Lower Bound Methods for Sign-rank

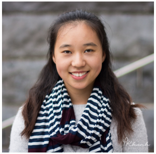
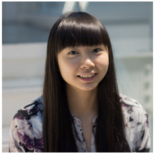
Communication Complexity and Applications III

Hamed Hatami

McGill University, School of Computer Science

Joint work with

- Pooya Hatami, William Pires, Ran Tao, Rosie Zhao, Lower Bound Methods for Sign-rank and their Limitations.

- Work in progress with Kaave Hosseini and Xiang Meng.

Sign Matrices as Binary Concept Classes

Matrix $A_{\mathcal{X}\times\mathcal{Y}}$ with ± 1 entries. Entry A_{xy} can represent:

- Person x likes/dislikes movie y.

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- Person x likes/dislikes movie y.

- Image x represents an object y. (Muffin, Chihuahua?)

- For person x, email y is spam/non-spam.

Geometric Representations

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Representation of this concept in $\mathbb{R}^4$:

$$A_{xy} = \text{sgn } \langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, -1) \rangle.$$

Sign-rank

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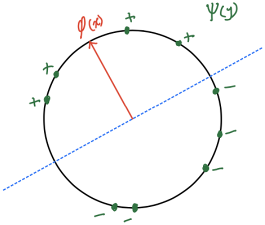
Definition (Sign-rank)

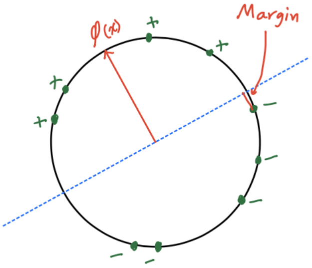
Sign-rank of a sign-matrix $A_{\mathcal{X}\times\mathcal{Y}}$ is the smallest d such that there are $\phi : \mathcal{X} \to \mathbf{S}^{d-1}$ and $\psi : \mathcal{Y} \to \mathbf{S}^{d-1}$ with

$$A_{xy} = \mathrm{sgn}\langle \phi(x), \psi(y) \rangle.$$



$$A_{xy} = 1 \iff \psi(y) \in \{z \mid \langle z, \phi(x) \rangle > 0\}.$$

Margin



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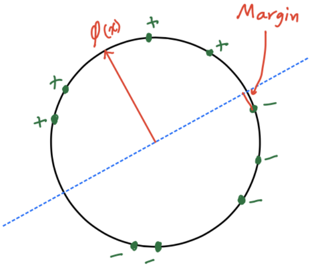

Definition (Margin)

- Margin of such a representation:

$$\inf_{x,y} |\langle \phi(x), \psi(y) \rangle|.$$

- Margin of A denoted by $\mathrm{m}(A)$: Largest possible margin over all representations in all dimensions.

Learning Theory: Low complexity concept classes

- Bounded VC-dimension (PAC learnable).
- Bounded Sign-rank (Linearization/Kernel Trick, low dimensional).
- Margin bounded away from zero (amenable to algorithms such as perceptron, Support vector machines).

Sign-rank Lower Bounds: What do we know?

Known Lower-bound Techniques

- Counting argument [AFR86, AMR16]: For $d \leq \frac{n}{2}$, there are only $2^{dn\log(n)}$ matrices of sign-rank d (out of all 2^{n^2} sign matrices).

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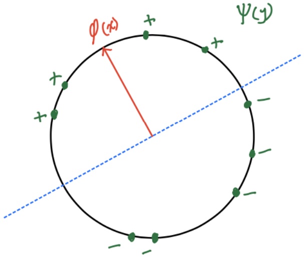
- Hence, most sign matrices have large sign-rank.
- Based of works of Milnor, Thom, Warren in 1960's on the number of connected components of real algebraic varieties.

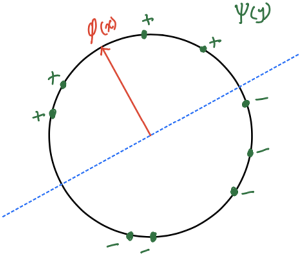
VC dimension



Theorem (VC dimension [Paturi-Simon 85])

$$\mathrm{rk}_{\pm}(A) \geq \mathsf{VC}(A).$$

Average Margin



- Forster based methods: "Small sign-rank $\Longrightarrow$ Large average margin"

$$\frac{1}{\mathsf{m}^{\mathsf{avg}}(A)} \leq \mathrm{rk}_{\pm}(A).$$

(Refinements of Forster's original bound were later developed by Linial, Shraibman, Sherstov, Razbrov, etc).

