DIVIDED DIFFERENCE OPERATORS FOR HESSENBERG REPRESENTATIONS

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Abstract. The equivariant cohomology ring of a regular semisimple Hessenberg variety in type A is a free module over the equivariant cohomology ring of a point. When equipped with Tymoczko's dot action, it becomes a twisted representation of the symmetric group, and the character of this representation is given by the chromatic quasisymmetric function of an indifference graph. In this note, we use divided difference operators to decompose this representation as a direct sum of subrepresentations in a way that categorifies the modular relation between chromatic quasisymmetric functions.

1. Introduction

TODO

2. Notation and computations

2.1. **Context.** Fix a natural number n. Let S_n be the symmetric group of order n, viewed as functions from the set $\{1, 2, \dots, n\}$ to itself, so that

$$(vw)(i) = v(w(i)) \quad \text{for } v, w \in S_n \text{ and } i = 1, 2, \dots, n. \tag{2.1}$$

The group S_n is generated by the adjacent transpositions,

$$s_i = (i \leftrightarrow i + 1) \quad \text{for } i = 1, 2, \dots, n - 1, \tag{2.2}$$

which satisfy the relations

$$\begin{gathered} s_i^2 = \mathrm{id}, \quad s_i s_{(i+1)} s_i = s_{(i+1)} s_i s_{(i+1)}, \\ s_i s_k = s_k s_i \quad \text{for } k \neq i \pm 1. \end{gathered} \tag{2.3}$$

We will sometimes write a permutation in one-line notation, as

$$w = [w(1), w(2), \dots, w(n)]. \tag{2.4}$$

The overarching context for all that follows will be the ring of functions

$$H = \mathrm{Fun}(S_n, \mathbb{C}[t_1, \dots, t_n]), \tag{2.5}$$

whose elements can be seen as S_n-indexed tuples of multivariate polynomials, or as certain complex-valued functions on $S_n \times \mathbb{C}^n$, with pointwise

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addition and multiplication. The ring H is equipped with two S_n-actions: on the left, Tymoczko's *dot* action, defined for $v \in S_n$ and $f \in H$ by

$$(v \cdot f)(w, t_1, \dots, t_n) = f(v^{-1}w, t_{v(1)}, \dots, t_{v(n)}), \tag{2.6}$$

and on the right, the *star* action, defined by

$$(f * v)(w, t_1, \dots, t_n) = f(wv^{-1}, t_1, \dots, t_n) \tag{2.7}$$

The left action and the right action commute, so that

$$(uv) \cdot f = u \cdot (v \cdot f), \quad (u \cdot f) * v = u \cdot (f * v), \quad (f * u) * v = f * (uv). \tag{2.8}$$

The elements fixed by the star action are those which satisfy

$$f(w, t_1, \dots, t_n) = f(\mathrm{id}, t_1, \dots, t_n) \quad \text{for all } w \in S_n. \tag{2.9}$$

As a subring, we identify them with the elements of $\mathbb{C}[t_1, \dots, t_n]$, since they are generated by the algebraically independent elements

$$(w, t_1, \dots, t_n) \mapsto t_i \quad \text{for } i = 1, 2, \dots, n. \tag{2.10}$$

The elements fixed by the dot action are those which satisfy

$$f(w, t_1, \dots, t_n) = f(\mathrm{id}, t_{w(1)}, \dots, t_{w(n)}) \quad \text{for all } w \in S_n. \tag{2.11}$$

As a subring, they are generated by the algebraically independent elements

$$r_i(w, t_1, \dots, t_n) = t_{w(i)} \quad \text{for } i = 1, 2, \dots, n, \tag{2.12}$$

so we will write this subring as $\mathbb{C}[r_1, \dots, r_n]$. For $w \in S_n$, we have

$$\begin{aligned} w \cdot t_i &= t_{w(i)} & t_i * w &= t_i \\ w \cdot r_i &= r_i & r_i * w &= r_{w^{-1}(i)} \end{aligned} \tag{2.13}$$

The intersection of $\mathbb{C}[t_1, \dots, t_n]$ and $\mathbb{C}[r_1, \dots, r_n]$ consists of the symmetric polynomials (in either set of variables), and the subring they generate together is

$$\frac{\mathbb{C}[t_1, \dots, t_n, r_1, \dots, r_n]}{\langle e_i(t_1, \dots, t_n) - e_i(r_1, \dots, r_n) \rangle_{i=1}^{n}}, \tag{2.14}$$

where e_i is the ith elementary symmetric polynomial in n variables.

