Strong XOR Lemma for Communication with Bounded Rounds

Huacheng Yu

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n-fold XOR function

For function $f : \mathcal{Z} \to \{0,1\}$, its n-fold XOR $f^{\oplus n} : \mathcal{Z}^n \to \{0,1\}$ is:

$$f^{\oplus n}(Z_1, \ldots, Z_n) = f(Z_1) \oplus \cdots \oplus f(Z_n)$$

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This talk: "CC of f" vs "CC of $f^{\oplus n}$"

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If this is the best possible, then

- moderately hard Boolean-valued $f \implies$ very hard Boolean-valued $f^{\oplus n}$

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Previous XOR lemmas:

- query complexity [Dru'12, BKLS'20]
- w/o n times more resource: circuit complexity [Yao'82], streaming alg [AN'21]
- w/o exponentially small adv: information complexity [BBCR'10]

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- w/o exponentially small adv: information complexity [BBCR'10]
- communication complexity & functions with small discrepancy [Shaltiel'03]
- ...

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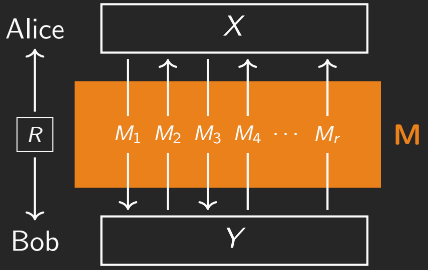
r-round communication for $f(X, Y)$:

- input pair (X, Y), public random bits R
- Alice speaks in odd rounds, Bob speaks in even rounds
- **M** determines the output



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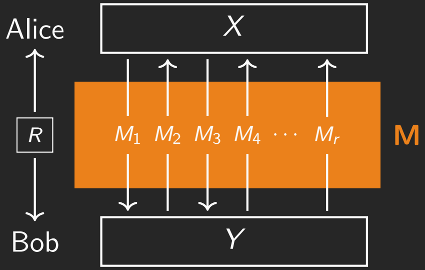

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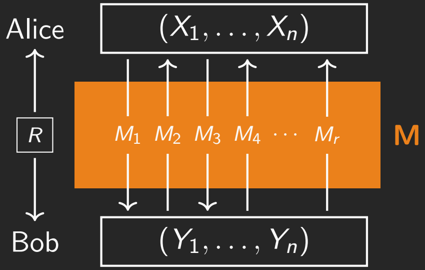

A strong XOR lemma for bounded-round communication

Let $\mathbf{R}_q^{(r)}(f)$ be the min communication cost to compute f in r rounds with prob q.

Theorem

For any f and r, we must have

$$\mathbf{R}^{(r)}_{1/2+2^{-n}}(f^{\oplus n}) \geq n \cdot \left(r^{-O(r)} \cdot \mathbf{R}^{(r)}_{2/3}(f) - 1\right).$$

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- "$-O(1)$" is needed: $f(X_i, Y_i) = X_{i,1} \oplus Y_{i,1}$ (XOR of 1st bit)

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[BBCR'10]: XOR lemma for info complexity $\Longleftarrow$ starting point of our proof
(with const adv instead of 2^{-n})

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Distributional strong XOR lemma

We also prove a strong XOR lemma w.r.t. a fixed input distribution μ:

Theorem

If every r-round C-bit comm. protocol computes f under input dist. μ w.p. at most

$$1/2 + \alpha/2,$$

then every r-round $o(r^{-1}nC)$-bit protocol computes $f^{\oplus n}$ under μ^n w.p. at most

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distributional strong XOR lemma + Yao's minimax + repetition $\Longrightarrow$ main theorem

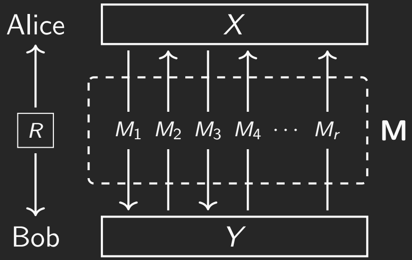
Outline

Rest of the talk, focus on distributional strong XOR lemma:

- alternative view of the XOR lemma for information complexity [BBCR'10]
- obtaining exponentially small advantage

Information complexity

input distribution μ + protocol defines a joint distribution π over $(X, Y, R, \mathbf{M})$...



