Non-negativity of a local classical solution to the Relativistic Boltzmann Equation without Angular Cut-off

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The relativistic Boltzmann equation

$$\partial_t F + \frac{p}{p^0} \cdot \nabla_x F = Q(F, F), \;\; x \in \Omega, \;\; p \in \mathbb{R}^3, \;\; t \geq 0$$

- $F = F(t, x, p)$: probability density in (position,velocity)
- $\Omega \subset \mathbb{R}^3$: domain in space
- $\frac{p}{p^0} \cdot \nabla_x F$: free transport term
- $p = (p^1, p^2, p^3) \in \mathbb{R}^3$, $p^\mu = (p^0, p)$ where $p^0 = \sqrt{1 + |p|^2}$ is the energy of a relativistic particle with momentum p.
- $Q(F, F)$ is the, local in (t,x), "collision operator":

$$Q(f, h) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \;\; v_\phi \sigma(g, \theta)[f(p')h(q') - f(p)h(q)],$$

where $v_\phi = v_\phi(p, q)$ is the Mϕller velocity given by

$$v_\phi(p, q) = \sqrt{\left|\frac{p}{p^0} - \frac{q}{q^0}\right|^2 - \left|\frac{p}{p^0} \times \frac{q}{q^0}\right|^2} = \frac{g\sqrt{s}}{p^0 q^0}.$$

Quantities in the collision operator

The post-collisional momentum are written for $\omega \in \mathbb{S}^2$ as

$$p' = \frac{p+q}{2} + \frac{g}{2}\Big(\omega + (\xi - 1)(p+q)\frac{(p+q)\cdot\omega}{|p+q|^2}\Big),$$

$$q' = \frac{p+q}{2} - \frac{g}{2}\Big(\omega + (\xi - 1)(p+q)\frac{(p+q)\cdot\omega}{|p+q|^2}\Big).$$

where $p^0 = \sqrt{1+|p|^2}$ and $q^0 = \sqrt{1+|q|^2}$, $\xi = \frac{p^0+q^0}{\sqrt{s}}$. The scattering angle, $\cos\theta$, is defined as

$$\cos\theta = 1 - 2\frac{g(p,p')^2}{g(p,q)^2}.$$

Known to be a well defined angle. The relative momentum g is

$$g = g(p,q) = \sqrt{2(p^0 q^0 - p\cdot q - 1)} \geq 0.$$

And $s = g^2 + 4$.

Hypothesis on the collision kernel

$$\sigma(g,\theta) = \Phi(g)\sigma_0(\theta), \quad \Phi(g) \geq 0, \quad \sigma_0(\theta) \geq 0,$$

where

$$\Phi(g) = C_\Phi g^\rho, \quad C_\Phi > 0.$$

Additionally $\theta \mapsto \sigma_0(\theta)$ is not locally integrable:

$$\sigma_0(\theta) \approx \theta^{-2-\gamma}, \quad \gamma \in (0,2), \quad \forall \theta \in (0, \pi/2].$$

We allow that

$$-\frac{3}{2} - \gamma < \rho < 2$$

where $\rho + \gamma \geq 0$ are called hard potentials,
and $\rho + \gamma < 0$ are called soft potentials.

- This full set of physical assumptions on the collision kernel proposed by Dudynski and Ekiel-Jezewska in 1988 in CMP.
- This collision kernel makes the relativistic Boltzmann equation into a non-local fractional diffusion equation.

Examples of relativistic kernels from physics literature

Mϕller Scattering: de Groot, van Leeuwen, van Weert (1980).

This is an approximation of electron-electron scattering.

$$\sigma = r_0^2 \frac{1}{u^2(u^2-1)^2} \frac{(2u^2-1)^2}{sin^4\theta} - \frac{2u^4 - u^2 - 1/4}{sin^2\theta} + \frac{1}{4}(u^2-1)^2,$$

where $u = \frac{\sqrt{s}}{2mc}$ and $r_0 = \frac{e^2}{4\pi mc^2}$.