2.2. **Divisibility conditions.** For an element $f \in H$ and a transposition $\gamma = (i \leftrightarrow k) \in S_n$, we will say that f *satisfies condition* γ if

$$f * (1 - \gamma) \quad \text{is a multiple of} \quad (r_i - r_k). \tag{2.15}$$

Note that, in particular, any multiple of $(r_i - r_k)$ satisfies condition γ:

$$\big((r_i - r_k)f\big) * (1 - \gamma) = (r_i - r_k)\big(f * (1 + \gamma)\big). \tag{2.16}$$

The set of elements which satisfy condition γ is closed under addition and multiplication, so they form a subring of H, which we call H_γ. More generally, for a set of transpositions $C \subseteq S_n$, we write

$$H_C = \textstyle\bigcap_{\gamma \in C} H_\gamma \tag{2.17}$$

for the subring of elements which satisfy all conditions in C. If C is the empty set, then H_C is the entire ring H. If C is the set of all transpositions in S_n, then H_C is the subring (2.14), generated by $t_1, \dots, t_n, r_1, \dots, r_n$. In particular, the elements $t_1, \dots, t_n$ and $r_1, \dots, r_n$ satisfy all divisibility conditions. If $w \in S_n$ and f satisfies condition γ, then

$$\begin{gathered} w \cdot f \text{ satisfies condition } \gamma \text{, and} \\ f * w \text{ satisfies condition } w^{-1}\gamma w, \end{gathered} \tag{2.18}$$

so each H_C is closed under the dot action, but usually *not* the star action.

2.3. **Divided differences.** For $i = 1, 2, \dots, n-1$, we define the ith divided difference operator $\partial_i : H_{s_i} \to H$ by

$$\partial_i(f) = \frac{f * (1 - s_i)}{r_i - r_{(i+1)}}. \tag{2.19}$$

Since $r_i - r_{(i+1)}$ is not a zero divisor in H, the quotient involved in this definition is unique if it exists, and the definition of H_{s_i} is that this quotient does exist. We record here some of the convenient computational properties of divided differences which follow directly from the definition.

Lemma 1. *For $f, g \in H_{s_i}$ and $w \in S_n$, we have:*

1. $\partial_i(f + g) = \partial_i(f) + \partial_i(g)$,
2. $\partial_i(fg) = \partial_i(f)\, g + (f * s_i)\, \partial_i(g)$,
3. $\partial_i(fg) = f\, \partial_i(g)$ *when* $\partial_i(f) = 0$,
4. $\partial_i(w \cdot f) = w \cdot \partial_i(f)$,
5. $\partial_i(f * s_i) = -\partial_i(f)$,
6. $\partial_i(f) * s_i = \partial_i(f)$,
7. $\partial_i(f) * w = \partial_k(f * w)$ *when* $w(k) = i$ *and* $w(k+1) = i + 1$,
8. $\partial_i(t_k) = 0$,
9. ∂_i *is* $\mathbb{C}[t_1, \dots, t_n]$*-linear,*
10. $\partial_i(r_i) = 1$,
11. $\partial_i(r_{(i+1)}) = -1$,
12. $\partial_i(r_k) = 0$ *when* $k \notin \{i, i+1\}$.