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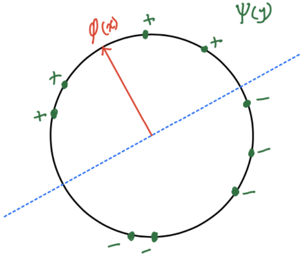

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- All these refinements prove upper-bounds on $\mathsf{m}^{\mathsf{avg}}(A)$.

Large Monochromatic rectangles

Theorem (Monochromatic rectangle [APPRRS 2005])

If $\mathrm{rk}_{\pm}(A) = d$*, then* $A_{n\times n}$ *contains an* $\frac{n}{2^{d+1}} \times \frac{n}{2^{d+1}}$ *monochromatic rectangle.*

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If $\mathrm{rk}_{\pm}(A) = d$*, then* $A_{n\times n}$ *contains an* $\frac{n}{2^{d+1}} \times \frac{n}{2^{d+1}}$ *monochromatic rectangle.*

By looking at all submatrices of A, and the size of the largest monochromatic rectangles in them, we define rect(A), and get

$$\log_2(\mathrm{rect}(A)) \lesssim \mathrm{rk}_{\pm}(A).$$

A comparison

Known lower bound techniques: $\mathrm{rk}_{\pm}(A)$ is (essentially) at least

$$\mathsf{VC}(A), \qquad \mathsf{m}^{\mathsf{avg}}(A)^{-1}, \qquad \log_2(\mathsf{rect}(A)).$$

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If the monochromatic rectangle ratio cannot provide a super-constant lower bound for the sign-rank of a matrix, then the other two methods will fail as well.

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- $$\sqrt{\mathsf{VC}(A)} \leq \mathsf{m}^{\mathsf{avg}}(A)^{-1} \leq \mathsf{rect}(A).$$
- *There exist $n \times n$ sign-matrices with $\mathsf{rect}(A) = O(1)$ and $\mathrm{rk}_{\pm}(A) \geq n^{\Omega(1)}$.*

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Problem

Construct an explicit sequence of sign-matrices A_n with

$$\text{rect}(A_n) = O(1) \qquad \text{and} \qquad \lim_n \mathrm{rk}_{\pm}(A_n) = \infty.$$

Two open problems

Problem I: Semi-algebraic matrices

Semi-algebraic matrices

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Matrices of sign-rank d are semi-algebraic: $\sum_{i=1}^{d} x_i y_i > 0$.

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Problem

Semi-algebraic $\equiv$ Bounded Sign-Rank?

A Simple Reformulation

Problem (Reformulation of Sign-rank $\stackrel{?}{\equiv}$ Semi-algebraic)

Is it true that for every d, there is $c_d \in \mathbb{N}$ such that

$$\mathrm{rk}_{\pm}(A), \mathrm{rk}_{\pm}(B) \leq d \Longrightarrow \mathrm{rk}_{\pm}(A \wedge B) \leq c_d?$$

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- Non-trivial for $d = 2$.
- Open for $d \geq 3$.
- Using Forster's method [Bun, Mande, Thaler'19]:

$$c_d \geq 2^{\log^2(d)}.$$

Second Reformulation

Definition (Intersection of Two Half-spaces)

For $[x_1, x_2] \in \mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}^d$ and $y \in \mathcal{Y} \subset \mathbb{R}^d$, define

$$\mathcal{I}_d([x_1, x_2], y) = \begin{cases} 1 & y \in H_{x_1} \cap H_{x_2} \\ -1 & \text{otherwise} \end{cases},$$

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Open for $d \geq 4$.

Problem II: Large Margin $\Rightarrow$ Low Sign-rank?

Problem ([Linial, Mendelson, Schechtman, Shraibman'07])

Does "large margin" imply bounded sign-rank:

$$\mathsf{m}(A) = \Omega(1) \Longrightarrow \mathrm{rk}_{\pm}(A) = O(1)?$$

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Equivalent formulations of LMSS:

$$\begin{matrix} \mathrm{m}(A), \mathrm{Disc}(A) = \Omega(1) \\ \mathrm{R}(A), \|A\|_{\gamma_2,\epsilon} = O(1) \end{matrix} \Longrightarrow \begin{matrix} \mathrm{rk}_{\pm}(A) = O(1) \\ \mathrm{UPP}(A) = O(1) \end{matrix} ?$$

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Problem (The CC formulation)

$$\mathrm{R}(A) = O(1) \Longrightarrow \mathrm{UPP}(A) = O(1)?$$

Conjecture (Towards a negative answer to LMSS)

