Mathematical analysis of non-self-adjoint eigenvalue problem for the bent waveguides

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BIRS workshop on Mathematical aspects of the physics with non-self-adjoint operators
Banff, Canada

10 - 15 July 2022

॥ त्वं ज्ञानमयो विज्ञानमयोऽसि ॥

What is a Waveguide?

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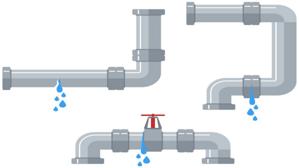

Water pipe

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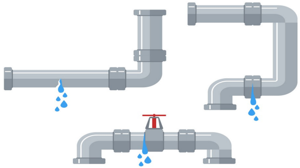

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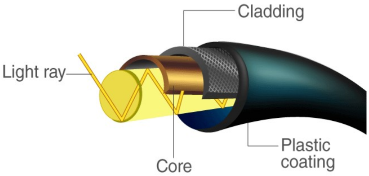

Optical fiber

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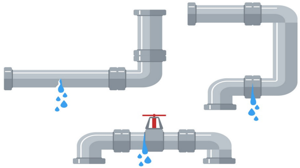

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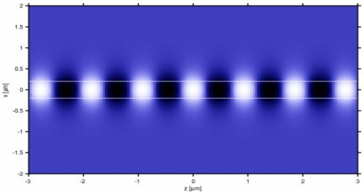
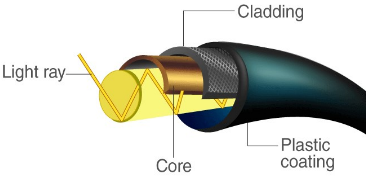


Straight mode profile



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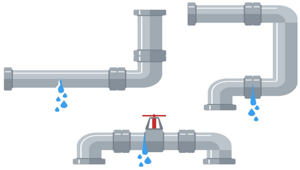

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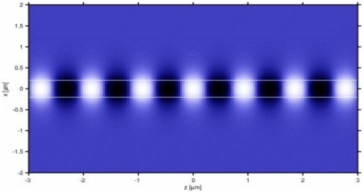

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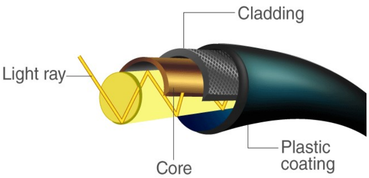

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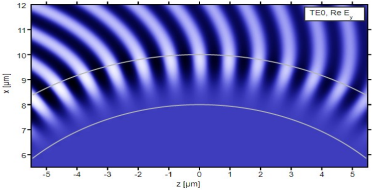

Bent mode profile

Ref. google images and `https://www.computational-photonics.eu/bends.html`

Mathematical setting for Straight Waveguides

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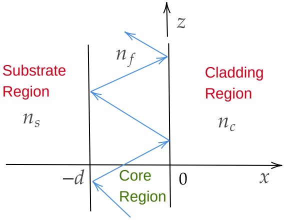

Straight Waveguide

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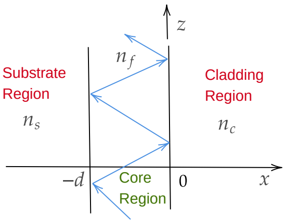

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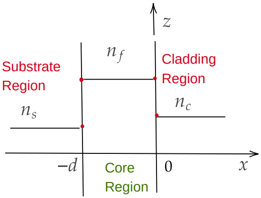
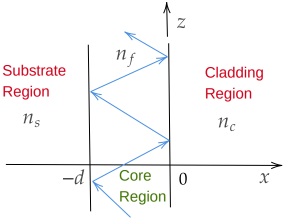


Refractive index profile

Mathematical setting for Straight Waveguides



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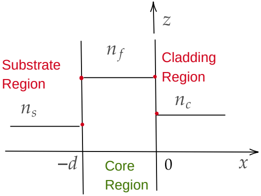

Refractive index profile

- Field, material properties are not varying in $y-$direction.