Proof. By direct computation. □

Furthermore, the divided difference operators satisfy the relations

$$\begin{gathered} \partial_i^2 = 0, \quad \partial_i \partial_{(i+1)} \partial_i = \partial_{(i+1)} \partial_i \partial_{(i+1)}, \\ \partial_i \partial_k = \partial_k \partial_i \quad \text{for } k \neq i \pm 1 \end{gathered} \tag{2.20}$$

on the appropriate domains, since

$$\partial_i\partial_{(i+1)}\partial_i(f) = \partial_{(i+1)}\partial_i\partial_{(i+1)}(f) \\ = \frac{f * \big(1 - s_i - s_{(i+1)} + s_i s_{(i+1)} + s_{(i+1)} s_i - s_i s_{(i+1)} s_i\big)}{\big(r_i - r_{(i+1)}\big)\big(r_i - r_{(i+2)}\big)\big(r_{(i+1)} - r_{(i+2)}\big)} \tag{2.21}$$

and

$$\partial_i\partial_k(f) = \partial_k\partial_i(f) = \frac{f * \big(1 - s_i - s_k + s_i s_k\big)}{\big(r_i - r_{(i+1)}\big)\big(r_k - r_{(k+1)}\big)}. \tag{2.22}$$

2.4. **Stability.** The divided difference operators aren't defined on all of H, and don't preserve the subrings H_C in general. But, motivated by the following lemma, we will say that a set of conditions C is s_i-*stable* if

$$s_i \in C \quad \text{and} \quad s_i C s_i = C. \tag{2.23}$$

Lemma 2. *If C is s_i-stable, then ∂_i is defined on H_C, and $\partial_i(H_C) \subseteq H_C$.*

Proof. Since $s_i \in C$, it's immediate ∂_i is defined on H_C. To show that $\partial_i(H_C) \subseteq H_C$, we will reduce to the case of three specific sets C, each of which can be checked on a small $\mathbb{C}[t_1, \dots, t_n]$-linear subspace of H_C, and provide an explicit basis for this subspace.

As a first reduction, it suffices to check that $\partial_i(H_C) \subseteq H_C$ for the *minimal* s_i-stable sets C, which are of the form

$$C(i, \gamma) = \{s_i,\ \gamma,\ s_i\gamma s_i\} \quad \text{for a transposition } \gamma \in S_n, \tag{2.24}$$

because for a general s_i-stable set C, we have

$$H_C = \textstyle\bigcap_{\gamma \in C} H_{C(i,\gamma)}. \tag{2.25}$$

As a second reduction, it suffices to check the three specific s_1-stable sets

$$\begin{aligned} C(1, s_1) &= \{(1 \leftrightarrow 2)\} \\ C(1, s_2) &= \{(1 \leftrightarrow 2), (1 \leftrightarrow 3), (2 \leftrightarrow 3)\} \\ C(1, s_3) &= \{(1 \leftrightarrow 2), (3 \leftrightarrow 4)\}, \end{aligned} \tag{2.26}$$

because each $C(i, \gamma)$ can be reduced to one of them, depending on the case:

Case $s_i = \gamma$: Let $w \in S_n$ be a permutation with

$$w(i) = 1, \qquad w(i+1) = 2. \tag{2.27}$$

Then, we have

$$\begin{aligned} \partial_i(H_{C(i,\gamma)}) = \partial_i(H_{C(1,s_1)} * w) &= \partial_1(H_{C(1,s_1)}) * w \\ &\subseteq^? H_{C(1,s_1)} * w = H_{C(i,\gamma)}. \end{aligned} \tag{2.28}$$

Case $s_i \neq \gamma$ and $\gamma \neq s_i\gamma s_i$: In this case, γ and $s_i\gamma s_i$ must be of the form $(i \leftrightarrow k)$ and $(i+1 \leftrightarrow k)$, in some order, for some $k \notin \{i, i+1\}$. Let $w \in S_n$ be a permutation with

$$w(i) = 1, \qquad w(i+1) = 2, \qquad w(k) = 3. \tag{2.29}$$

Then, we have

$$\begin{aligned}\partial_i(H_{C(i,\gamma)}) = \partial_i(H_{C(1,s_2)} * w) &= \partial_1(H_{C(1,s_2)}) * w \\ &\subseteq^? H_{C(1,s_2)} * w = H_{C(i,\gamma)}\end{aligned} \tag{2.30}$$

