Nonlocal gradients in Nonlinear Elasticity

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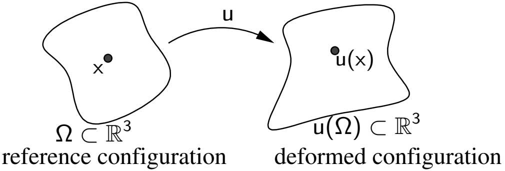
(joint with José C. Bellido and Javier Cueto)

Classical Nonlinear Elasticity (A.-L. Cauchy, G. Green)



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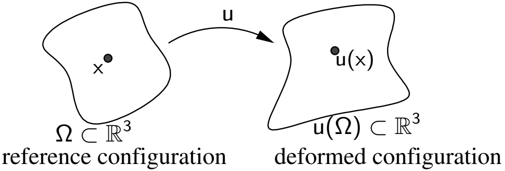

Total energy of elastic deformation

$$\underbrace{\int_\Omega W(Du(x))\,\mathrm{d}x}_{\text{elastic}} - \underbrace{\int_\Omega f \cdot u\,\mathrm{d}x}_{\text{external force}}.$$

where $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ stored energy function.

Peridynamics

Silling 00 proposed a reformulation of classical continuum mechanics. In its *bond based* variant the elastic energy is

$$\mathcal{I}(u) = \int_\Omega \int_\Omega w(x - x', u(x) - u(x'))\, \mathrm{d}x' \, \mathrm{d}x.$$

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Features:

- non-local: points at a positive distance exert a force upon each other.
- absence of gradients.
- main example: $\displaystyle\int_\Omega \int_\Omega \frac{|u(x) - u(x')|^p}{|x - x'|^\alpha} \, \mathrm{d}x' \, \mathrm{d}x$
- deformations with discontinuities do not require a separate treatment.

Existence of minimizers

The existence theory for models based on

$$\int_{\Omega}\int_{\Omega} w(x - x', u(x) - u(x'))\,\mathrm{d}x'\,\mathrm{d}x$$

is relatively well-understood, via direct method of Calculus of Variations. (Bellido & C.M-C. 14)

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Based on previous results:

Lower semicontinuity: Boulanger, Elbau, Pontow & Scherzer 11

Coercivity: Bourgain, Brezis & Mironescu 02,

Ponce 04,

Andreu, Mazón, Rossi & Toledo 08, 09,

Aksoylu & Mengesha 10,

Aksoylu & Parks 11,

Hinds & Radu 12,

Hurri-Syrjänen & Vähäkangas 13.

Nonlocal → local as horizon $\delta \to 0$

(Bellido, C.M.-C., Pedregal 15)

$$\frac{1}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(x,\delta)} w(x-x', u(x)-u(x'))\,\mathrm{d}x'\,\mathrm{d}x \xrightarrow{\Gamma} \int_\Omega W(Du(x))\,\mathrm{d}x.$$

Use Bourgain, Brezis & Mironescu 01, Ponce 04.

Apparently, we recover the classical model, but this limit passage retrieves very few stored energies W. No Mooney-Rivlin is recovered via this method. (Bellido, Cueto & C.M.-C. 20)

Models based on nonlocal gradients

Based on Mengesha & Spector 15, Mengesha & Du 15, Shieh & Spector 15, 18, we adopt the model

$$\mathcal{I}(u) = \int_{\Omega} W(\mathcal{G}u(x))\,\mathrm{d}x$$

where $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a typical stored-energy function in hyperelasticity, and $\mathcal{G}u$ is a *nonlocal gradient*:

$$\mathcal{G}u(x) = \int_{\Omega} \frac{u(x) - u(x')}{|x - x'|} \otimes \frac{x - x'}{|x - x'|}\,\rho(x - x')\,\mathrm{d}x'.$$

for a suitable kernel ρ.

We adopt the functional setting of Shieh & Spector 15, 18: $\Omega = \mathbb{R}^n$,

$$\rho(x - x') = \frac{c_{n,s}}{|x - x'|^{n+s-1}}$$

for $0 < s < 1$, so

$$\mathcal{G}u(x) = D^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x')}{|x - x'|^{n+s}} \otimes \frac{x - x'}{|x - x'|} \, \mathrm{d}x'$$

is Riesz' s-fractional gradient.

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The functional space

$$H^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : D^s u \in L^p(\mathbb{R}^n)\}$$

coincides with Bessel potential space.

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We have analogues of:

- Sobolev–Gagliardo–Nirenberg: $\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|D^s u\|_{L^p(\mathbb{R}^n)}$.
- Rellich–Kondrachov: $H^{s,p}(\mathbb{R}^n)$ with $u = u_0$ in $\mathbb{R}^n \setminus \Omega$ is compactly embedded in $L^p(\mathbb{R}^n)$.