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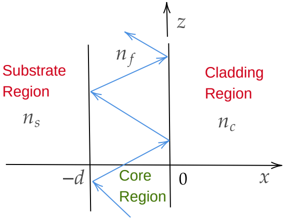

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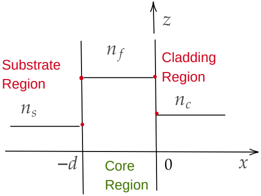

Refractive index profile

- Field, material properties are not varying in $y-$direction.
- Field ansatz

$$E = (E_x(x), E_y(x), E_z(x))e^{\imath(\omega t - \beta_S z)},$$

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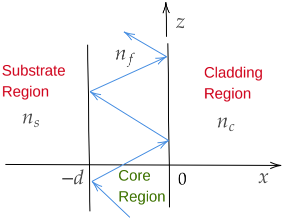

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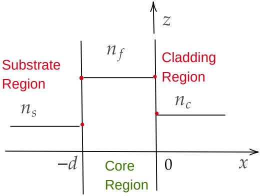

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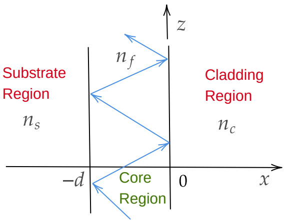

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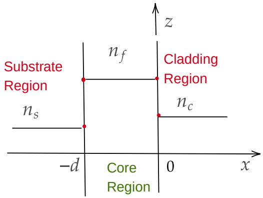

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- For TE mode, governing equation is

$$L_S(E_y) = \frac{1}{k^2}\frac{d^2 E_y}{dx^2} + n^2(x)E_y = \frac{\beta_S^2}{k^2}E_y,$$

$$|E_y| \to 0 \text{ as } x \to \pm\infty, E_y \text{ is outgoing waves.}$$

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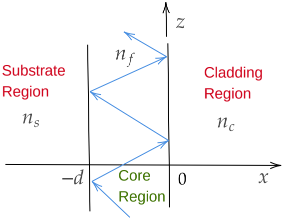

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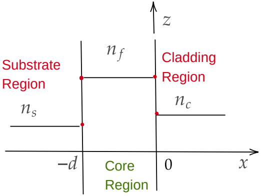

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- β_S is propagation constant (unknown).

Ref: P. Joly and C. Poirier, Mathematical analysis of electromagnetic open waveguides, ESAIM: Mathematical Modelling and Numerical Analysis, 29(1995), pp. 505 – 575

Mathematical setting for Bent Waveguides

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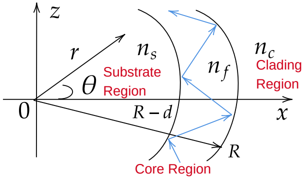
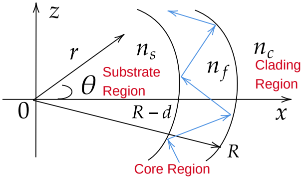


Bent Waveguide

Mathematical setting for Bent Waveguides



Bent Waveguide

- Field ansatz

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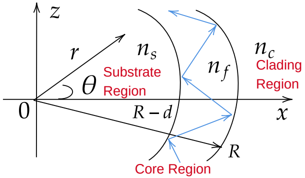

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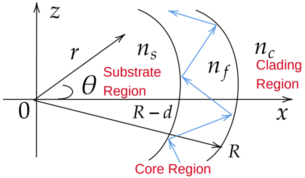

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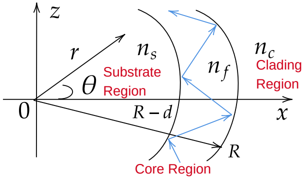

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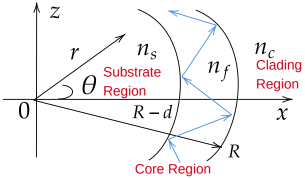

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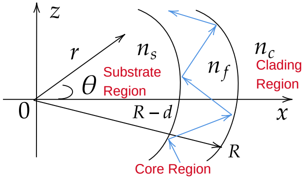

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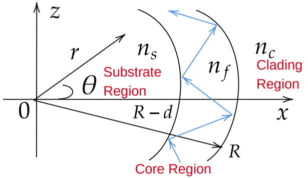

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$$r^2 \frac{d^2\psi}{dr^2} + r\frac{d\psi}{dr} + (n^2(r)k^2r^2 - \gamma^2 R^2)\psi = 0,$$

for $\psi = E_y$ or $\psi = H_y$.

Ref: K.R. Hiremath, M.Hammer, R. Stoffer, L. Prkna, and J. Čtyroký, Analytic approach to dielectric optical bent slab waveguides. Optical and quantum electronics, 37(1),2005, pp.37-61.

Eigenvalue problem for Bent Waveguides

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Define an operator L_R as

$$L_R\psi := \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi, \qquad 0 < r < \infty. \tag{1}$$

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Associated boundary conditions for the guided modes are given by

$$|\psi(r)| \to 0 \quad \text{as} \quad r \to 0,$$
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where refractive index profile

$$n(r) = \begin{cases} n_s, & 0 < r < R - d, \\ n_f, & R - d \le r \le R, \\ n_c, & R < r < \infty, \end{cases}$$

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and d core width, $k = \frac{2\pi}{\lambda}$ vacuum wavenumber, $\gamma = \beta - \imath\alpha$ unknown propagation constant, and R bent radius parameter.