Case $s_i \neq \gamma$ **and** $\gamma = s_i \gamma s_i$**:** In this case, γ must be of the form $(j \leftrightarrow k)$ for some $\{j, k\}$ disjoint from $\{i, i+1\}$. Take $w \in S_n$ such that

$$w(i) = 1, \quad w(i+1) = 2, \quad w(j) = 3, \quad w(k) = 4. \tag{2.31}$$

Then, we have

$$\begin{aligned}\partial_i(H_{C(i,\gamma)}) = \partial_i(H_{C(1,s_3)} * w) &= \partial_1(H_{C(1,s_3)}) * w \\ &\subseteq^? H_{C(1,s_3)} * w = H_{C(i,\gamma)}\end{aligned} \tag{2.32}$$

As an independent reduction, it suffices to check that $\partial_i(H_C) \subseteq H_C$ on a certain $\mathbb{C}[t_1, \ldots, t_n]$-linear subspace of H_C. Specifically, let $\langle C \rangle$ be the subgroup of S_n generated by C, and let $1_{\langle C \rangle} \in H_C$ be defined by

$$1_{\langle C \rangle}(w, t_1, \ldots, t_n) = \begin{cases} 1 & \text{if } w \in \langle C \rangle \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

Then, $1_{\langle C \rangle} H_C$ is the $\mathbb{C}[t_1, \ldots, t_n]$-linear subspace of functions in H_C which are zero outside of $\langle C \rangle \times \mathbb{C}^n$. The unit element in H_C decomposes as

$$1 = \textstyle\sum_w w \cdot 1_{\langle C \rangle}, \tag{2.34}$$

where the sum is over an arbitrary choice of representatives w for the cosets $w\langle C \rangle \subseteq S_n$. For an element $f \in H_C$ and a permutation $w \in S_n$, let

$$f_w = 1_{\langle C \rangle}\,(w^{-1} \cdot f) \in 1_{\langle C \rangle} H_C. \tag{2.35}$$

Then, the element f can be decomposed as

$$f = \textstyle\sum_w w \cdot f_w, \tag{2.36}$$

so that

$$\partial_i(f) = \textstyle\sum_w w \cdot \partial_i(f_w). \tag{2.37}$$

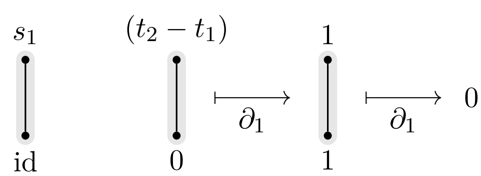
Thus, $\partial_i(H_C) \subseteq H_C$ is implied by $\partial_i(1_{\langle C \rangle} H_C) \subseteq H_C$.

Given all of these reductions, the remaining task is to compute ∂_1 on a $\mathbb{C}[t_1, \ldots, t_n]$-linear basis of $1_{\langle C \rangle} H_C$ for C in $\{C(1, s_1),\ C(1, s_2),\ C(1, s_3)\}$.

Case $C = C(1, s_1)$**:** The following two elements form a basis:

$$1_{\langle C \rangle}(r_1 - t_1), \qquad 1_{\langle C \rangle} \tag{2.38}$$

and the computation of ∂_1 on them is illustrated below:

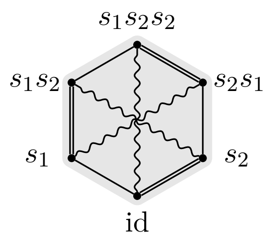


(2.39)