Israel, J. Math. Phys. (1963)

$$\sigma = \frac{m}{2g} b(\theta)$$

Other example kernels such as Short range interactions, Compton Scattering (de Groot, van Leeuwen, van Weert (1980)), and Neutrino Gas (Dijkstra, van Leeuwen (1978)).

Relativistic Maxwellian equilibrium: Jüttner Solutions

The equlibrium states are characterized as a particle distribution which maximizes the entropy subject to constant mass, momentum, and energy. They are given by

$$J(p) = \frac{e^{-\frac{cp^0}{k_B T}}}{4\pi c k_B T K_2(\frac{c^2}{k_B T})},$$

where k_B is Boltzmann constant, T is the temperature, and K_2 stands for the Bessel function $K_2(z) = \frac{z^2}{2} \int_1^\infty dt\ e^{-zt}(t^2 - 1)^{\frac{3}{2}}$.

The Jüttner solution with normalized constants is

$$J(p) = \frac{e^{-p^0}}{4\pi}.$$

(short incomplete) list of Relativistic Boltzmann results

- Bichteler (1967 local wellposedness for a bounded cross-section)
- Dudyński and Ekiel-Jeżewska (1988-89 L^2 solutions to linearized equation)
- Glassey and Strauss (1993 smooth solutions on $\mathbb{T}^3$ for the hard potentials)
- Glassey and Strauss (1995 whole space result for hard potentials)
- Strain (2010 global existence and stability for soft potentials, $\mathbb{T}^3$)
- Strain (2010 global Newtonian limit near vacuum)
- Speck and Strain (2011, Hilbert expansion to relativistic fluids)
- Guo and Strain (2012 two-species Vlasov-Maxwell-Boltzmann near eq.)
- Strain and Zhu (2012 soft potentials in $\mathbb{R}^3$)
- Lee and Rendall (2013 global existence for spatially homogeneous and hard potentials, both Minkowski and Robertson-Walker spacetime)
- Duan and Yu (2017 global existence and stretched exponential decay for soft potentials, $\mathbb{T}^3$)
- Wang (2018, initial smallness in $L^1_x L^\infty_p$ and mass, energy, entropy)
- Nishimura (2018, initial smallness in $L^\infty_x L^1_{p,loc}$ and mass, energy, entropy)
- Bae, Jang, and Yun (2021, global wellposedness for quantum statistics)
- Jang and Strain (2022, global wellposedness without angular cutoff)

The (Newtonian) Boltzmann equation (1872)

$$\partial_t F + v \cdot \nabla_x F = Q(F,F), \quad x \in \Omega, \quad v \in \mathbb{R}^3, \quad t \geq 0$$

- $\Omega \subset \mathbb{R}^3$: domain in space
- $v \cdot \nabla_x F$: free transport term
- $Q(F,F)$: collision operator, local in (t,x), quadratic operator

$$Q(F,G)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(v-v_*,\sigma)[F(v'_*)G(v')-F(v_*)G(v)].$$

- Pre-post collisional velocities (v, v_*) and (v', v'_*) satisfy

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,$$
$$v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

Newtonian “cancellation lemma”

Collision kernel: $B(v - v_*, \sigma) \approx |v - v_*|^\alpha \theta^{-2-\nu}$ for $\alpha > -3$ and $\nu \in (0, 2)$. Alexandre-Desvillettes-Villani-Wennberg (2000):

$$\int_{\mathbb{R}^3} dv \int_{\mathbb{S}^2} d\sigma \; B(v - v_*, \sigma)(F(v') - F(v)) = (F * S)(v_*)$$

$$S(z) = C_3 |z|^\alpha, \quad 0 < C_3 < \infty.$$

This holds for a general class of functions $F(v)$. This is based upon the change of variables $v' \to v$ with Jacobian determinant

$$\left| \frac{dv'}{dv} \right| = \frac{1}{4} (\cos(\theta/2))^2 \geq \frac{1}{8} > 0 \quad \left(0 \leq \theta \leq \frac{\pi}{2} \right)$$

This was a big part of the foundation for estimates of the non-local fractional diffusion of the Newtonian Boltzmann equation.