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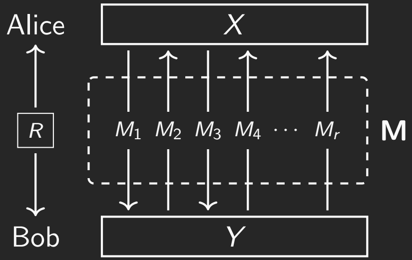

(internal) information cost of π: $I(X; \mathbf{M} \mid Y, R) + I(Y; \mathbf{M} \mid X, R)$

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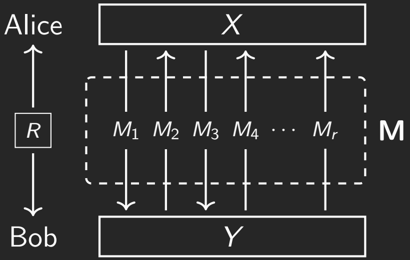

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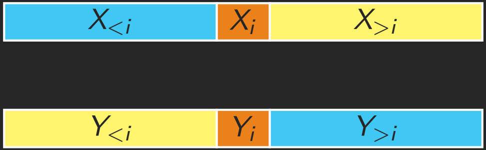
information complexity of f under μ: min information cost to compute f

XOR lemma for information complexity [BBCR'10]

[BBCR'10]: if info complexity of $f^{\oplus n}$ under μ^n is $\leq I$,
then info complexity of f under μ is $\leq I/n + O(1)$

(assuming success probability 1 for now)



fix π for $f^{\oplus n}$ with info cost I; Protocol τ for $f(x, y)$:

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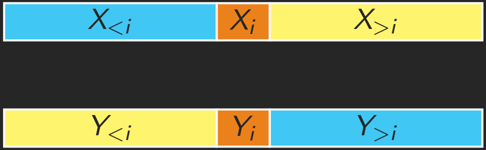

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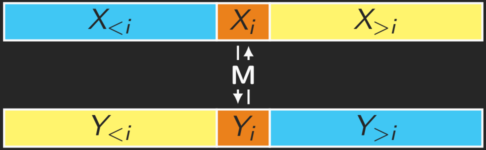

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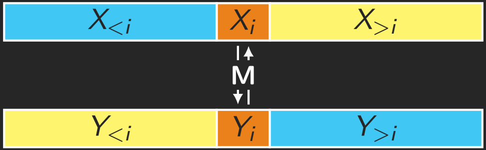

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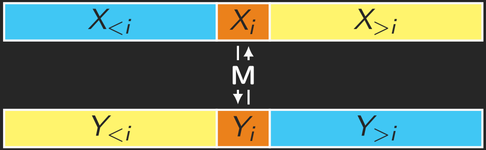

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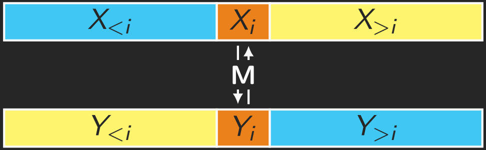

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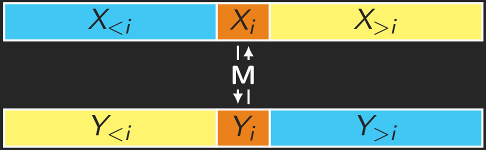

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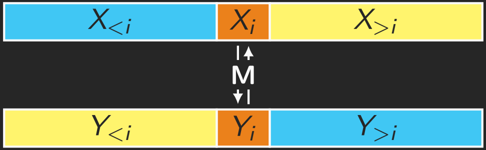

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sum up both terms: τ computes f with info cost $I/n + O(1)$

XOR lemma for information complexity [BBCR'10]

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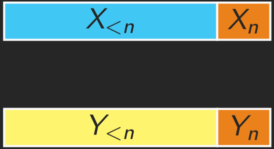
an alternative view of their proof:

- fix π for $f^{\oplus n}$ with info cost I
- "decompose" π into π_n for f and info cost I_1 and
  $\pi_{<n}$ for $f^{\oplus n-1}$ with info cost I_2

  such that $I_1 + I_2 = I + O(1)$

Decomposition of π



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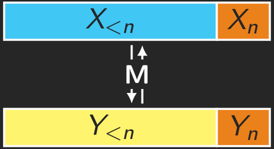

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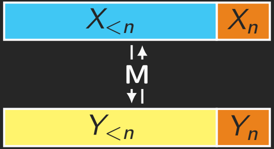

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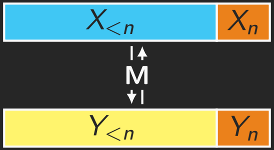

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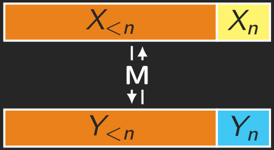

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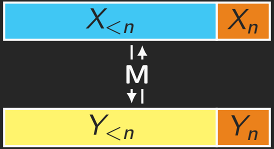

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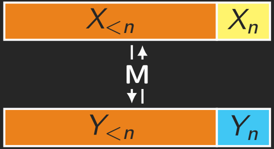

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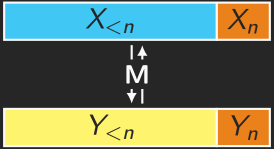

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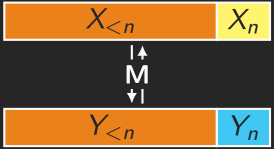

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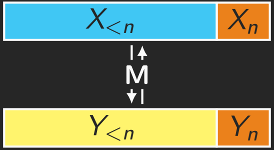
- run π and Bob sends $f(X_n, Y_n)$

Cost: $I(X_{<n}; \mathbf{M} \mid X_n, Y, R)$ (1st term)

1st terms in costs sum up to $I(X; \mathbf{M} \mid Y, R) + 1$ by chain rule

Decomposition of π

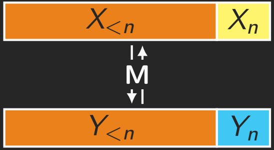


Input: 1 pair

Protocol π_n:

- view input as X_n and Y_n
- publicly sample $Y_{<n}$
- Alice priv. samples $X_{<n}$ cond. on $Y_{<n}$
- run π and A. sends $f^{\oplus n-1}(X_{<n}, Y_{<n})$

Cost: $I(X_n; \mathbf{M} \mid Y, R) + 1$ (1st term)



Input: $n-1$ pairs

Protocol $\pi_{<n}$:

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- run π and Bob sends $f(X_n, Y_n)$

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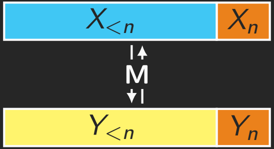
1st terms in costs sum up to $I(X; \mathbf{M} \mid Y, R) + 1$ by chain rule

2nd term is similar; info costs of $\pi_{<n}$ and π_n sum up to $I + O(1)$

Decomposition of π



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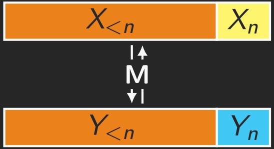

Input: $n-1$ pairs

Protocol $\pi_{<n}$

iteratively decomposing $\pi_{<n}$ gives n protocols for f

- i-th last: the original protocol when it embeds input into (X_i, Y_i)

Another view of decomposition of π

for the same underlying distribution of $(X, Y, R, \mathbf{M})$, we view different parts of it as inputs, public randomness, transcript (private randomness not important)

- π: inputs (X, Y), public randomness R, transcript $\mathbf{M}$
- π_n: inputs (X_n, Y_n), public rand. $(R, Y_{<n})$, transcript $(\mathbf{M}, f^{\oplus n-1}(X_{<n}, Y_{<n}))$
- $\pi_{<n}$: inputs $(X_{<n}, Y_{<n})$, public randomness (R, X_n), transcript $(\mathbf{M}, f(X_n, Y_n))$