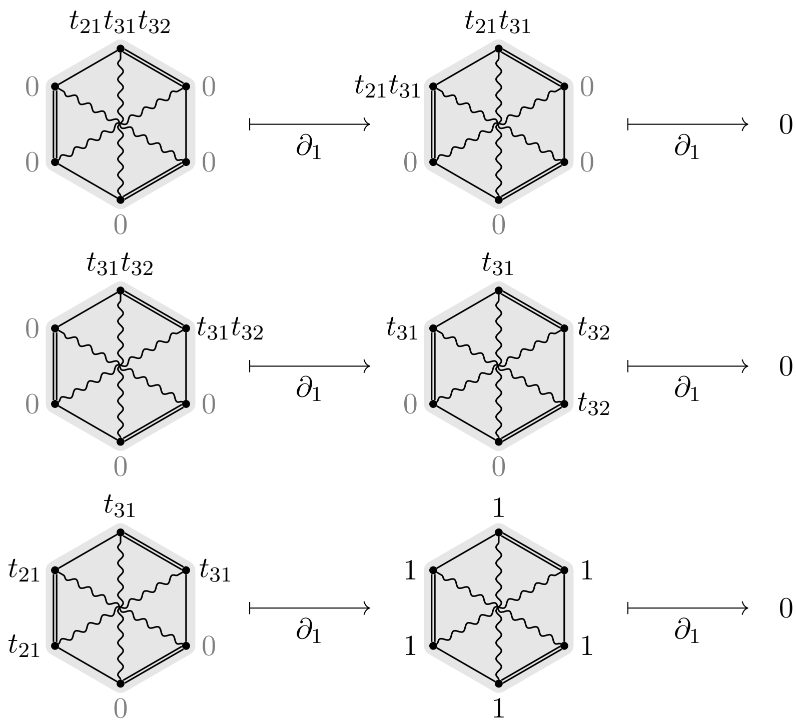
Case $C = C(1, s_2)$: The following six elements form a basis:

$$\begin{array}{ll} 1_{\langle C\rangle}(r_2 - t_1)(r_1 - t_1)(r_1 - t_2), & 1_{\langle C\rangle}(r_2 - t_1)(r_1 - t_1), \\ 1_{\langle C\rangle}(r_1 - t_1)(r_1 - t_2), & 1_{\langle C\rangle}(t_3 - r_3), \\ 1_{\langle C\rangle}(r_1 - t_1), & 1_{\langle C\rangle}, \end{array} \tag{2.40}$$

and the computation of ∂_1 on these elements is illustrated below, using the abbreviation $t_{ik} = (t_i - t_k)$:



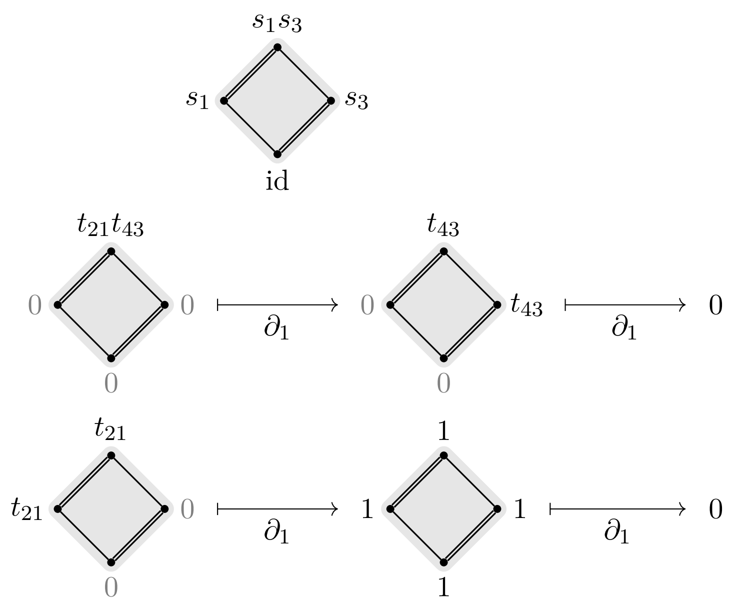
(2.41)