Newtonian "cancellation lemma" in general

- Consider the difference

$$u = \vartheta v' + (1-\vartheta)\, v, \quad \vartheta \in [0,1].$$

- The change of variable $u \to v$ has Jacobian determinant:

$$\left|\frac{du}{dv}\right| = \left(1 - \frac{\vartheta}{2}\right)^2 \left\{ \left(1 - \frac{\vartheta}{2}\right) + \frac{\vartheta}{2}\cos\theta \right\} \geq \left(1 - \frac{\vartheta}{2}\right)^3 > 0,$$

  since $\cos\theta \geq 0$ on $0 \leq \theta \leq \pi/2$.

- Various versions of this change of variables has served as the foundation for virtually all estimates of the non-local fractional diffusion for the Newtonian Boltzmann equation.
- We show in the special relativistic situation that this analogous change of variables is generally not well defined.

Relativistic Carleman representation

For $G = G(p, q, p', q')$ we have

$$\int_{\mathbb{R}^3} \frac{dq}{q^0} \int_{\mathbb{R}^3} \frac{dq'}{q'^0} \int_{\mathbb{R}^3} \frac{dp'}{p'^0} s\sigma(g,\theta)\delta^{(4)}(p'^\mu + q'^\mu - p^\mu - q^\mu)G$$

$$= \int_{\mathbb{R}^3} \frac{dp'}{p'^0} \int_{E^q_{p'-p}} \frac{d\pi_q}{8\bar{g}q^0} s\sigma(g,\theta)G,$$

where G has a sufficient vanishing condition so that the integrals are well-defined. Here $E^q_{p'-p}$ is the two-dimensional hypersurface for relativistic collisions which is defined as

$$E^q_{p'-p} = \{q \in \mathbb{R}^3 : (p'^\mu - p^\mu)(p_\mu + q_\mu) = 0\}.$$

And the measure is defined by

$$d\pi_q = dq\ u(p^0 + q^0 - p'^0)\delta\left(\frac{\bar{g}}{2} + \frac{q^\mu(p_\mu - p'_\mu)}{\bar{g}}\right).$$

Here $u(x) = 0$ if $x < 0$, and $u(x) = 1$ if $x \geq 0$. Also $\bar{g} = g(p, p')$ measures the difference between p and p'.

Proof of the lack of a relativistic "cancellation lemma"

Formally write down the following relativistic quantity:

$$\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta)(J(q) - J(q')) = \tilde{\zeta}_1^B(p) - \tilde{\zeta}_2^B(p).$$

Here $\tilde{\zeta}_1^B(p)$ and $\tilde{\zeta}_2^B(p)$ are written in a relativistic Carleman representation. Recall the relativistic Maxwellian (a Schwartz function) is

$$J(p) = \frac{e^{-p^0}}{4\pi}.$$

Merely assuming the collision kernel: $\sigma(g,\theta) = \text{constant}$. Then

$$\tilde{\zeta}_1^B(p) < \infty, \quad \text{but} \quad \tilde{\zeta}_2^B(p) = \infty.$$

We conclude that there is no such "cancellation lemma" for the relativistic Boltzmann equation, and this statement is independent of the coordinates choosen. (Jang and S, 2022, Ann. PDE, 10.1007/s40818-022-00137-2)

Linearization of relativstic Boltzmann equation

We consider the time evolution of perturbations

$$F(t,x,p) = J(p) + \sqrt{J(p)}f(t,x,p).$$

The perturbation $f = f(t,x,p)$ evolves via the equation

$$\partial_t f + \frac{p}{p^0}\cdot\nabla_x f + Lf = \Gamma(f,f),$$

where the non-linear collision operator is

$$\Gamma(f,h) = J^{-\frac{1}{2}}Q(\sqrt{J}f,\sqrt{J}h),$$

and the linearized collision operator is given by

$$Lf = -\Gamma(f,\sqrt{J}) - \Gamma(\sqrt{J},f).$$

Global wellposedness nearby equilibrium without cutoff

Theorem (Jang and S, Ann. PDE (2022))

Fix $N \geq 2$, which represents the total number of spatial derivatives. Fix $\gamma \in (0,1)$. Choose

$$f_0 = f_0(x,p) \in H^N_l(\mathbb{T}^3 \times \mathbb{R}^3)$$

for any fixed $l \geq 0$ which satisfies the conservation laws.
There is a small $\eta_0 > 0$ such that if

$$\|f_0\|_{H^N_l(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \eta_0,$$

then there exists a unique global solution, $f(t,x,p)$, to the relativistic Boltzmann equation which satisfies

$$f(t,x,p) \in L^\infty_t([0,\infty); H^N_l(\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^2_t((0,\infty); I^{\rho,\gamma}_{l,N}(\mathbb{T}^3 \times \mathbb{R}^3)).$$

Question.