Exponentially small advantage

given a protocol computing $f^{\oplus n}$ w.p. $2/3$ under μ^n with cost $o(nC)$

then there is a protocol computing f w.p. $2/3$ under μ with cost $\leq C$

Exponentially small advantage

To prove strong XOR lemma, need to show:

given a protocol computing $f^{\oplus n}$ w.p. $1/2 + \alpha^{o(n)}/2$ under μ^n with cost $o(nC)$

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Main challenge: design a decomposition that increases the advantage

Benefit of the alternative view

let $\mathrm{adv}(f \mid \mathbf{W}) := |2\Pr[f = 1 \mid \mathbf{W}] - 1| \in [0, 1]$ be the advantage for f cond. on $\mathbf{W}$

- given $\mathbf{W}$, one can predict f w.p. $1/2 + \mathrm{adv}(f \mid \mathbf{W})/2$
- $\mathrm{adv}(b_1 \oplus b_2) = \mathrm{adv}(b_1) \cdot \mathrm{adv}(b_2)$

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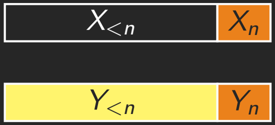
End of π_n, Alice knows $(X_n, Y_{<n}, R, \mathbf{M})$



: input, : public

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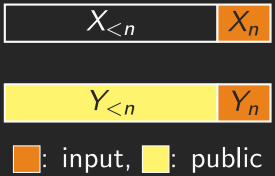

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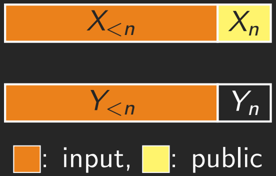
End of $\pi_{<n}$, Bob knows $(X_n, Y_{<n}, R, \mathbf{M})$



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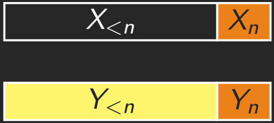

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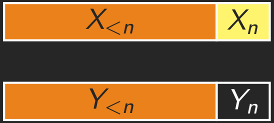

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Key obs: $f(X_n, Y_n)$ and $f^{\oplus n-1}(X_{<n}, Y_{<n})$ are independent cond. on $(X_n, Y_{<n}, R, \mathbf{M})$

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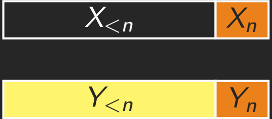

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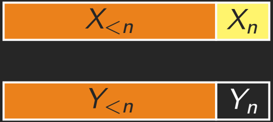

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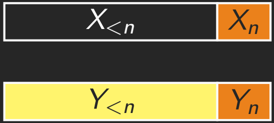

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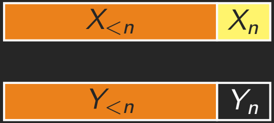

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Since $f^{\oplus n}(X, Y) = f^{\oplus n-1}(X_{<n}, Y_{<n}) \oplus f(X_n, Y_n)$,

$$\mathrm{adv}(f(X_n, Y_n) \mid X_n, Y_{<n}, R, \mathbf{M}) \cdot \mathrm{adv}(f^{\oplus n-1}(X_{<n}, Y_{<n}) \mid X_n, Y_{<n}, R, \mathbf{M})$$
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Pointwise equality for advantage

let $\mathrm{adv}(f \mid \mathbf{W}) := |2\Pr[f = 1 \mid \mathbf{W}] - 1| \in [0,1]$ be the advantage for f cond. on $\mathbf{W}$

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- decomposition increases the advantage

High-level proof strategy

Proof strategy:

1. given π for $f^{\oplus n}$, decompose into π_n for f and $\pi_{<n}$ for $f^{\oplus n-1}$
2. prove:
    2.1 if π_n has “high cost”: $\pi_{<n}$ has much “lower cost” than π
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3. if π has "low cost" and non-trivial adv: iterative decomposition gives a good protocol for f

"Exponential version" of info cost

Strong XOR lemma is false for info complexity

- compute $f^{\oplus n}$ exactly w.p. $1/n$; output random bit w.p. $1 - 1/n$

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Open problems

Given a protocol π computing $f^{\oplus n}$ with const prob

- obtain a protocol computing f w.p. $1 - O(1/n)$?

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Thank you for listening!