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An 1-D eigenvalue problem with weight function $\frac{R}{r}$, eigenvalues $\frac{\gamma^2}{k^2}$, and defined on

$$\mathbb{V}_R = \{\psi |\ \psi, \psi_r \in \mathcal{L}^2[0,\infty)\},$$

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- Variable coefficient

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- Variable coefficient
- Parameter

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- Variable coefficient
- Parameter
- Weighted problem

Eigenvalue problem for Bent Waveguides

$$L_R\psi := \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi, \qquad 0 < r < \infty.$$

An 1-D eigenvalue problem with weight function $\frac{R}{r}$, eigenvalues $\frac{\gamma^2}{k^2}$, and defined on

$$\mathbb{V}_R = \{\psi |\ \psi, \psi_r \in \mathcal{L}^2[0,\infty)\},$$

where $\mathcal{L}^2[0,\infty)$ space of square-integrable functions and

$$<\psi,\phi>_r = \int_0^\infty \psi(r)\phi^*(r)\frac{R}{r}dr, \qquad \|\psi\|_r{}^2 = \int_0^\infty \psi(r)\psi^*(r)\frac{R}{r}dr,$$

where $*$ represents the complex conjugate.

- Variable coefficient
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The adjoint operator is

$$L_R^*\psi = \left(\frac{r}{Rk^2}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi + \underbrace{\left(\frac{\psi}{rRk^2} - \frac{2}{Rk^2}\psi_r\right)}_{\text{Cause of Non-self-adjointness}} = \frac{\gamma^{*2}}{k^2}\frac{R}{r}\psi, \tag{3}$$

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and $\gamma = \beta - \imath\alpha$ is complex proved by $\alpha \neq 0$.

Ref: R. Kumar, and K.R. Hiremath, Non-self-adjointness of bent optical waveguide eigenvalue problem, Journal of Mathematical Analysis and Applications, 512(1), 2022 p.126024.

Waveguides eigenvalue problems

Waveguides eigenvalue problems

Bent Waveguide

Waveguides eigenvalue problems

Bent Waveguide

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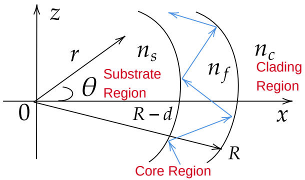

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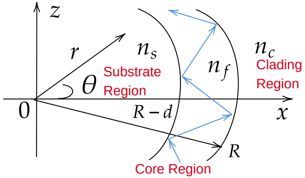

$r = Re^{\frac{x}{R}}$

$\longrightarrow$

$R \to \infty$

Waveguides eigenvalue problems

Bent Waveguide

Straight Waveguide

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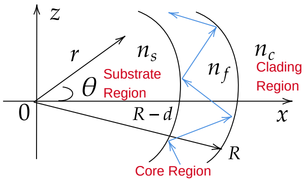

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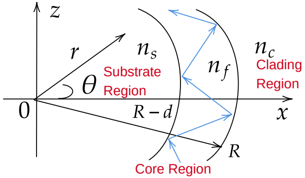

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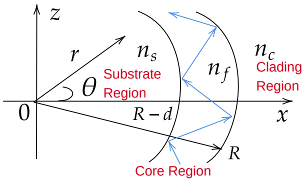

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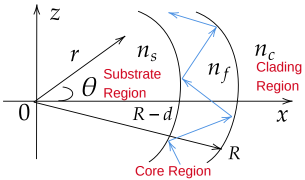

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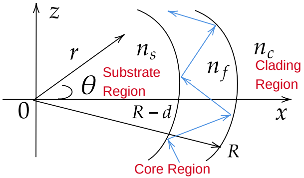

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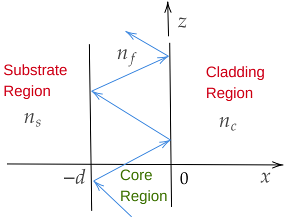

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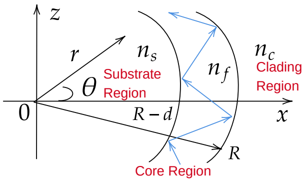

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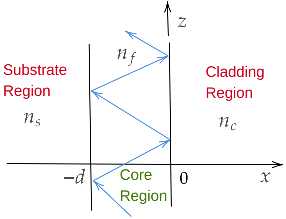

Non-self-adjoint problem $\xrightarrow{R}$ Self-adjoint problem

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Waveguides eigenvalue problems

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Straight Waveguide $(L_S, \frac{\beta_S^2}{k^2}, \phi)$

Waveguides eigenvalue problems

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- Eigenvalues are real.

