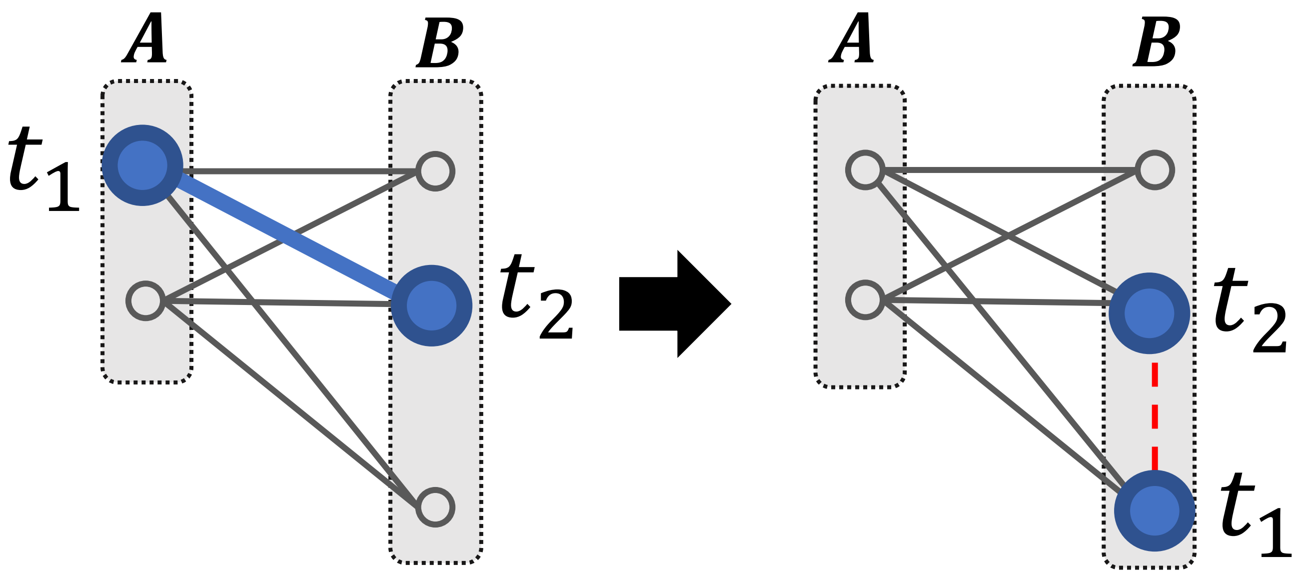
	$RISR_d$	
	TS	TJ
Bipartite grapgh	[Lokshtanov et al., 2019] $d = 0$: PSPACE-c [Our Results] $\boldsymbol{d \geq 1}$:**P**	[Lokshtanov et al., 2019] $d = 0$: NP-c [Our Results] $\boldsymbol{d \geq 1}$:**PSPACE-c**

Example

Reconfiguration rule TJ (Token Sliding)

d=1, $\boldsymbol{RISR_1}$

$\boldsymbol{Token\ t_1, t_2}$

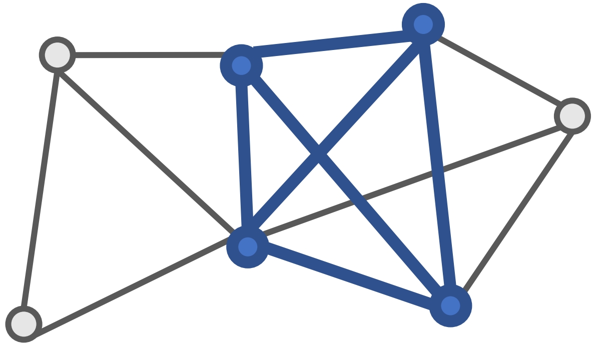


Related work and Our Results

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**
Bipartite graphs	[Lokshtanov et al., 2019] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:P**	[Lokshtanov et al., 2019] $d = 0$: NP-c [Our Results] **$d \geq 1$:PSPACE-c**

Chordal graph

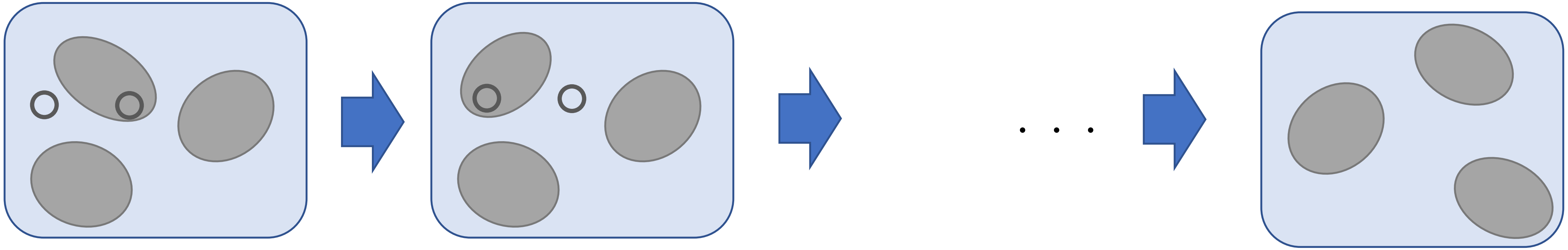
- Every connected regular induced subgraph in a chordal graph is a complete graph [Asahiro et al., 2014].



Chordal graph

- Every connected regular induced subgraph in a chordal graph is a complete graph [Asahiro et al., 2014]

