Semi-Classical-Fourier-Integral-Operator-Valued Pseudodifferential Operators and Scattering in a Strong Magnetic Field

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For $n \geq 3$ and $b > 0$ let $H(b) = H_0(b) + bV$, where the electric potential V satisfies

$$V = V(x, y, z) \in C_c^\infty(\mathbb{R}^n)$$

$$H_0(b) = \sum_{j=1}^{d} \left[\left(D_{x_j} - \frac{b\mu_j}{2} y_j \right)^2 + \left(D_{y_j} + \frac{b\mu_j}{2} x_j \right)^2 \right] - \Delta_z$$

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Let $h = \dfrac{1}{\sqrt{b}}$ be the semi-classical parameter.

The wave operators for the pair $(H_0(b), H(b))$ defined by

$$W_\pm = \text{s-}\lim_{t \to \pm\infty} U(t) U_0(-t) \text{ in } L^2(\mathbb{R}^n)$$

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We can therefore define the scattering operator S by setting

$$S = W_+^* W_-.$$

For $E > 0$, there exists an operator

$$S(E,b) : L^2\left(\mathbb{R}^{2d} \times \mathbb{S}^{n-2d-1}\right) \to L^2\left(\mathbb{R}^{2d} \times \mathbb{S}^{n-2d-1}\right)$$

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$\mathcal{T}(\lambda b, b) = S(E,b) - I$ is called the scattering amplitude at energy E

$$\mathcal{T}(\lambda b, b) = \frac{2\pi}{h^3} \tilde{U}^* \tilde{\mathcal{F}}_0 \left(\frac{\lambda}{h^2} \right) R_- \left[h^2 \Delta_z, \chi_1 \right] \int_0^{T_0} e^{-\frac{i}{h} t \mathcal{P}_N(\lambda)} dt$$
$$\left[h^2 \Delta_z, \chi_2 \right] R_+ \tilde{\mathcal{F}}_0 \left(\frac{\lambda}{h^2} \right)^* \tilde{U} + \mathcal{O}\left(h^{N-2} \right),$$

where $T_0 >> 0$ and $\tilde{U}$ is a metaplectic (i.e., a change of variable) operator.

We construct a parametrix (i.e., an approximation) to $e^{-\frac{i}{h}t\mathcal{P}_N}$ for small t of the form

$$\frac{1}{(2\pi h)^{n-2d}}\int_{\mathbb{R}^{n-2d}} e^{\frac{i}{h}(\phi(t,y,z,\eta;\theta)-z'\cdot\theta)}\upsilon(t,y,z,\eta;\theta)d\theta,$$

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Thus $\tau\in S^{\frac{1}{2}}$ and it is an h-FIO-valued symbol.