Case $C = C(1, s_3)$: The following four elements form a basis:

$$\begin{aligned} &1_{\langle C\rangle}(r_1 - t_1)(r_3 - t_3), \quad && 1_{\langle C\rangle}(r_3 - t_3), \\ &1_{\langle C\rangle}(r_1 - t_1), && 1_{\langle C\rangle}, \end{aligned} \tag{2.42}$$

and the computation of ∂_1 on these elements is illustrated below, again using the abbreviation $t_{ik} = (t_i - t_k)$:



(2.43)

This completes the proof of Lemma 2. ☐

2.5. **Decompositions.** Now that we have a handle on the domain and range of the divided difference operators, we can use them to obtain decomposition of some of the subrings H_C.

Theorem 3. *Let C be an s_i-stable set of divisibility conditions. Then, we have the direct sum decomposition*

$$H_C = H_C^{s_i} \oplus (r_i - r_{(i+1)})H_C^{s_i} \tag{2.44}$$

as subgroups of H equipped with the dot action, where

$$H_C^{s_i} = \{f \in H_C \mid f * s_i = f\}. \tag{2.45}$$

Proof. Since C is s_i-stable, the subgroup $\{\mathrm{id}, s_i\} \subseteq S_n$ of order two acts on the right on H_C by the star action, and we have the natural decomposition

$$H_C = H_C^{s_i} \oplus H_C^{-s_i}, \tag{2.46}$$

where we write

$$H_C^{-s_i} = \{f \in H_C \mid f * s_i = -f\} \tag{2.47}$$

for the (-1)-eigenspace of the map $f \mapsto f * s_i$. We claim that

$$H_C^{-s_i} = (r_i - r_{(i+1)})H_C^{s_i}, \tag{2.48}$$

because we further claim that the maps

$$H_C^{-s_i} \underset{(r_i - r_{(i+1)})/2}{\overset{\partial_i}{\rightleftarrows}} H_C^{s_i} \tag{2.49}$$

form a bijection. This follows from the facts that

$$\begin{gathered}
\partial_i(H_C^{-s_i}) \subseteq \partial_i(H_C) \subseteq H_C^{s_i}, \\
(r_i - r_{(i+1)})H_C^{s_i} \subseteq H_C^{-s_i}, \\
\partial_i((r_i - r_{(i+1)})f)/2 = f * (1 + s_i)/2 = f \quad \text{when } f \in H_C^{s_i}, \\
(r_i - r_{(i+1)})\partial_i(f)/2 = f * (1 - s_i)/2 = f \quad \text{when } f \in H_C^{-s_i},
\end{gathered} \tag{2.50}$$

which can be verified by direct computation. □

The proof of Theorem 3 yields an additional fact:

Corollary 4. *The kernel and image of $\partial_i : H_C \to H_C$ are equal when C is s_i-stable. They consist of those elements $f \in H_C$ such that $f * s_i = f$.*

Motivated by the following decomposition, we define the notion of almost-stability. We will say that a set of divisibility conditions C_1 is *almost-s_i-stable* if $s_i \in C_1$ and there is a unique transposition γ such that $\gamma \in C_1$ but $s_i\gamma s_i \notin C_1$. If this is the case, note that the sets

$$C_0 = C_1 \setminus \{\gamma\}, \qquad C_2 = C_1 \cup \{s_1\gamma s_1\}, \tag{2.51}$$

obtained from C_1 by adding or removing a single element, are s_i-stable.

Theorem 5. *Let C_1 be an almost-s_i-stable set of divisibility conditions, so that $C_0 = C_1 \setminus \{\gamma\}$ and $C_2 = C_1 \cup \{s_i\gamma s_i\}$ are s_i-stable. If $\gamma = (i \leftrightarrow k)$, then we have the direct sum decomposition*

$$H_{C_1} = H_{C_2}^{s_i} \oplus (r_i - r_k)H_{C_0}^{s_i} \tag{2.52}$$

as subgroups of H equipped with the dot action. Otherwise, $\gamma = (i + 1 \leftrightarrow k)$ and we have the direct sum decomposition

$$H_{C_1} = H_{C_2}^{s_i} \oplus (r_k - r_{(i+1)})H_{C_0}^{s_i}. \tag{2.53}$$

Proof. We will assume that $\gamma = (i \leftrightarrow k)$, since the case where $\gamma = (i+1 \leftrightarrow k)$ follows by using the transformation

$$H_{C_1} * s_i = H_{s_iC_1s_i}. \tag{2.54}$$

We claim that the decomposition is given by the following maps:

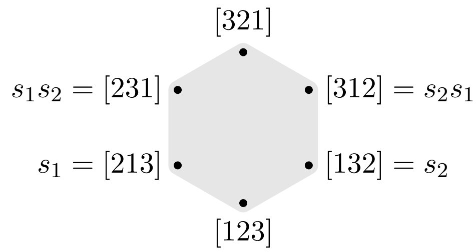
$$H_{C_2}^{s_i} \underset{\partial_i \circ (r_k - r_{(i+1)})}{\overset{\text{id}}{\rightleftarrows}} H_{C_1} \underset{(r_i - r_k)}{\overset{\partial_i}{\rightleftarrows}} H_{C_0}^{s_i} \tag{2.55}$$

To prove this, we need to check that:

- All four maps land in the stated codomain. Using the inclusions
$$\begin{aligned} H_{C_2} &\subseteq H_{C_1} \subseteq H_{C_0} \\ (r_i - r_k)H_{C_0} &\subseteq H_{C_1} \\ (r_k - r_{(i+1)})H_{C_1} &\subseteq H_{C_2}, \end{aligned} \tag{2.56}$$
we indeed have that
$$\begin{aligned} \mathrm{id}(H_{C_2}^{s_i}) &\subseteq H_{C_2} \subseteq H_{C_1} \\ \partial_i(H_{C_1}) &\subseteq \partial_i(H_{C_0}) \subseteq H_{C_0}^{s_i} \\ \partial_i\big((r_k - r_{(i+1)})H_{C_1}\big) &\subseteq \partial_i(H_{C_2}) \subseteq H_{C_2}^{s_i} \\ (r_i - r_k)H_{C_0}^{s_i} &\subseteq (r_i - r_k)H_{C_0} \subseteq H_{C_1}. \end{aligned} \tag{2.57}$$
- The two compositions $H_{C_1} \to H_{C_1}$ are complementary idempotents. Using the computational properties from Lemma 1, we have
$$\begin{aligned} \partial_i\big((r_k - r_{(i+1)})f\big) + (r_i - r_k)\partial_i(f) &= f \\ \partial_i\Big((r_k - r_{(i+1)})\partial_i\big((r_k - r_{(i+1)})f\big)\Big) &= \partial_i\big((r_k - r_{(i+1)})f\big) \\ (r_i - r_k)\partial_i\big((r_i - r_k)\partial_i(f)\big) &= (r_i - r_k)\partial_i(f) \end{aligned} \tag{2.58}$$
for every $f \in H_{C_1}$, as required.
- The compositions $H_{C_0}^{s_i} \to H_{C_0}^{s_i}$ and $H_{C_2}^{s_i} \to H_{C_2}^{s_i}$ are the identity. Again using Lemma 1, we have
$$\partial_i\big((r_i - r_k)f\big) = f + (r_{(i+1)} - r_k)\partial_i(f) = f \tag{2.59}$$
for every $f \in H_{C_0}^{s_i}$ and
$$\partial_i\big((r_k - r_{(i+1)})f\big) = f + (r_i - r_k)\partial_i(f) = f \tag{2.60}$$
for every $f \in H_{C_2}^{s_i}$, as required. □

