Theoretical and Applied Aspects for nonlocal Models
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A nonlocal notion of calibration

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1. Setting

Let $\Omega \subset \mathbb{R}^d$ be an open reference set with finite Lebesgue measure, and let Ω^c be its complement. Let also $K\colon \mathbb{R}^d \to [0,+\infty)$ be an even kernel such that

$$\int_{\mathbb{R}^d} K(x)\min(1,|x|)\mathrm{d}x < +\infty.$$

For $u\colon \mathbb{R}^d \to [0,1]$, we define the functional

$$J_K(u;\Omega) := \frac{1}{2}\int_\Omega\int_\Omega K(y-x)\,|u(y)-u(x)|\,\mathrm{d}y\mathrm{d}x + \int_\Omega\int_{\Omega^c} K(y-x)\,|u(y)-u(x)|\,\mathrm{d}y\mathrm{d}x,$$

This may be regarded as a **nonlocal total variation** of u in Ω; in particular, when $u = \chi_E$ is the characteristic function of a set $E \subset \mathbb{R}^d$, J_K may be understood as a **nonlocal perimeter** of E in Ω.
Nonlocal perimeters were firstly introduced by L. Caffarelli & al. in [3] to describe phase field models that feature long-range space interactions. In that work, K is a fractional kernel, that is $K(x) = |x|^{-d-s}$, with $s \in (0,1)$.

2. Nonlocal Plateau's problem

We consider a Plateau-type problem for the nonlocal energy J_K, that is, we consider its minimisation under prescribed boundary conditions.

Theorem 1 [1]. Let $E_0 \subset \mathbb{R}^d$ be a set such that $J_K(\chi_{E_0};\Omega) < +\infty$, and let

$$\mathscr{F} := \left\{ v\colon \mathbb{R}^d \to [0,1] : v = \chi_{E_0} \text{ in } \Omega^c \right\}.$$

There exists $E \subset \mathbb{R}^d$ such that $\chi_E \in \mathscr{F}$ and

$$J_K(\chi_E;\Omega) \le J_K(v;\Omega) \qquad \text{for all } v \in \mathscr{F}.$$

Idea of the proof. In general, sequences with uniformly bounded energy are not precompact in L^1. However, weak convergence in L^p is enough to show the existence of a minimizer, because J_K is convex and L^p-lower semicontinuous for $p \in [1,+\infty)$. A generalised Coarea Formula grants that there exists a minimizer which is a characteristic function. □

Problem. Once existence of solutions is on hand, how to establish the minimality of a certain competitor?
In [5], a nonlocal counterpart of the well-known principle based on the concept of calibration was introduced to this purpose.

3. Definition of nonlocal calibration

Definition [5]. We say that $\zeta\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a **nonlocal calibration** for $u\colon \mathbb{R}^d \to [0,1]$ if the following hold:

1. $|\zeta(x,y)| \le 1$ for a.e. $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$;
2. the integrals

$$I_r(x) := \int_{B(x,r)^c} K(y-x)\,(\zeta(y,x)-\zeta(x,y))\,\mathrm{d}y \qquad \text{with } r>0 \text{ and } x \in \mathbb{R}^d,$$

satisfy

$$\lim_{r\to 0^+} I_r = 0 \quad \text{w.r.t. the } L^1(\Omega)\text{-norm.} \tag{1}$$

3. for a.e. $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $u(x) \ne u(y)$,

$$\zeta(x,y)(u(y)-u(x)) = |u(y)-u(x)|\,. \tag{2}$$

4. Some remarks about the definition

- Requirement 1. plays the role of the so called "size condition".
- Equation (1) may be regarded as a vanishing divergence condition.
- Let $u = \chi_E$ with E a set and let ζ be a calibration for u. Heuristically, ζ gives the sign of the inner product between the vector $y - x$ and the inner normal to E at the "crossing point" (see Figure 1).

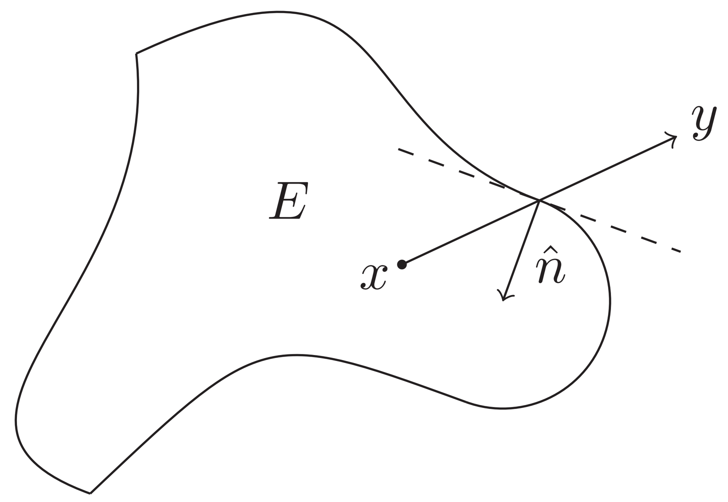

Figure 1: If ζ is a calibration for the set E (i.e. for χ_E) and x, y are as in the picture, then $\zeta(x,y) = -1$.

5. Minimality *via* calibrations

The existence of a calibration is a sufficient condition for a function u to minimise the energy J_K w.r.t compact perturbations:

Theorem 2 [5]. Let $\mathscr{F}$ be as in Thm. 1. If some $u \in \mathscr{F}$ admits a calibration, then

$$J_K(u;\Omega) \le J_K(v;\Omega) \qquad \text{for all } v \in \mathscr{F}.$$

Idea of the proof. Thanks to a sort of nonlocal divergence theorem, it can be proved that $J_K(v;\Omega) \ge b_0$ for all $v \in \mathscr{F}$, where b_0 is a constant that depends only on the boundary datum χ_{E_0}. Then one shows by means of (2) that the lower bound is attained by u. □

6. Applications & related works

- **Halfspaces are the unique minimisers of J_K in a ball.** A simple example in which the criterion above can be used is the following:

Theorem 3 [5]: . Let B the a ball with centre in the origin and, for $\hat{n} \in \mathbb{S}^{d-1}$, let $H_{\hat{n}} := \left\{ x \in \mathbb{R}^d : x \cdot \hat{n} > 0 \right\}$. Then, $\zeta(x,y) := \operatorname{sign}((y-x)\cdot\hat{n})$ is a calibration for $\chi_{H_{\hat{n}}}$, and for any $v\colon \mathbb{R}^d \to [0,1]$ such that $v(x) = \chi_{H_{\hat{n}}}(x)$ for $\mathscr{L}^d$-a.e. $x \in B^c$, it holds $J_K(\chi_{H_{\hat{n}}};B) \le J_K(v;B)$.
Moreover, for any other minimiser u satisfying the same constraint, it holds $u(x) = \chi_{H_{\hat{n}}}(x)$ $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.

- **Asympotics of the rescalings of J_K.** The optimality of halfspaces can be exploited to prove that suitable rescalings of J_K Γ-converge to the standard perimeter as the scaling vanishes.
- **Nonlocal perimeters on Carnot groups.** The notion of nonlocal calibration and the related minimality of halfspaces have been recently generalised to the context of Carnot groups by A. Carbotti & al. in [4].
- **An alternative definition of calibration.** An similar notion of nonlocal calibration has been independently proposed by X. Cabré in [2].

References

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