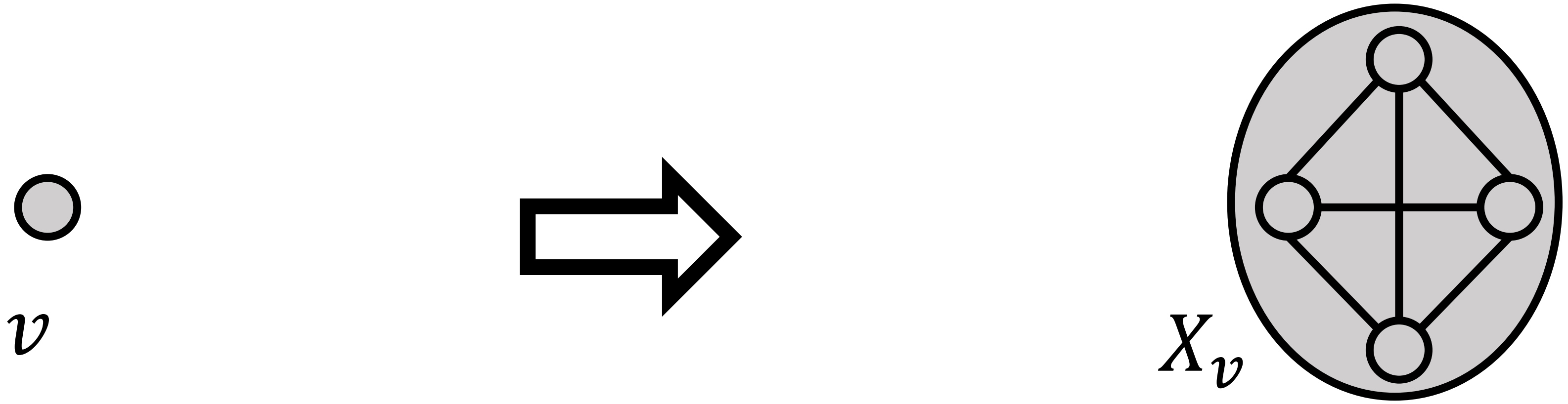
	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**



	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

- We give a reduction from independent set reconfiguration on chordal graph under TS.
- For each $v \in V(H)$, we take a set X_v of $d + 1$ vertices.

Example, d=3

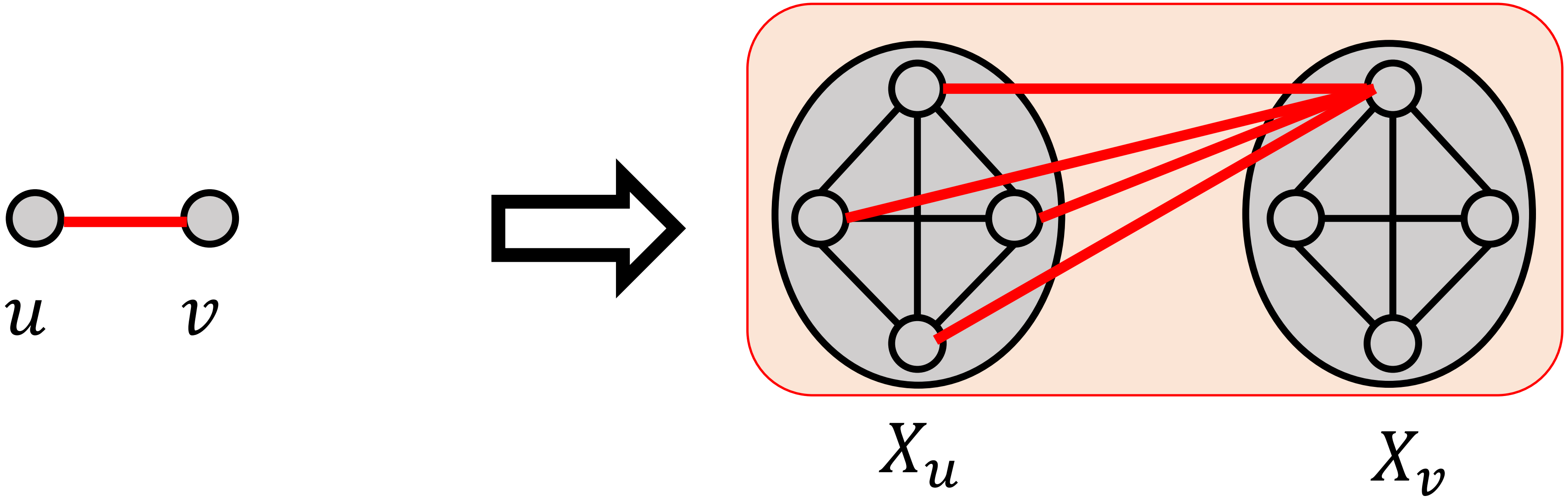


Graph $H = (V(H), E(H))$ New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3



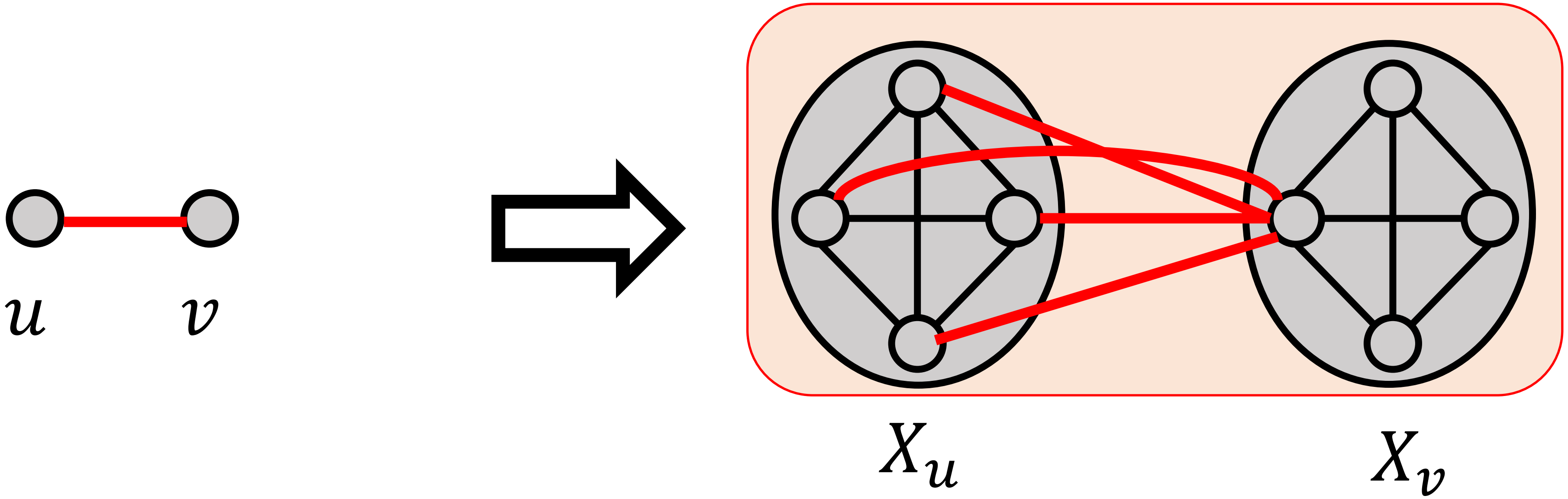
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3



