The diamagnetic inequality for the Dirichlet-to-Neumann operator

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The diamagnetic inequality for Laplacian

Let $\vec{a} = (a_1, \ldots, a_d)$ with $a_k \in L_{2,\mathrm{loc}}(\mathbb{R}^d)$ for all $k \in \{1, \ldots, d\}$.
Set $H(\vec{a}) = (\nabla - i\vec{a})^*(\nabla - i\vec{a})$.
Then

$$|e^{-tH(\vec{a})} f| \leq e^{t\Delta} |f|$$

for all $t > 0$ and $f \in L_2(\mathbb{R}^d)$.

The same result holds in presence of a real-valued potential V, i.e., with operators $H(\vec{a}) + V$ and $-\Delta + V$.

The setting (1)

Let $\Omega \subset \mathbb{R}^d$ bounded open with Lipschitz boundary Γ.
Let $c_{kl}, b_k, c_k, a_0 \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$.
Ellipticity condition: there exists a $\mu > 0$ such that

$$\mathrm{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and almost every $x \in \Omega$.

The setting (2)

Consider form $\mathfrak{a}\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$

$$\mathfrak{a}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_l u)\,\overline{\partial_k v} + \sum_{k=1}^{d} \int_\Omega \Big(b_k\,u\,\overline{\partial_k v}\Big) + c_k\,(\partial_k u)\,\overline{v} + \int_\Omega a_0\,u\,\overline{v}.$$

Define $\mathcal{A}\colon W^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\langle \mathcal{A}u, v\rangle_{W^{-1,2}(\Omega)\times W_0^{1,2}(\Omega)} = \mathfrak{a}(u,v).$$

Let $\psi \in L_2(\Gamma)$ and $u \in W^{1,2}(\Omega)$ with $\mathcal{A}u \in L_2(\Omega)$.
Definition: u has **weak conormal derivative** ψ if

$$\mathfrak{a}(u,v) - (\mathcal{A}u, v)_{L_2(\Omega)} = (\psi, \mathrm{Tr}\, v)_{L_2(\Gamma)} \quad \text{for all } v \in W^{1,2}(\Omega).$$

Notation $\partial_\nu^{\mathfrak{a}} u = \psi$.

The Dirichlet-to-Neumann operator $\mathcal{N}$

Assumption: 0 is not a Dirichlet eigenvalue.
Definition: A function $u \in W^{1,2}(\Omega)$ is called $\mathcal{A}$-**harmonic** if

$$\mathfrak{a}(u, v) = 0 \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

For all $\varphi \in H^{1/2}(\Gamma)$ there is a unique $\mathcal{A}$-harmonic $u \in W^{1,2}(\Omega)$ such that $\operatorname{Tr} u = \varphi$.
IF u has a weak conormal derivative, then we say

$$\varphi \in D(\mathcal{N}) \text{ and } \mathcal{N}\varphi = \partial_\nu^{\mathfrak{a}} u.$$

The operator $-\mathcal{N}$ is the generator of a C_0-semigroup.

Two form methods

Define $\mathfrak{b}\colon H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{C}$ by

$$\mathfrak{b}(\varphi, \xi) := \mathfrak{a}(u, v),$$

where $u, v \in W^{1,2}(\Omega)$ are $\mathcal{A}$-harmonic with $\operatorname{Tr} u = \varphi$ and $\operatorname{Tr} v = \xi$, respectively.

Then $\mathfrak{b}$ is a densely defined continuous elliptic form and $\mathcal{N}$ is the associated operator.

The operator $\mathcal{N}$ is the operator associated with $\mathfrak{b}$ in the following sense: Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if and only if $\varphi \in D(\mathfrak{b})$ and

$$\mathfrak{b}(\varphi, \xi) = (\psi, \xi)_{L_2(\Gamma)} \quad \text{for all } \xi \in D(\mathfrak{b}).$$

Second form method

Let V and H be Hilbert spaces.
Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form.
Let $j\colon V \to H$ be a continuous operator with dense range.
Suppose $\mathfrak{a}$ is j-elliptic, that is, there are $\mu > 0$ and $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \, \|j(u)\|_H^2 \geq \mu \, \|u\|_V^2 \quad \text{for all } u \in V.$$

The operator A **associated with** $(\mathfrak{a}, j)$ is defined as follows:
Let $x, f \in H$. Then $x \in D(A)$ and $Ax = f$ if and only if there exists a $u \in V$ such that $j(u) = x$ and

$$\mathfrak{a}(u,v) = (f, j(v))_H \quad \text{for all } v \in V.$$

Theorem (Arendt–tE). The operator A is well defined and $-A$ is the generator of a holomorphic C_0-semigroup in H.