Non-negativity of F?

Non-negativity and local wellposedness theorems

Theorem (Jang and S, Non-negativity)

Fix an integer $N \geq 2$. Let $F = J + \sqrt{J} f$ be a solution of the relativistic Boltzmann equation under the non-cutoff hypothesis with initial condition $\|f_0\|_{H^N_x L^2_p}$ that is sufficiently small. (Such a solution exists locally-in-time by the theorem below.) Suppose $F_0 = J + \sqrt{J} f_0 \geq 0$ initially on $\mathbb{T}^3_x \times \mathbb{R}^3_p$. Then, we have $F \geq 0$ on $[0, T] \times \mathbb{T}^3_x \times \mathbb{R}^3_p$.

Theorem (Jang and S, Local wellposedness)

For any sufficiently small $M_0 > 0$, there exists a time $T_0 = T_0(M_0) > 0$ and $M_1 > 0$ such that if $\|f_0\|^2_{H^N_x L^2_p} \leq M_1$, then there exists a unique solution $F = J + \sqrt{J} f$ to the relativistic Boltzmann equation on $[0, T_0) \times \mathbb{T}^3 \times \mathbb{R}^3$ such that

$$\sup_{0 \leq t \leq T_0} \mathcal{M}(f(t)) \leq M_0,$$

where the energy norm is defined as

$$\mathcal{M}(f(t)) \overset{def}{=} \|f(t)\|^2_{H^N_x L^2_p} + \int_0^t \|f(\tau)\|^2_{H^N_x I^{\rho,\gamma}_p}.$$

Sequence of approximated solutions

- For the proofs of both non-negativity and local-wellposedness, we consider the solution to the Boltzmann equation as a limit of approximate solutions $\{F^n\}_{n\geq 0}$ of the form

$$\begin{cases} \partial_t F^{n+1} + \hat{p} \cdot \nabla_x F^{n+1} = Q(F^n, F^{n+1}), \\ F^0(t,x,p) = J(p) \stackrel{\text{def}}{=} \frac{1}{4\pi} e^{-p^0}, \\ F^{n+1}|_{t=0} = F_0 \geq 0. \end{cases}$$

- This construction of the sequence of approximated solutions is slightly different from that of Jang-S (Ann. PDE, 2022), and we need a proof for the local existence again using this sequence.
- Want to emphasize that a lot of computations that are "trivial" in the Newtonian case become very serious challenging algebraic and conceptual difficulties in the special relativistic situation.

For the proof of non-negativity

- Fix any $\lambda > 0$. Taking a sufficiently large $\kappa > 0$ such that $\frac{\lambda}{2\kappa} > T$, we define for $n \geq 0$

$$h^n(t,x,p) = J_\kappa^{-1} F^n(t,x,p),$$

with

$$J_\kappa = J_\kappa(t,p) \stackrel{\text{def}}{=} e^{-(\lambda - \kappa t)p^0}.$$

- Then a sequence of approximated solutions $\{h^n\}_{n\geq 0}$ satisfies

$$\partial_t h^{n+1} + \hat{p} \cdot \nabla_x h^{n+1} + \kappa p^0 h^{n+1} = \Gamma^\kappa(h^n, h^{n+1})$$

where

$$\Gamma^\kappa(f,h) = J_\kappa(t)^{-1} Q(J_\kappa(t) f,\, J_\kappa(t) h).$$

- This uses the maximum principle approach of Alexandre, Morimoto, Ukai, Xu and Yang from (ARMA, 2010)

Main strategy for the proof of non-negativity

- Consider the convex function

$$\beta(s) = \frac{1}{2}(s_-)^2 = \frac{1}{2}s\,(s_-),$$

  with $s_- = \min\{s, 0\}$.