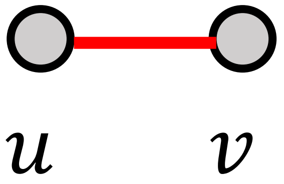
Graph $H = (V(H), E(H))$

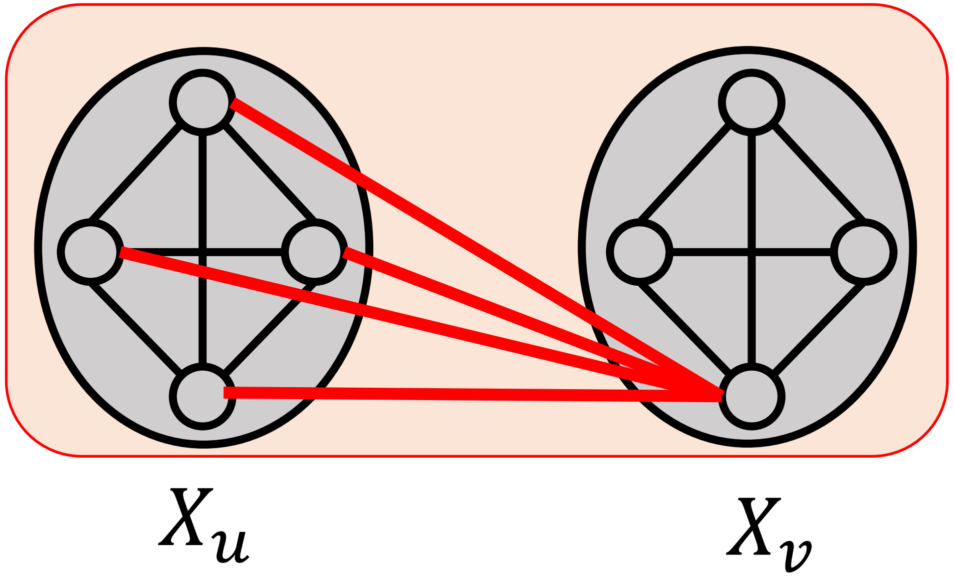
New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3





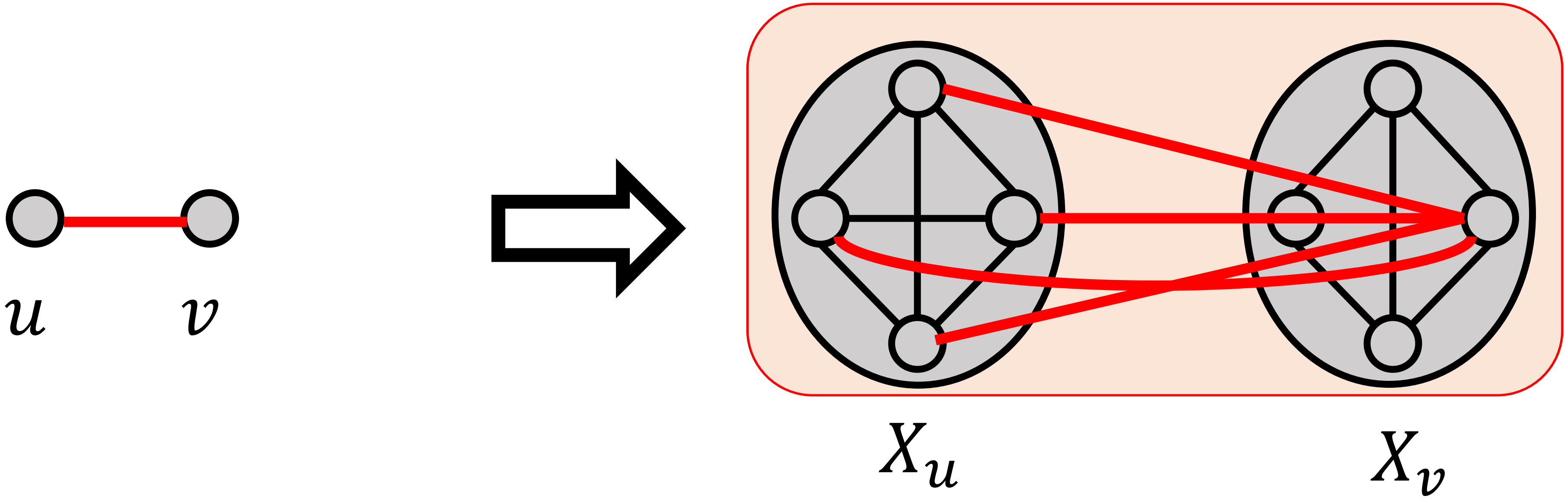
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3



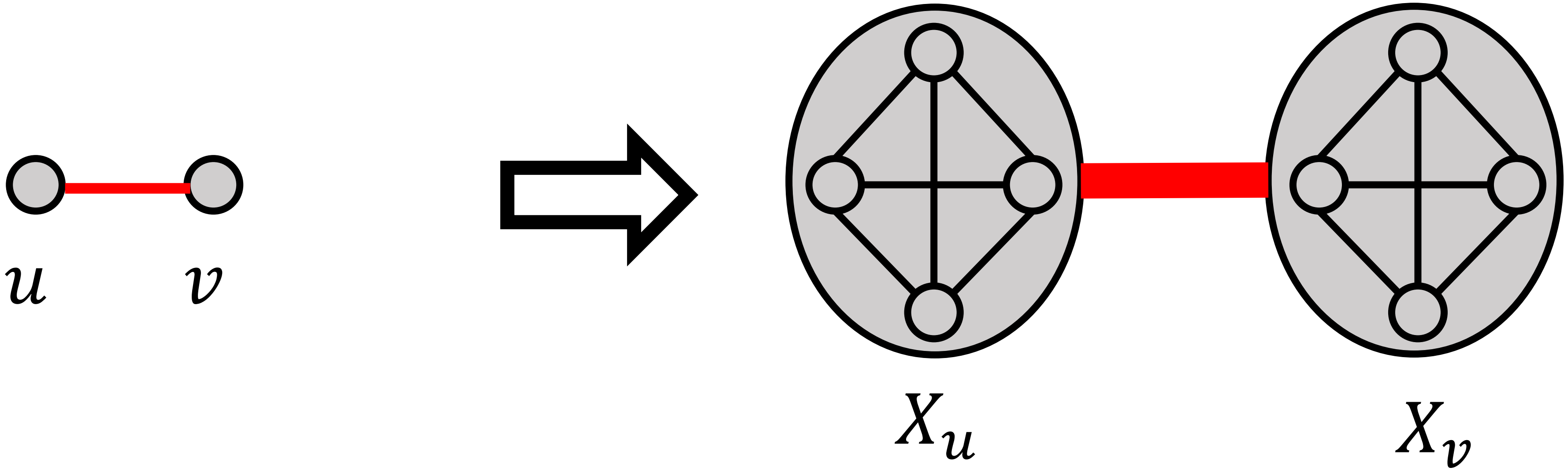
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3