Let $Q_d : \{0,1\}^d \times \{0,1\}^d \to \{-1,1\}$ be the (sign) adjacency matrix of the d-dimensional hypercube:

$$Q_d(x,y) = -1 \iff \|x-y\|_1 = 1.$$

Is it true

$$\lim_{d\to\infty} \mathrm{rk}_{\pm}(Q_d) = \infty?$$

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$$\mathsf{R}(Q_d) = O(1).$$

- If the above Conj is true, then

$$\mathsf{m}(A) = \Omega(1) \not\Longrightarrow \mathrm{rk}_\pm(A) = O(1).$$

Summary

Problem (Intersection of Half-spaces)

Is it true that

$$\mathrm{rk}_{\pm}(\mathcal{I}_d) < c_d?$$

Problem (Hypercubes)

Let Q_d be the (sign) adjacency matrix of the d-dimensional hypercube. We have

$$\lim_{d \to \infty} \mathrm{rk}_{\pm}(Q_d) = \infty?$$

Beyond the reach of discussed lower bound techniques!

We have $\mathrm{rect}(\mathcal{I}_d) = O(1)$ and $\mathrm{rect}(Q_d) = O(1)$.

A separation of Margin vs Sign-rank for partial functions (Joint work with Kaave and Xiang)

The statement for Partial Functions

Problem

Are there partial matrices A with

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- Not known for total functions (hypercube is a candidate).
- Partial functions: Canonical candidate

$$f : \mathbf{S}^{d-1} \times \mathbf{S}^{d-1} \to \{-1, 1, *\}$$

$$f(x, y) = \begin{cases} 1 & \langle x, y \rangle \geq \epsilon \\ -1 & \langle x, y \rangle \leq -\epsilon \\ * & \text{otherwise} \end{cases}.$$

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For $\epsilon < 1$, every completion of f has sign-rank at least d.

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For $\epsilon < 1$, every completion of f has sign-rank at least d.

- Sharpness: $g(x,y) := \mathrm{sgn}\langle x, y\rangle$ has sign-rank d.

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- Note $\mathrm{R}(f) = O(1)$ and $\mathrm{UPP}(f) = \log_2(d) \pm O(1)$.
- The proof is short but uses Borsuk-Ulam: Every continuous $\phi : \mathbf{S}^{d-1} \to \mathbb{R}^{d-1}$ satisfies $\phi(x) = \phi(-x)$ for some x.

A related problem

$$f : \mathbf{S}^{d-1} \times \mathbf{S}^{d-1} \to \{-1, 1, *\}$$

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Conjecture ([Alon, Hanneke, Holzman, Moran'21])

Every completion of f to a total function have VC dimension $\geq c_d$ with $\lim_{d\to\infty} c_d = \infty$.

Discretization: Large-Gap-Hamming Distance

$$G : \{0,1\}^d \times \{0,1\}^d \to \{-1,1,*\}$$

$$G(x,y) = \begin{cases} 1 & \langle x,y \rangle \geq d\epsilon \\ -1 & \langle x,y \rangle \leq -d\epsilon \\ * & \text{otherwise} \end{cases} .$$

Theorem (HH,Hosseini,Meng'22)

Sign-rank of G is $\Omega(d/\log^2 d)$.

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$$G : \{0,1\}^d \times \{0,1\}^d \to \{-1, 1, *\}$$

$$G(x,y) = \begin{cases} 1 & \langle x, y \rangle \geq d\epsilon \\ -1 & \langle x, y \rangle \leq -d\epsilon \\ * & \text{otherwise} \end{cases} .$$

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- The unbounded-error randomized CC of G is $\Omega(\log(d))$ (Sharp by Newman's lemma).

Conclusion: More Open Problems

Open Problems

- Recall the conjecture (hypercubes):

$$\mathsf{m}(A) = \Omega(1), \quad \|A\|_{\gamma_2,\epsilon} = O(1) \not\Longrightarrow \mathrm{rk}_{\pm}(A) = O(1).$$

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Theorem ([HH,Hatami,Pires,Tao,Zhao'22])

We have

$$\mathrm{rk}_{\pm}(A) \le 4^{\mathrm{D}^{\mathrm{EQ}}(A)}.$$

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The above conjecture is true for XOR-functions.

- The proof uses Green and Sanders' quantitative version of Cohen's idempotent theorem. If the conj is true, then it characterizes idempotents of the algebra of Schur multipliers.

Thank You For Your Attention!