- Our goal is to prove $\beta(h_-^{n+1}) = 0$ under the following induction hypothesis:

  1. Suppose that there exists a $\lambda > 0$ such that, for all $n \in \mathbb{N}$, we have

  $$\sup_{n\in\mathbb{N}} \left\| e^{\lambda p^0} F^n(t, x, p) \right\|_{L^\infty([0,T]\times\mathbb{T}^3_x;L^2(\mathbb{R}^3_p))} \leq M,$$

     where $M > 0$ is independent of n.

  2. Suppose that $F^n \geq 0$.

L^2-type energy inequality for h_-^{n+1}

Obtain with algebraic weight function $\varphi(x)$ that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\beta(h^{n+1})\varphi(x)^{-2}dxdp+\kappa\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}p^0\beta(h^{n+1})\varphi(x)^{-2}dxdp$$

$$\leq \int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\Gamma^\kappa(h^n,h_-^{n+1})h_-^{n+1}\varphi(x)^{-2}dxdp$$

$$+\int_{\mathbb{T}^3}dx\int_{\mathbb{R}^3}dp\int_{\mathbb{R}^3}dq\int_{\mathbb{S}^2}d\omega\ v_\o\sigma(g,\theta)J_\kappa(t,q)h_-^{n+1}(p)\varphi(x)^{-2}$$

$$\times\left(h^n(q')h_+^{n+1}(p')-h^n(q)h_+^{n+1}(p)\right)$$

$$+C\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\beta(h^{n+1})\varphi(x)^{-2}dxdp$$

$$=\mathcal{A}_1+\mathcal{A}_2+C\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\beta(h^{n+1})\varphi(x)^{-2}dxdp.$$

Further decompositions

$\mathcal{A}_1$ can further be decomposed into $\mathcal{A}_1 = \mathcal{B}_1 + \mathcal{B}_2$

$$\mathcal{B}_1 \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta) J_\kappa(t,q) h^n(q) h^{n+1}_\varphi(p) \times \left(h^{n+1}_\varphi(p') - h^{n+1}_\varphi(p)\right),$$

and

$$\mathcal{B}_2 \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} dx \; \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta) \left(J_\kappa(t,q') - J_\kappa(t,q)\right) \times h^n(q) h^{n+1}_\varphi(p) h^{n+1}_\varphi(p').$$

New representation and decomposition of $\mathcal{B}_1$

$$
\begin{aligned}
\mathcal{B}_1 = -\frac{1}{2}\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta)J_{\kappa}(t,q)h^n(q)\frac{\tilde{s}\Phi(\tilde{g})\tilde{g}^4}{s\Phi(g)g^4} \\
\times \left(h_{\varphi}^{n+1}(p') - h_{\varphi}^{n+1}(p)\right)^2 \\
-\frac{1}{2}\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta)J_{\kappa}(t,q)h^n(q) \\
\times \left(1 - \frac{\tilde{s}\Phi(\tilde{g})\tilde{g}^4}{s\Phi(g)g^4}\right)|h_{\varphi}^{n+1}(p)|^2 \\
+\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta)J_{\kappa}(t,q)h^n(q) \\
\times \left(1 - \frac{\tilde{s}\Phi(\tilde{g})\tilde{g}^4}{s\Phi(g)g^4}\right)h_{\varphi}^{n+1}(p)h_{\varphi}^{n+1}(p') \\
\stackrel{\text{def}}{=} \mathcal{B}_{1,a} + \mathcal{B}_{1,b} + \mathcal{B}_{1,c}.
\end{aligned}
$$

- $\mathcal{B}_{1,a}$ is non-positive.
- $\left(1 - \frac{\tilde{s}\Phi(\tilde{g})\tilde{g}^4}{s\Phi(g)g^4}\right) \geq 0$ always pointwise.
- Thus $\mathcal{B}_{1,b} \leq 0$ since $|h_{\varphi}^{n+1}(p)|^2 \geq 0$ and by assumption $h^n \geq 0$.