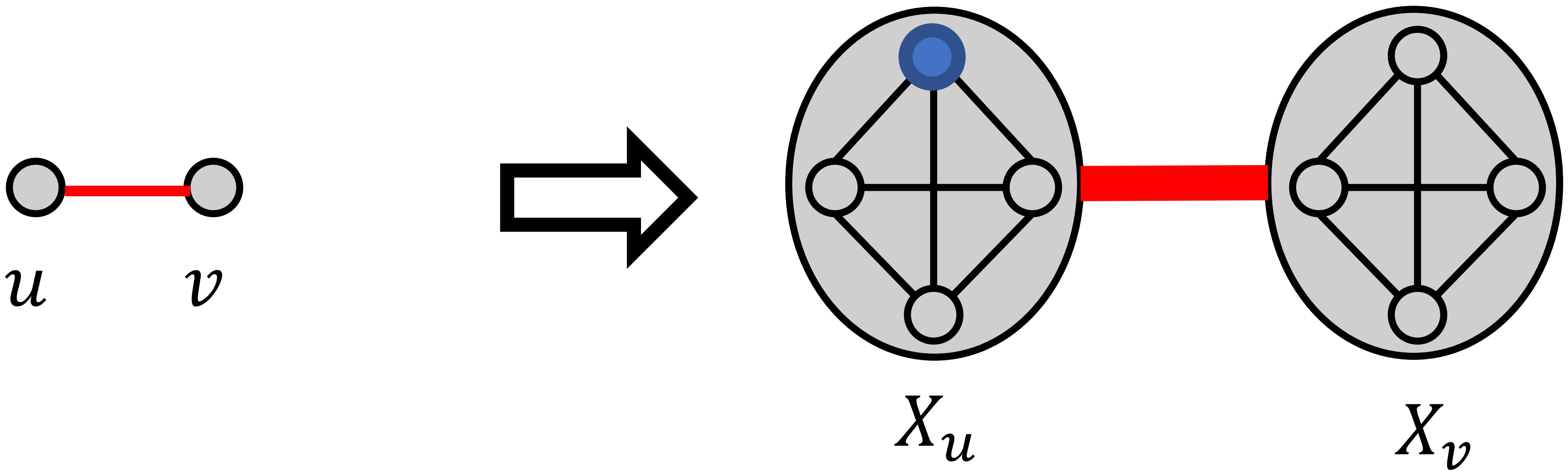


Graph $H = (V(H), E(H))$ New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3



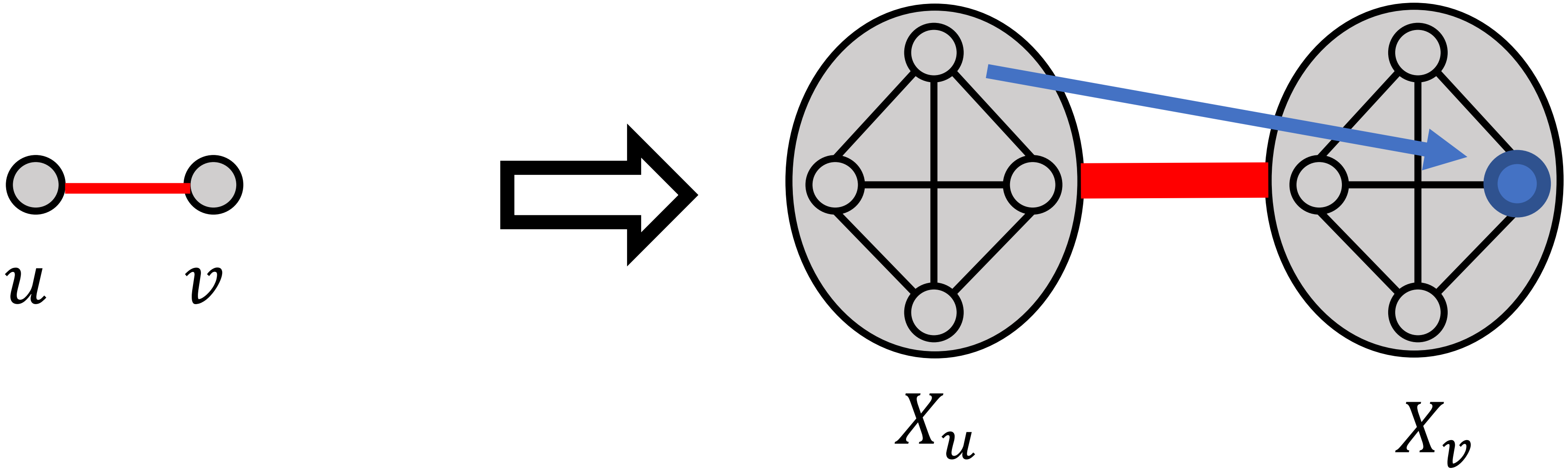
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

