Existence of classical solutions for the non-cutoff Boltzmann equation with irregular initial data

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Recent Progress in Kinetic and Integro-Differential Equations
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Joint work with Christopher Henderson (University of Arizona)
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Introduction

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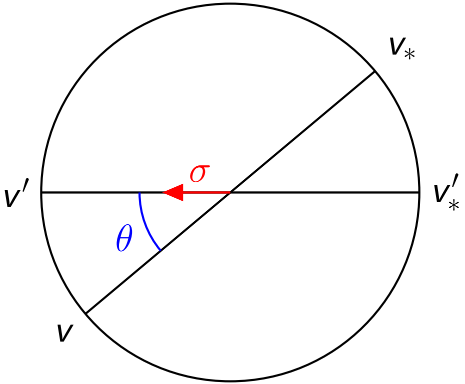
Our goal is to treat initial data that may have **low regularity**, **slow decay for large** $|v|$, **vacuum regions**, and **no decay for large** $|x|$.

Boltzmann collision operator

$$Q(f,g)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) \left[f(v_*')g(v') - f(v_*)g(v) \right] \, \mathrm{d}\sigma \, \mathrm{d}v_*$$

Gain Loss

Pre- and post-collisional velocities are related as follows:



$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma$$

$$v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

$$\cos\theta = \sigma \cdot \frac{v - v_*}{|v - v_*|}$$

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We take the *non-cutoff* collision kernel: for $\gamma > -3$ and $s \in (0,1)$,

$$B(|v - v_*|, \sigma) = |v - v_*|^\gamma b(\cos\theta),$$

where

$$b(\cos\theta) \approx \theta^{-2-2s} \quad \text{as } \theta \to 0.$$

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Singularity at $\theta = 0$ induces fractional differentiation of order $2s$ in the v variable.

Soft potentials

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There are fewer results dealing with γ close to -3.

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- Smoothing via iteration of hypoelliptic estimates [Chen-He, ARMA 2012].
- Conditional regularity in terms of mass/energy/entropy bounds, culminating in [Imbert-Silvestre, JAMS 2022].

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In the space homogeneous case, local existence for irregular initial data has been understood for some time: see e.g. [Desvillettes-Wennberg, CPDE 2005], [Chen-He, ARMA 2011].

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The only results that need less than four derivatives are restricted to the case $s \in (0, \frac{1}{2})$: [Alexandre-Morimoto-Ukai-Xu-Yang, KRM 2013], [another theorem in Henderson-Wang, SIMA, to appear].

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Then there exists $T > 0$ depending on $\|f_{\rm in}\|_{L^\infty_q}$ and a classical solution f of the Boltzmann equation such that $\langle v \rangle^q f \in L^\infty([0, T] \times \mathbb{R}^6)$ and f is locally Hölder continuous of order $2s+$ (in the kinetic scaling).

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The solution agrees with $f_{\rm in}$ *in a weak sense (integration against test functions).*

If, in addition, $f_{\rm in}$ is continous, then we can show $f_{\rm in} = \lim_{t \downarrow 0} f(t)$.

What do we mean by classical solution?

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- If f_{in} has more decay in v, then f has more regularity: for any multi-index $k = (k_t, k_x, k_v)$, there exists $q(k)$ so that $D^k f$ exists pointwise whenever $\langle v \rangle^{q(k)} f_{\text{in}} \in L^\infty$.

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- If $f_{\rm in}$ decays faster than any polynomial, then the solution f is C^∞.

Local positivity assumption

In our main theorem, we need to assume

$$f_{\text{in}} \geq \delta \text{ in } B_r(x_0) \times B_r(v_0),$$

for some $\delta, r > 0$ and $(x_0, v_0) \in \mathbb{R}^6$. (This is automatically true if f_{in} is continuous and not identically zero, but our f_{in} may be discontinuous.)

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Without quantitative lower bounds for f, we cannot access the smoothing properties of $Q(f, \cdot)$.

Weak solutions

Theorem (Henderson-S-Tarfulea, preprint 2022)

Suppose the initial data $f_{\mathrm{in}} \geq 0$ *satisfies*

- $(1 + |v|^q) f_{\mathrm{in}} \in L^\infty(\mathbb{R}^6)$ *for some* $q > \gamma + 2s + 3$.

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Weak solution: for any compactly supported $\varphi \in C^1_{t,x} C^2_v([0,T) \times \mathbb{R}^6)$,

$$\int_{\mathbb{R}^6} f_{\mathrm{in}} \varphi \,\mathrm{d}v\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}^6} [f(\partial_t + v \cdot \nabla_x)\varphi + W(f,f,\varphi)] \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t,$$

where $W(f,f,\varphi) =$

$$\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) f(v) f(v_*) [\varphi(v'_*) + \varphi(v') - \varphi(v_*) - \varphi(v)] \,\mathrm{d}\sigma\,\mathrm{d}v_*$$

is Maxwell's weak form of the collision operator.

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This would likely require new *a priori* regularity estimates that do not need as strong positivity properties for the solution f.

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- Apply [Henderson-S-Tarfulea, CVPDE 2020] to conclude f^ε satisfies good lower bounds for positive times.
- Regularity estimates as in [Imbert-Silvestre, JAMS 2022] that are sufficient to take the limit in ε.

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A common solution is to divide f by $e^{(\rho-\kappa t)|v|^2}$, but this requires Gaussian decay for f_{in}. More intricate methods (e.g. [Morimoto-Yang 2015, Henderson-Wang 2021]) also require relatively high polynomial decay.

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Unlike their result, our estimates hold only up to a finite time T, but we obtain an upper bound for $\|\langle v\rangle^q f\|_{L^\infty}$ depending only on the initial data and T.

Side note about barriers

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We previously established an existence result for the Landau equation in a similar spirit to this one. [Henderson-S-Tarfulea, Annales IHP 2020].

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We previously established an existence result for the Landau equation in a similar spirit to this one. [Henderson-S-Tarfulea, Annales IHP 2020].

In that study, barrier arguments were more convenient because the Landau collision operator $Q_L(f,g)$ is local in g. To bound $Q_L(f,g)$ at the point where f and g cross, one only needs information about g at the crossing point.

More subtle decay estimates

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We want to continue f^ε up to a uniform time interval, but the available continuation criteria require the qualitative assumption of rapid decay.

Even though $f_{\mathrm{in}}^\varepsilon$ decays faster than any polynomial rate, no available prior results ensure the same is true for f^ε for $t > 0$, except when $\gamma > -\frac{3}{2}$.

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$$Q(f, \langle v\rangle^{-q}) \leq C\|\langle v\rangle^{q_0} f\|_{L^\infty} \langle v\rangle^{-q} \quad \text{for } q_0 \leq q \leq q_0 + |\gamma|.$$

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By carefully iterating this argument, after finitely many steps we reach any $q > 0$ such that $\|\langle v\rangle^q f_{\mathrm{in}}\|_{L^\infty} < \infty$.

Regularity estimates

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In particular, when $\gamma + 2s < 0$, we have to modify the change of variables used to pass from local to global regularity estimates.

These estimates give us enough compactness to take $\varepsilon \to 0$ and obtain f.

Uniqueness

Nothing in the above argument guarantees that the solution is unique.

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For a uniqueness result for non-cutoff Boltzmann in a related setting, see the talk from Andrei Tarfulea on Wednesday.

Thank you!