In our case, if $\mathfrak{a}$ is Tr-elliptic, then the operator $\mathcal{N}$ is the operator associated with $(\mathfrak{a}, \operatorname{Tr})$.

The magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$

Let $\vec{a} := (a_1, \ldots, a_d)$ with $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k \in \{1, \ldots, d\}$. Set

$$D_k := \partial_k - i a_k$$

Consider form $\mathfrak{a}(\vec{a}) \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$

$$\mathfrak{a}(\vec{a})(u, v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl} \, (D_l u) \, \overline{D_k v} + \sum_{k=1}^{d} \int_\Omega \Big(b_k \, u \, \overline{D_k v} \Big) + c_k \, (D_k u) \, \overline{v} + \int_\Omega a_0 \, u \, \overline{v}.$$

Assumption: 0 is not a Dirichlet eigenvalue.

Define similarly that an element of $W^{1,2}(\Omega)$ is $\mathcal{A}(\vec{a})$-harmonic and the magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$.

Formally, if $\varphi \in D(\mathcal{N}(\vec{a}))$ and $u \in W^{1,2}(\Omega)$ is $\mathcal{A}(\vec{a})$-harmonic with trace $\operatorname{Tr} u = \varphi$, then

$$\mathcal{N}(\vec{a})\varphi = \partial_\nu^{\mathfrak{a}(\vec{a})} u = \sum_{k,l=1}^{d} \nu_k \operatorname{Tr} (c_{kl} \, \partial_l u) - i \sum_{k,l=1}^{d} \nu_k \operatorname{Tr} (c_{kl} \, a_l \, u) + \sum_{k=1}^{d} \nu_k \operatorname{Tr} (b_k \, u)$$

The diamagnetic inequality

Let $T_{\vec{a}} = (T_{\vec{a}}(t))_{t>0}$ and $T = (T(t))_{t>0}$ be the semigroups generated by $-\mathcal{N}(\vec{a})$ and $-\mathcal{N}$ on $L_2(\Gamma)$, respectively.

Theorem (tE–Ouhabaz). Suppose $\mathfrak{a}$ is accretive and there exist $\mu, \omega > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \, \|\operatorname{Tr} u\|^2_{L_2(\Gamma)} \geq \mu \, \|u\|^2_{W^{1,2}(\Omega)} \quad \text{for all } u \in W^{1,2}(\Omega).$$

Then

$$|T_{\vec{a}}(t)\varphi| \leq T(t)|\varphi|$$

for all $t > 0$ and $\varphi \in L_2(\Gamma)$.

Kernel bounds

Suppose Ω is of class $C^{1+\kappa}$ for some $\kappa > 0$.
Suppose also that $c_{kl} = c_{lk} \in C^\kappa(\Omega, \mathbb{R})$, $b_k = c_k = 0$ and $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$.
Suppose $a_0 \geq 0$ a.e. on Ω.
Then $T_{\vec{a}}$ has a kernel $K_{\vec{a}}$ and there exists a constant $c > 0$ such that

$$|K_{\vec{a}}(t, z, w)| \leq \frac{c\,(t \wedge 1)^{-(d-1)}\, e^{-\lambda_1 t}}{\left(1 + \frac{|z - w|}{t}\right)^d}$$

for all $z, w \in \Gamma$ and $t > 0$, where λ_1 is the first eigenvalue of the operator $\mathcal{N}(\vec{a})$.