- We add all possible edges between X_u and X_v if $\{u, v\} \in E(H)$

Example, d=3



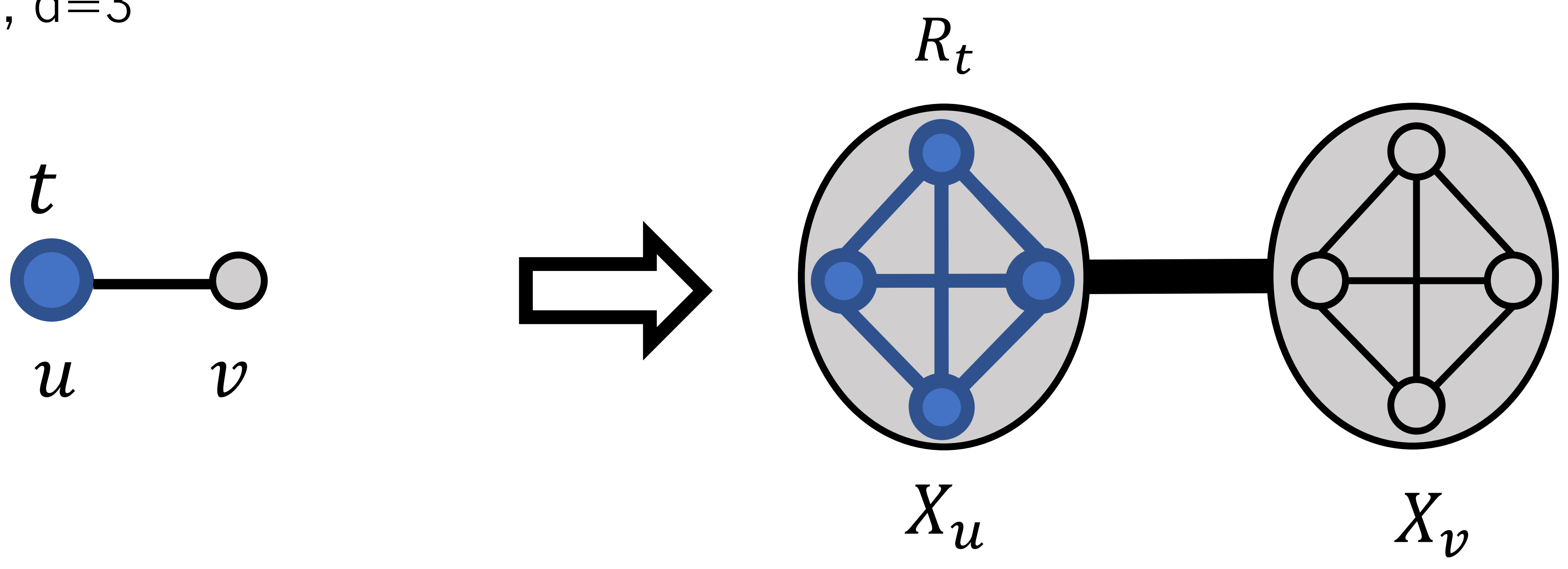
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

If token $\boldsymbol{t}$ move in graph H, then 3-regular set R_t from X_u to X_v

Example, d=3



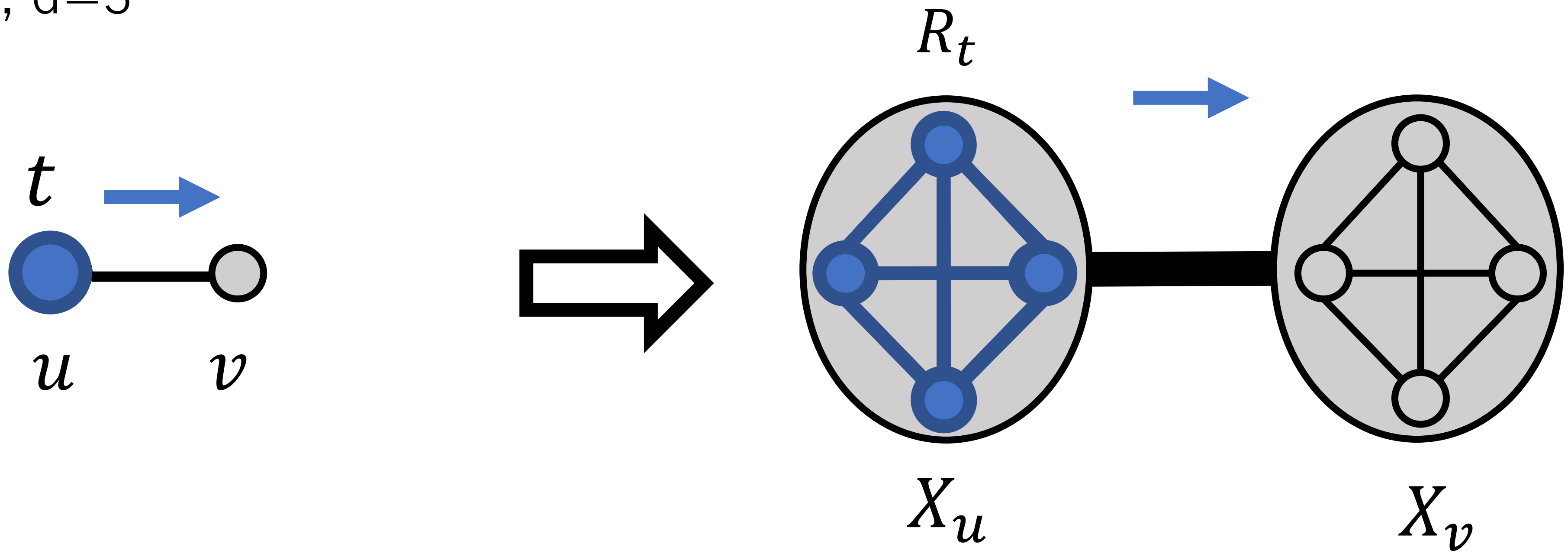
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

If token $\boldsymbol{t}$ move in graph H, then 3-regular set R_t from $X_u$to X_v