Waveguides eigenvalue problems

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Waveguides eigenvalue problems

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Answers? How?

Relationship between real and imaginary part of eigenvalues

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Asymptotic expansion of the ψ for $r \to \infty$

$$\psi \sim A_c\sqrt{\frac{2}{\pi n_c kr}}\exp -\imath\left(n_c kr - \gamma R\frac{\pi}{2} - \frac{\pi}{4}\right). \tag{5}$$

Ref: K.R. Hiremath, M.Hammer, R. Stoffer, L. Prkna, and J. Čtyroký, Analytic approach to dielectric optical bent slab waveguides. Optical and quantum electronics, 37(1),2005, pp.37-61.

Relationship between real and imaginary part of eigenvalues

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Using asymptotic expansion in Eq. (4), one gets

$$\beta = |A_c|^2 \frac{1}{\alpha R \pi} \exp(\alpha R \pi), \tag{6}$$

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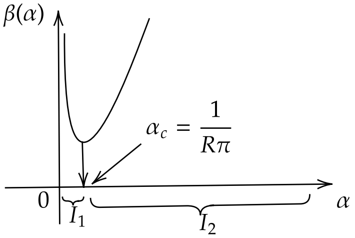

Shows the relationship between phase propagation constant β and attenuation constant α. At $\alpha_c = \frac{1}{R\pi}, \beta$ changes its behavior. For finite R, $\beta \nrightarrow 0$.

General result on finite number of eigenvalues for a operator

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Theorem (Compactness criterion)

Let $T : \mathbb{X} \to \mathbb{Y}$ be a linear operator where both $\mathbb{X}$ and $\mathbb{Y}$ are normed space.

General result on finite number of eigenvalues for a operator

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Let $T : \mathbb{X} \to \mathbb{Y}$ *be a linear operator where both* $\mathbb{X}$ *and* $\mathbb{Y}$ *are normed space. Then* T *is compact iff it maps every bounded sequence* $\{\psi_n\}$ *in* $\mathbb{X}$ *onto a sequence* $\{T\psi_n\}$ *in* $\mathbb{Y}$ *which has a convergent subsequence.*

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Theorem (Accumulation point of eigenvalues set)

Let $T : \mathbb{X} \to \mathbb{X}$ be any compact linear operator where $\mathbb{X}$ is a normed space.

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Theorem (Accumulation point of eigenvalues set)

Let $T : \mathbb{X} \to \mathbb{X}$ be any compact linear operator where $\mathbb{X}$ is a normed space. Then set of the eigenvalues of the operator T is countable and the only possible accumulation point is zero.

Ref. E. Kreyszig, Introductory functional analysis with applications. Vol. 17. John Wiley and Sons, 1991.

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Theorem (Finiteness of number of eigenvalues)

Let $T : \mathcal{D}(T) \subseteq \mathbb{X} \to \mathbb{X}$ be a bounded linear operator where domain $\mathcal{D}(T)$ of operator T is a compact normed space, and $\mathbb{X}$ is a Banach space.

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Finite number of eigenvalues for operator L_R

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The operator $L_R : \mathbb{V}_R \subseteq \mathcal{L}^2[0,\infty) \to \mathcal{L}^2[0,\infty)$ defined on the Banach space. According to this previous theorem, if domain $\mathbb{V}_R$ is compact set and L_R is bounded then operator L_R will be compact operator.

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A subset J of $(\mathcal{L}^p(\mathbb{R}^n), \|.\|_p), 1 \leq p < \infty$ is totally bounded in $\mathcal{L}^p(\mathbb{R}^n)$ iff the following conditions hold

1. *J is bounded i.e. there exist an $M > 0$ such that $\|f\|_p \leq M$ for every $f \in J$.*
2. *For each $\epsilon > 0$, there is a $\eta > 0$ such that $|t| < \eta$ and $f \in J$ implies $\int_{\mathbb{R}^n} |f(t + x) - f(x)|^p dx \leq \epsilon^p$.*
3. $\lim_{n \to \infty} \int_{|x|>n} |f(x)|^p dx = 0$ *for every $f \in J$.*

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Totally bounded ($\mathbb{V}_R$) and Completeness ($\mathcal{L}^2[0,\infty)$) $\implies$ Relatively compact ($\mathbb{V}_R$).

Finite number of eigenvalues for operator L_R

The operator $L_R : \mathbb{V}_R \subseteq \mathcal{L}^2[0,\infty) \to \mathcal{L}^2[0,\infty)$ defined on the Banach space. According to this previous theorem, if domain $\mathbb{V}_R$ is compact set and L_R is bounded then operator L_R will be compact operator.