Hölder continuous kernel bounds

Same assumptions. In addition suppose that $d \geq 3$.
Then for all $\varepsilon, \tau' \in (0,1)$, $\tau > 0$ there exist $c, \nu > 0$ such that

$$|K_{\vec{a}}(t,z,w) - K_{\vec{a}}(t,z',w')|$$

$$\leq c\,(t \wedge 1)^{-(d-1)} \left(\frac{|z-z'| + |w-w'|}{t + |z-w|}\right)^{\nu} \frac{1}{\left(1 + \frac{|z-w|}{t}\right)^{d-\varepsilon}} \,(1+t)^{\nu}\, e^{-\lambda_1 t}$$

for all $z, w, z', w' \in \Gamma$ and $t > 0$ with $|z-z'| + |w-w'| \leq \tau\, t + \tau'\, |z-w|$.

Sketch of proof

The diamagnetic inequality is obtained by proving the invariance of the closed convex set

$$\{(\varphi, \psi) \in L_2(\Gamma) \times L_2(\Gamma) : |\varphi| \leq \psi\}$$

for the semigroup

$$\begin{pmatrix} T_{\vec{a}}(t) & 0 \\ 0 & T(t) \end{pmatrix}_{t>0} .$$

Invariance of closed convex sets

Let V and $\widetilde{H}$ be Hilbert spaces with V densely and continuously embedded in $\widetilde{H}$.
Let $\mathfrak{a} \colon V \times V \to \mathbb{C}$ be a continuous accretive sesquilinear form.
Suppose $\mathfrak{a}$ is elliptic, that is i-elliptic, where i is the inclusion map.
Let $\widetilde{S}$ be the associated semigroup.
Let $\widetilde{C} \subset \widetilde{H}$ be a non-empty closed convex set and let $\widetilde{P} \colon \widetilde{H} \to \widetilde{C}$ be the projection.

Theorem (Ouhabaz). The following are equivalent.

- $\widetilde{C}$ is invariant under $\widetilde{S}$, that is $\widetilde{S}_t \widetilde{C} \subset \widetilde{C}$ for all $t > 0$.
- $\widetilde{P} V \subset V$ and $\operatorname{Re} \mathfrak{a}(\widetilde{P} u, u - \widetilde{P} u) \geq 0$ for all $u \in V$.
- $\widetilde{P} V \subset V$ and $\operatorname{Re} \mathfrak{a}(u, u - \widetilde{P} u) \geq 0$ for all $u \in V$.

Invariance of closed convex sets

Let V and H be Hilbert spaces.
Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form.
Let $j\colon V \to H$ be a continuous operator with dense range.
Suppose $\mathfrak{a}$ is j-elliptic and accretive.
Let S be the semigroup associated with $(\mathfrak{a}, j)$.
Let $C \subset H$ be a non-empty closed convex set and let $P\colon H \to C$ be the projection.

Theorem (Arendt–tE). The following are equivalent.

- C is invariant under S, that is $S_t C \subset C$ for all $t > 0$.
- For all $u \in V$ there exists a $w \in V$ such that $P(j(u)) = j(w)$ and $\operatorname{Re} \mathfrak{a}(w, u - w) \geq 0$.
- For all $u \in V$ there exists a $w \in V$ such that $P(j(u)) = j(w)$ and $\operatorname{Re} \mathfrak{a}(u, u - w) \geq 0$.

Invariance of closed convex sets

Proposition. Let $C \subset H$ be a non-empty closed convex set and let $P \colon H \to C$ be the projection.
Let $\widetilde{C} \subset \widetilde{H}$ be a non-empty closed convex set and let $\widetilde{P} \colon \widetilde{H} \to \widetilde{C}$ be the projection.
Suppose $\mathfrak{a}$ is j-elliptic and accretive.
Suppose $\widetilde{C}$ is invariant under the semigroup $\widetilde{S}$ and

$$P \circ j = j \circ \widetilde{P} \quad \text{on } V.$$

Then C is invariant under the semigroup S.

Our situation

$V = W^{1,2}(\Omega)$.
$H = L_2(\Gamma)$.
$\widetilde{H} = L_2(\Omega)$.
$j = \mathrm{Tr} : W^{1,2}(\Omega) \to L_2(\Gamma)$.
S semigroup generated by Dirichlet-to-Neumann operator.
$\widetilde{S}$ semigroup on $L_2(\Omega)$ with Neumann boundary conditions.

We need in addition to prove a diamagnetic inequality for differential operators in divergence form with lower-order terms and Neumann boundary conditions on Ω.

The latter was done by Hundertmark and Simon for the Laplacian.

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