Example, d=3

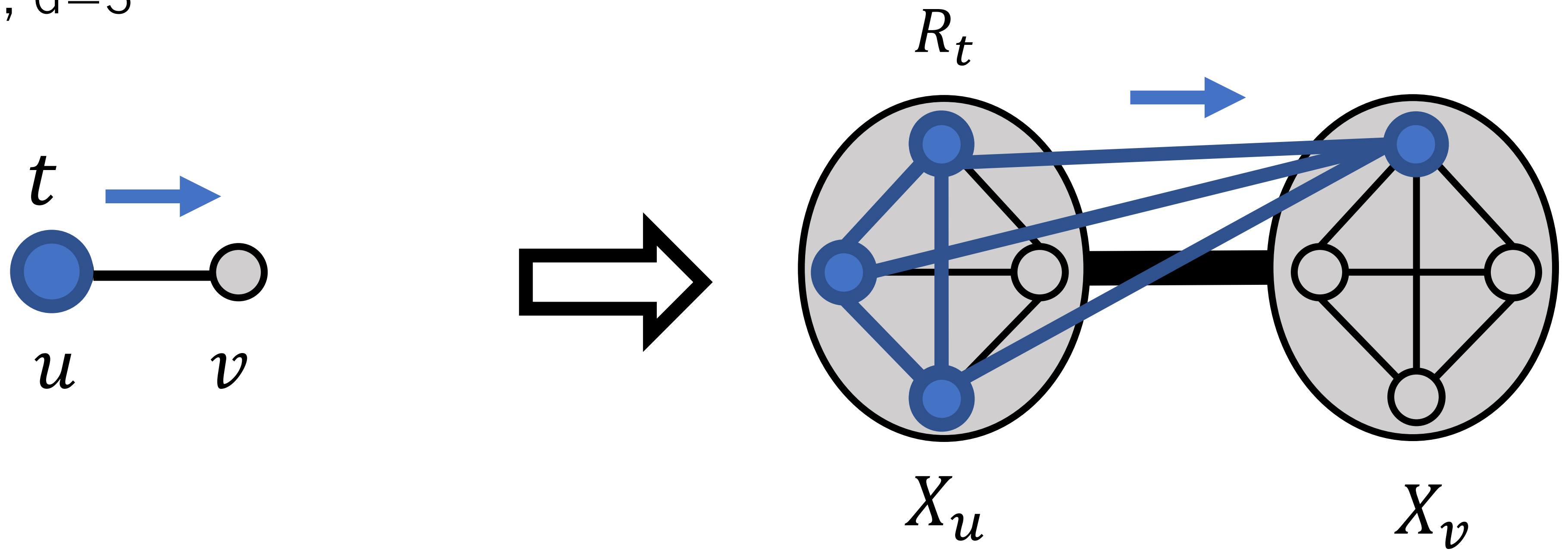


Graph $H\ = (V(H), E(H))$ New graph $G\ = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

If token $\boldsymbol{t}$ move in graph H, then 3-regular set R_t from $X_u$to X_v

Example, d=3

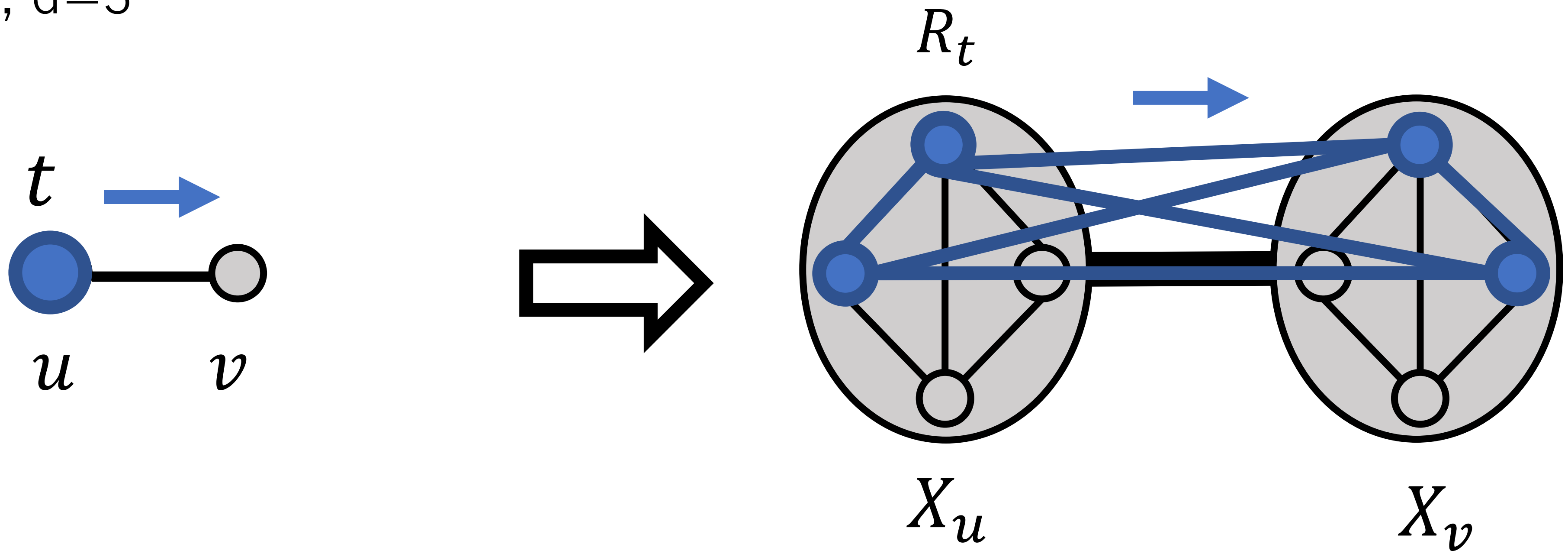


Graph $H = (V(H), E(H))$ New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d=0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d=0$:P [Our Results] **$d \geq 1$:PSPACE-c**

If token $\boldsymbol{t}$ move in graph H, then 3-regular set R_t from $X_u$to X_v