Main estimates

The goal is to prove the following inequalities:

- $\mathcal{A}_2, \mathcal{B}_{1,a}, \mathcal{B}_{1,b} \leq 0$.
- $\mathcal{B}_{1,c} \lesssim \int_{\mathbb{T}^3} dx \, \|h_\varphi^{n+1}\|^2_{L^2_{\frac{\rho+\gamma}{2}}} \|h^n\|_{L^2}$.
- For any $\epsilon \in (0, 1-\gamma)$, we have

$$\mathcal{B}_2 \lesssim \int_{\mathbb{T}^3} dx \, \|h_\varphi^{n+1}\|^2_{L^2_{\frac{\rho+\gamma+\epsilon}{2}}} \|h^n\|_{L^2_{\frac{\rho+\gamma}{2}}}.$$

Final step

- The previus inequalities will lead us to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\beta(h^{n+1})\varphi(x)^{-2}dxdp + \frac{\kappa}{2}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}p^0\beta(h^{n+1})\varphi(x)^{-2}dxdp \lesssim C\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\beta(h^{n+1})\varphi(x)^{-2}dxdp,$$

  where the constant C now depends on M,δ,ρ,γ, and ϵ, if we choose $\delta > 0$ sufficiently small.
- Then by the Grönwall inequality, we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\beta(h^{n+1})\varphi(x)^{-2}dxdp \leq 0,$$

  since $\beta(h_0^{n+1}) = 0$.
- Thus $\beta(h^{n+1}) = \frac{1}{2}(h_-^{n+1})^2 = 0$ which implies $h^{n+1} \geq 0$.

Key methods

- Derive the Carleman dual representation for $\mathcal{B}_1$.
- Obtain the following equivalent *center-of-momentum* representation

$$\mathcal{B}_1 = \int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o}\sigma(g,\theta) J_\kappa(t,q) h^n(q) h_\varphi^{n+1}(p') \times \left[\frac{\tilde{s}\Phi(\tilde{g})\tilde{g}^4}{s\Phi(g)g^4} (h(p) - h_\varphi^{n+1}(p')) + h_\varphi^{n+1}(p) \left(1 - \frac{\tilde{s}\Phi(\tilde{g})\tilde{g}^4}{s\Phi(g)g^4} \right) \right].$$

- Sum this representation with the original representation.
- Use the dyadic decomposition of the region near the angular singularity $\bar{g} \approx 0$ for both $\mathcal{B}_1$ and $\mathcal{B}_2$.

Non-negativity lemma

Lemma

Let $\{F^n\}_{n\in\mathbb{N}}$ be a sequence of the approximated solutions. Suppose that there exists a $\lambda > 0$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{n\in\mathbb{N}} \left\| e^{\lambda p^0} F^n(t,x,p) \right\|_{L^\infty([0,T]\times\mathbb{T}^3_x;L^2(\mathbb{R}^3_p))} \le M,$$

where $M > 0$ is independent of n. Suppose that $F^n \ge 0$. Then, we have $F^{n+1} \ge 0$ on $[0,T] \times \mathbb{T}^3_x \times \mathbb{R}^3_p$ if $F^{n+1}(0,x,v) \ge 0$ on $\mathbb{T}^3_x \times \mathbb{R}^3_p$.

- This lemma implies our non-negativity theorem as long as the solution F in the strong pointwise limit $F^n \to F$ exists.

Local existence and uniqueness of such a solution F

- Further consider the perturbation around the relativistic Maxwellian J as $F^n = J + \sqrt{J} f^n$
- Obtain the following linearized system for f^n (slightly different from that of Jang-Strain (Ann. PDE, 2022):

$$\begin{cases} \partial_t f^{n+1} + \hat{p} \cdot \nabla_x f^{n+1} = \Gamma(f^n, \sqrt{J}) + \Gamma(\sqrt{J}, f^{n+1}) + \Gamma(f^n, f^{n+1}), \\ f^0(t,x,p) = 0, \\ f^{n+1}|_{t=0} = f_0. \end{cases}$$