The dual operator of D^s is the s-fractional divergence div^s, so that integration by parts hold (Mengesha & Spector 15)

$$\int_{\mathbb{R}^n} D^s u(x) \cdot \phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} u(x)\,\mathrm{div}^s\,\phi(x)\,\mathrm{d}x.$$

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We have the functional-analytic tools to start an existence theory parallel to the classical theory.

Existence theory

For W polyconvex: Bellido, Cueto & C.M.-C. 20.

For W quasiconvex: Kreisbeck & Schönberger 21.

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Two methods of proof:

- Adapt the proofs of classical case.
- Exploit the fact that every nonlocal gradient is a gradient: For every $u \in H^{s,p}$ there exists $v \in W^{1,p}$ such that $Dv = D^s u$, and vice versa.

A model for bounded domains

Main drawback of model

$$\int_{\mathbb{R}^n} W(D^s u(x))\,\mathrm{d}x, \qquad u = u_0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

Interactions are assumed over whole $\mathbb{R}^n$; energy is calculated over whole $\mathbb{R}^n$.

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Go back to general nonlocal gradient

$$\mathcal{G}u(x) = \int_\Omega \frac{u(x) - u(x')}{|x - x'|} \otimes \frac{x - x'}{|x - x'|}\, \rho(x - x')\,\mathrm{d}x'.$$

Choose

$$\rho(\tilde{x}) = \frac{c_{n,s}}{|\tilde{x}|^{n+s-1}} w_\delta(\tilde{x})$$

with $w_\delta \in C_c^\infty(B(0,\delta))$ cut-off function, so ρ is a *truncated Riesz kernel*.

Framework

Nonlocal gradient

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \otimes \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}} \, dy.$$

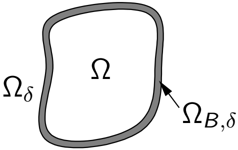
Energy

$$\int_\Omega W(D_\delta^s u(x)) \, \mathrm{d}x.$$

Domain for u: $\Omega_\delta = \Omega + B(0,\delta)$ 'nonlocal clousure'.

Domain for $D_\delta^s u$: Ω 'nonlocal interior'.

Boundary conditions: $u = u_0$ in $\Omega_{B,\delta} = \Omega_\delta \setminus \Omega$ 'nonlocal boundary'.



Funcional analysis framework

Functional space:

$$H^{s,p,\delta}(\Omega) = \left\{ u \in L^p(\Omega_\delta) : D_\delta^s u \in L^p(\Omega) \right\}.$$

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Subspace of homogeneous Dirichlet boundary conditions:

$$H_0^{s,p,\delta}(\Omega_{-\delta}) = \left\{u \in H^{s,p,\delta}(\Omega) : u = 0 \text{ in } \Omega_\delta \setminus \Omega_{-\delta}\right\}.$$

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Poincaré inequality:

$$\|u\|_{L^p(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)}, \qquad u \in H_0^{s,p,\delta}(\Omega_{-\delta}).$$

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$$\|u\|_{L^p(\Omega)} \le C \left\|D_\delta^s u\right\|_{L^p(\Omega)}, \qquad u \in H_0^{s,p,\delta}(\Omega_{-\delta}).$$

Compactness: $H_0^{s,p,\delta}(\Omega_{-\delta})$ is compactly embedded in $L^p(\Omega)$.

Main tools

- Integration by parts:

$$\text{Fractional: } \int_{\mathbb{R}^n} D^s u \cdot \phi = - \int_{\mathbb{R}^n} u \operatorname{div}^s \phi.$$

$$\text{Nonlocal: } \int_{\Omega} D^s_\delta u \cdot \phi = - \int_{\Omega} u \operatorname{div}^s_\delta \phi.$$

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- Fundamental Theorem of Calculus:

$$\text{Classical:} \quad u(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} Du(y) \cdot \frac{x-y}{|x-y|^n} \, dy.$$

$$\text{Fractional:} \quad u(x) = c_{n,-s} \int_{\mathbb{R}^n} D^s u(y) \cdot \frac{x-y}{|x-y|^{n-s+1}} \, dy.$$

$$\text{Nonlocal:} \quad u(x) = \int_{\Omega} D^s_\delta u(y) \cdot V^s_\delta(x-y) \, dy.$$

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- Nonlocal and classical gradients:

Fractional: Every $D^s u$ is a Dv and vice versa.

Nonlocal: Every $D^s_\delta u$ is a Dv and vice versa.

Existence theory

For polyconvex W: Bellido, Cueto & C.M.-C. 22.

For quasiconvex W: Cueto, Kreisbeck & Schönberger 22.

Recent attempts of unifying theories based on nonlocal gradients

D'Elia, Gulian, Olson & Karniadakis 21

D'Elia, Gulian, Mengesha & Scott 22