Example, d=3

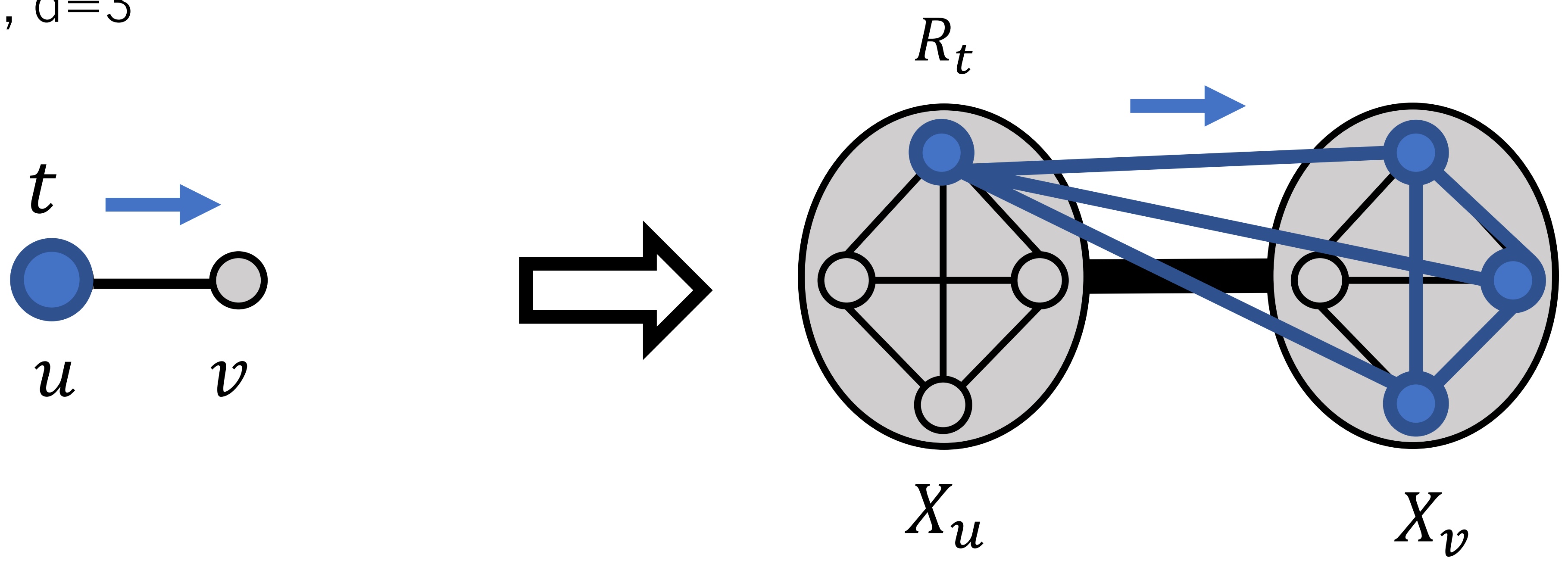


Graph $H = (V(H), E(H))$ New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**

If token $\boldsymbol{t}$ move in graph H, then 3-regular set R_t from $X_u$to X_v

Example, d=3

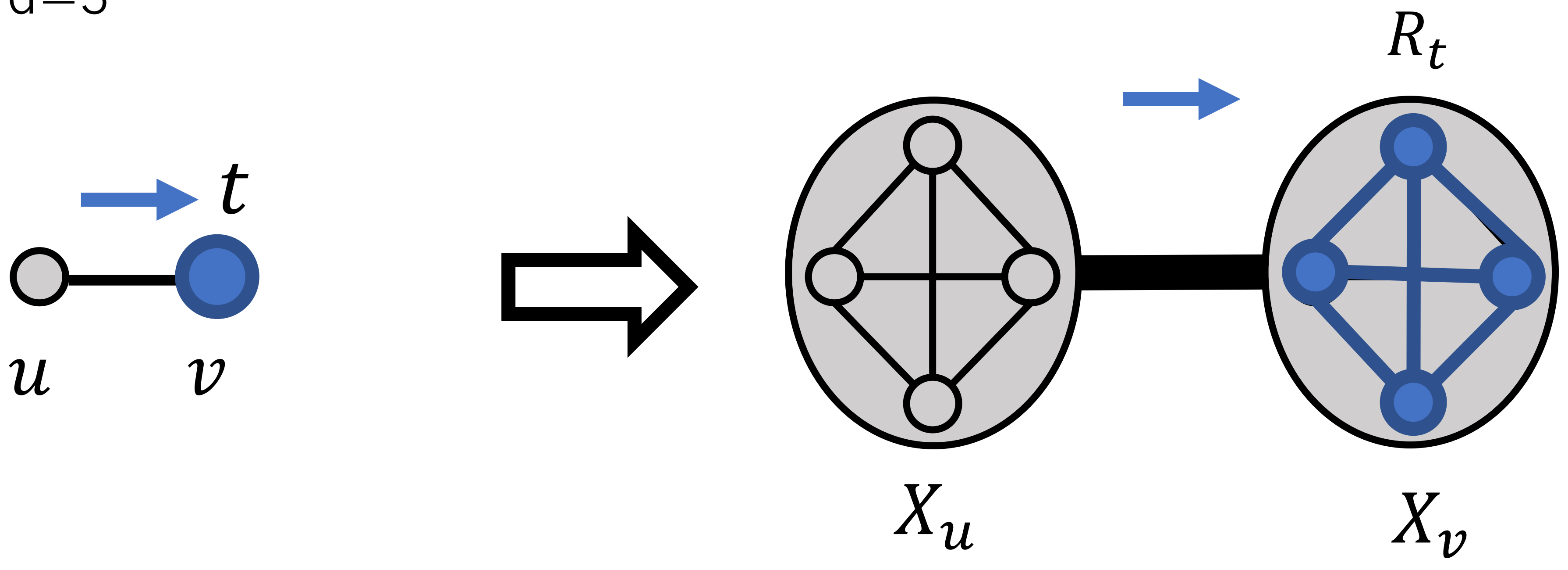


Graph $H = (V(H), E(H))$ New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] $\boldsymbol{d \geq 1}$**:PSPACE-c**

If token $\boldsymbol{t}$ move in graph H, then 3-regular set R_t from X_u to X_v