- Operators are defined as:

$$\Gamma(f,h) \stackrel{\text{def}}{=} J^{-1/2} Q(\sqrt{J} f, \sqrt{J} h).$$

$$\mathcal{K} f = \zeta_{\mathcal{K}}(p) f - \Gamma(f, \sqrt{J}),$$

$$\begin{aligned} \mathcal{N} f &= -\Gamma(\sqrt{J}, f) - \zeta_{\mathcal{K}}(p) f \\ &= \zeta(p) f - \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o} \sigma(g,\omega)(f(p') - f(p)) \sqrt{J(q')} \sqrt{J(q)}. \end{aligned}$$

- Then the weights satisfy the following asymptotics; for any $\varepsilon \in (0, \gamma/2)$, there exists a finite constant $C_\varepsilon > 0$ such that we have

$$|\zeta_{\mathcal{K}}(p)| \lesssim C_\varepsilon (p^0)^{\frac{\rho}{2} + \varepsilon} \quad \text{and} \quad \zeta(p) \approx (p^0)^{\frac{\rho+\gamma}{2}}.$$

Preliminary estimates for the operators $\Gamma, \mathcal{K}, \mathcal{N}$

Lemma (Jang-S (Ann. PDE, 2022))

Suppose that $|\alpha| \leq N$ *with* $N \geq 2$ *and* $l \geq 0$. *Then we have the estimate*

$$\left| \left(w^{2l} \partial^\alpha \Gamma(f, h), \partial^\alpha \eta \right) \right| \lesssim \|f\|_{H^N_x L^2_p} \|h\|_{H^N_x I^{\rho,\gamma}_p} \|\partial^\alpha \eta\|_{L^2_x I^{\rho,\gamma}_p},$$

where $(\cdot, \cdot)$ *is the* L^2 *inner product in both* x *and* p.

Lemma (Jang-S (Ann. PDE, 2022))

For any fixed $\epsilon > 0$ *small enough, we have that*

$$|\langle \Gamma(f, \sqrt{J}), h \rangle| \lesssim \|f\|_{L^2_{p, \frac{\rho}{2} - \epsilon}} \|h\|_{I^{\rho,\gamma}_p},$$

where $\langle \cdot, \cdot \rangle$ *is the* L^2 *inner product in* p.

Preliminary estimates for the operators $\Gamma, \mathcal{K}, \mathcal{N}$

Lemma (Jang-S (Ann. PDE, 2022))

We have the uniform coercive lower bound estimate:

$$\langle \mathcal{N} f, f \rangle \gtrsim \|f\|^2_{I_p^{\rho,\gamma}},$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product in p.

A direct consequence:

Corollary

We have the uniform coercive lower bound estimate. For any $\varepsilon \in (0, \frac{\gamma}{2})$, there exists a finite constant $C_\varepsilon > 0$ such that

$$\langle \mathcal{N} f + \zeta_{\mathcal{K}} f, f \rangle \gtrsim \|f\|^2_{I_p^{\rho,\gamma}} - C_\varepsilon \|f\|^2_{L^2_{p,\frac{\rho}{2}+\varepsilon}}.$$

Main energy estimates for the local wellposedness

Lemma

Let $\{f^n\}$ be the sequence of iterated approximate solutions for the linearized relativistic Boltzmann equation. Then there exists a short time $T > 0$ such that for $\|f_0\|^2_{H^N_x L^2_p}$ sufficiently small, there exist uniform-in-n constants $C_0 > 0$ and $\beta > 0$ such that

$$\sup_{n\geq 0} \sup_{0\leq t\leq T} \mathcal{M}(f^n(t)) \leq 2C_0 \|f_0\|^2_{H^N_x L^2_p} e^{\beta T},$$

where the energy norm is defined as

$$\mathcal{M}(f(t)) \overset{def}{=} \|f(t)\|^2_{H^N_x L^2_p} + \int_0^t \|f(\tau)\|^2_{H^N_x I^{\rho,\gamma}_p}.$$

- This uniform bound and compactness will establish the local existence of a strong solution.
- The uniqueness also follows, since the strong limit solves the same linearized Boltzmann equation as Jang-Strain, Ann. PDE (2022).

Thank you!

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