Theorem (Kolmogorov–Riesz theorem)

A subset J of $(\mathcal{L}^p(\mathbb{R}^n), \|.\|_p), 1 \leq p < \infty$ is totally bounded in $\mathcal{L}^p(\mathbb{R}^n)$ iff the following conditions hold

1. *J is bounded i.e. there exist an $M > 0$ such that $\|f\|_p \leq M$ for every $f \in J$.*
2. *For each $\epsilon > 0$, there is a $\eta > 0$ such that $|t| < \eta$ and $f \in J$ implies $\int_{\mathbb{R}^n} |f(t+x) - f(x)|^p dx \leq \epsilon^p$.*
3. $\lim_{n\to\infty} \int_{|x|>n} |f(x)|^p dx = 0$ *for every $f \in J$.*

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L_R, Non-self-adjoint operator having finite number of distinct eigenvalues.

Ref. H. H. Olsen, H. Holden, and E. Malinnikova, An improvement of the Kolmogorov–Riesz compactness theorem. Expositiones Mathematicae, 37(1), 2019, pp.84-91.

Orthogonality of eigenfunctions

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Consider

$$L_R\psi := \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi. \tag{7}$$

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For another eigenfunction ϕ and eigenvalue $\delta \neq \gamma$

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L_R, Non-self-adjoint operator with orthogonal eigenfunctions

Summary

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- Finiteness of the eigenvalues set.

Namaste, Thank You

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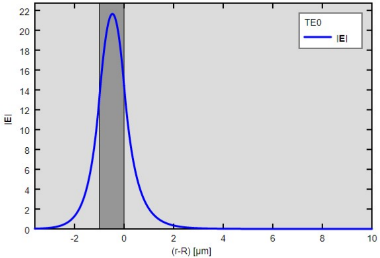
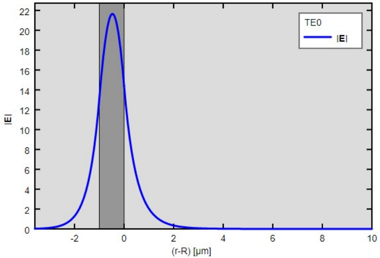
Asymptotic expansion of eigenfunctions

Asymptotic expansion of eigenfunctions



Bent mode profile for large R

Asymptotic expansion of eigenfunctions



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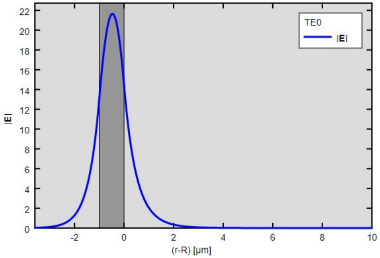

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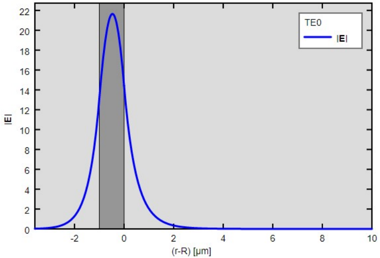

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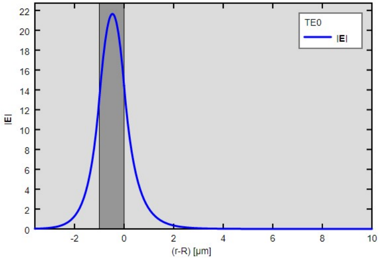

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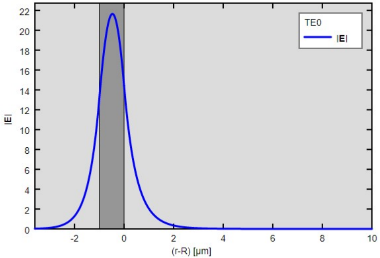

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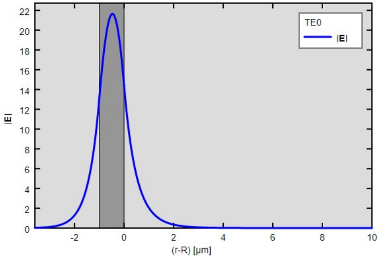

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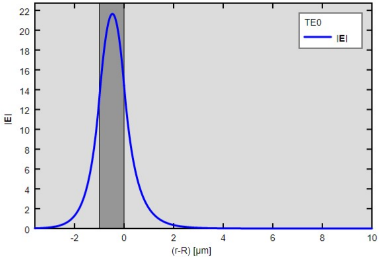

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