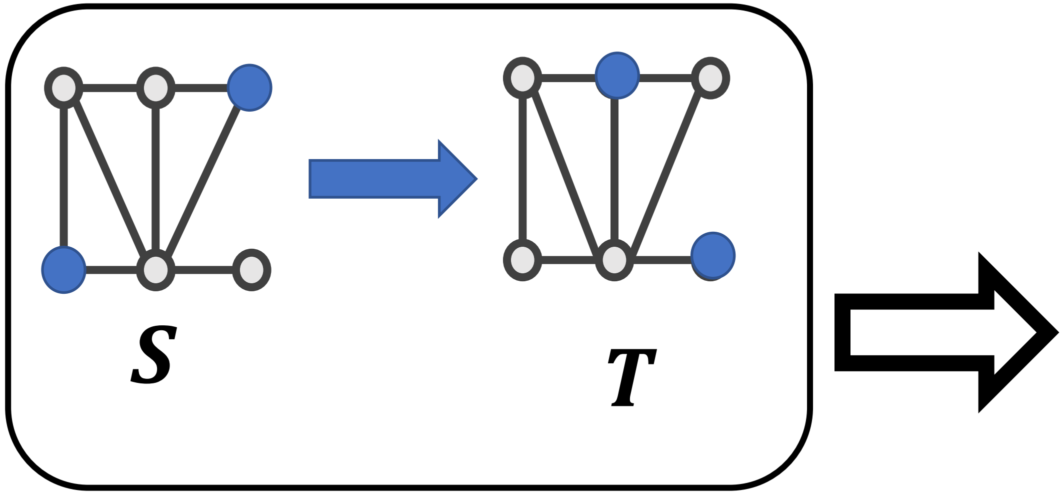
Example, d=3



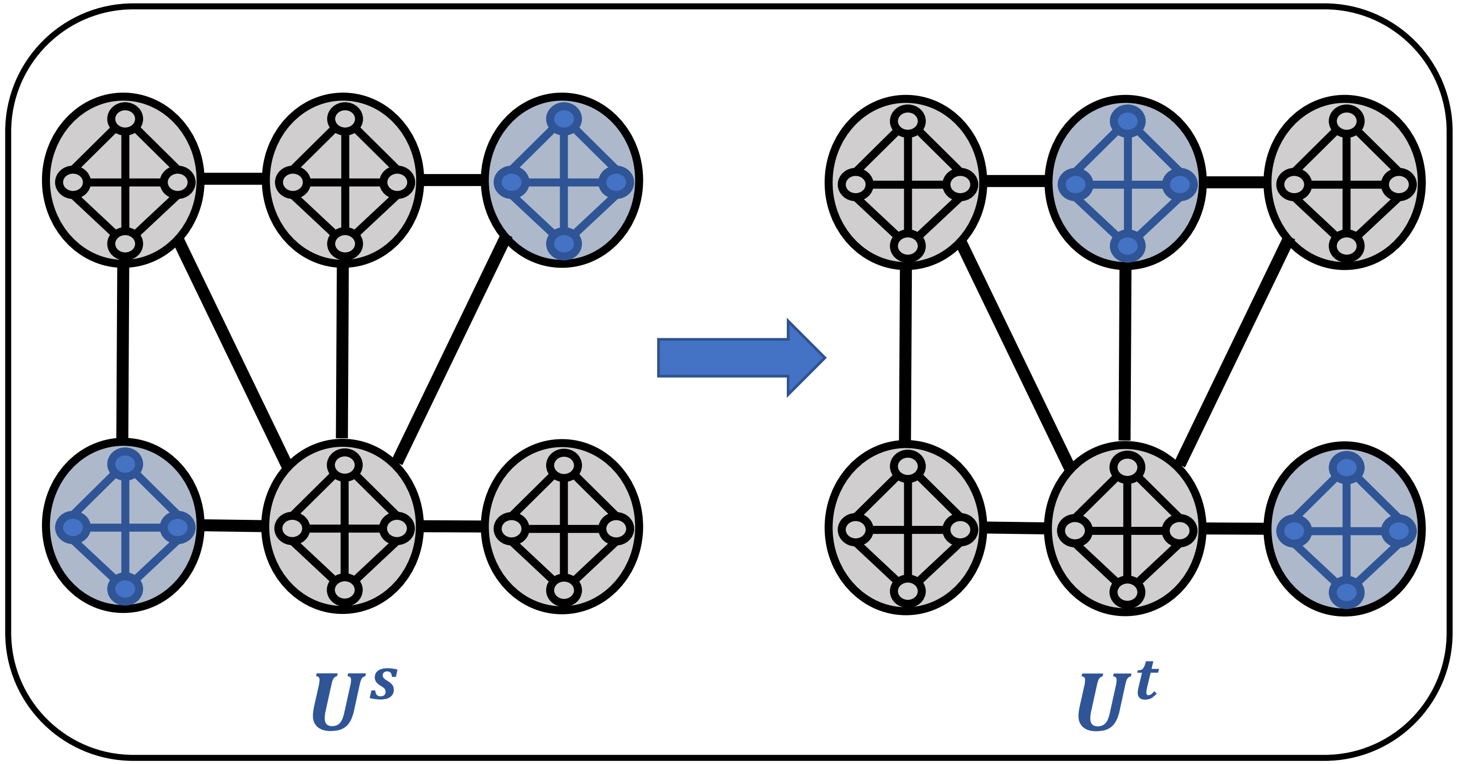
Graph $H = (V(H), E(H))$

New graph $G = (V(G), E(G))$

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] $d \geq 1$:**PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] $d \geq 1$:**PSPACE-c**



Independent set reconfiguration on chordal graph under TS



$RISR_d$ **on chordal graph under TS**

Related work and Our Results

	$RISR_d$	
	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:PSPACE-c**	[Kamiński et al., 2012] $d = 0$:P [Our Results] **$d \geq 1$:PSPACE-c**
Bipartite graphs	[Lokshtanov et al., 2019] $d = 0$: PSPACE-c [Our Results] **$d \geq 1$:P**	[Lokshtanov et al., 2019] $d = 0$: NP-c [Our Results] **$d \geq 1$:PSPACE-c**

Related work and Our Results

	$RISR_d$		$CRISR_d$ $(d \geq 2)$	
	TS	TJ	TS	TJ
Chordal graphs	[Belmonte et al., 2021] $d = 0$: PSPACE-c [Our Results] $d \geq 1$:PSPACE-c	[Kamiński et al., 2012] $d = 0$:P [Our Results] $d \geq 1$:PSPACE-c	[Ito et al., 2015] P	
Bipartite graphs	[Lokshtanov et al., 2019] $d = 0$: PSPACE-c [Our Results] $d \geq 1$:P	[Lokshtanov et al., 2019] $d = 0$: NP-c [Our Results] $d \geq 1$:PSPACE-c	[Our Results] P	[Our Results] PSPACE-c

$\boldsymbol{CRISR_d}$

Connected *d-Regular Induced Subgraph Reconfiguration under R* ($\boldsymbol{CRISR_d}$)	
Input:	A graph G and connected d-regular set $\boldsymbol{U^s}$ and $\boldsymbol{U^t}$ of G
Question:	Is there an R-sequence between U^s and U^t?

Reconfiguration rule (R ∊ TJ,TS)

- TJ:Token Jumping
- TS:Token Sliding