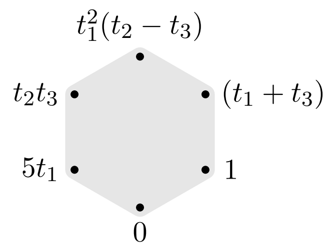
Appendix A. Examples for $n = 3$

To clarify the notation and conventions used in this note, we illustrate them here for the case $n = 3$. In everything that follows, we draw the elements of S_3 as six vertices arranged in a hexagon, with the identity permutation at the bottom, the long word $w_0 = s_1s_2s_1 = s_2s_1s_2 = [321]$ at the top, and the other vertices labeled like this:



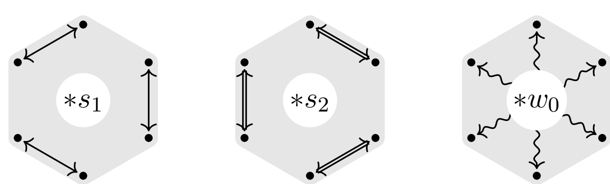
(A.1)

Then, an element of H can be represented as a hexagon with a polynomial in $\mathbb{C}[t_1, t_2, t_3]$ attached to each vertex, like this:



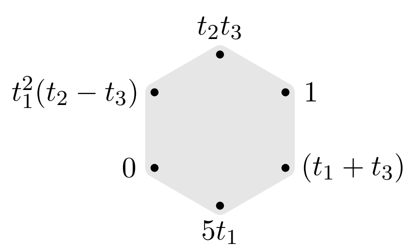
(A.2)

The star action of S_3 on H on the right permutes the polynomials attached to the vertices. For the three transpositions $s_1, s_2, w_0 \in S_3$, the effect is:



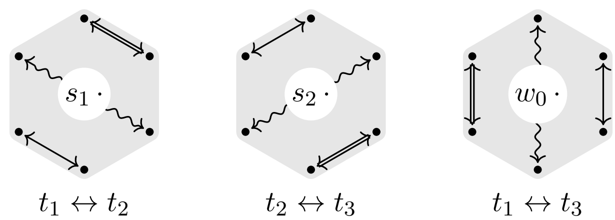
(A.3)

The three actions above are drawn with different kinds of arrows; each kind corresponds to a divisibility condition which could be imposed on elements of H. For the element $f \in H$ of (A.2), the element $(f * s_1)$ is:



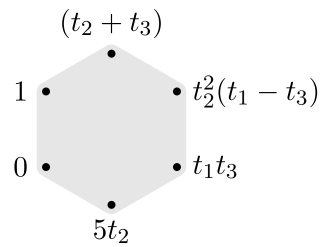
(A.4)

The dot action of S_3 on H on the left relabels the variables t_1, t_2, t_3 in addition to permuting the polynomials attached to the vertices. For the three transpositions $s_1, s_2, w_0 \in S_3$, the effect is:



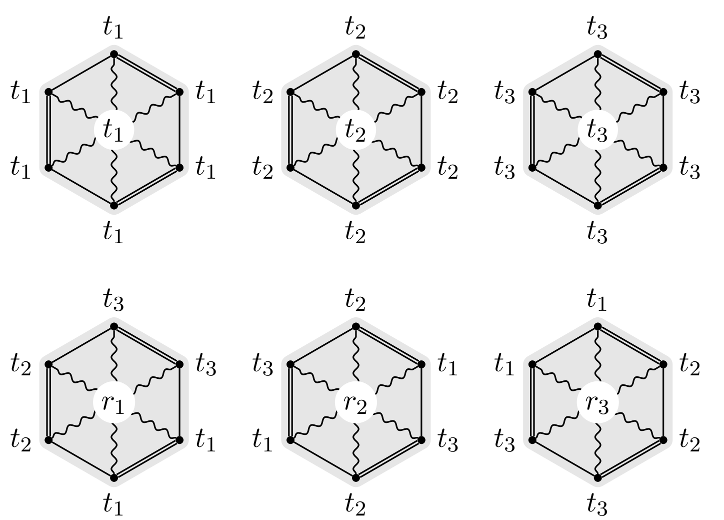
(A.5)

For the element $f \in H$ of (A.2), the element $(s_1 \cdot f)$ is:



(A.6)

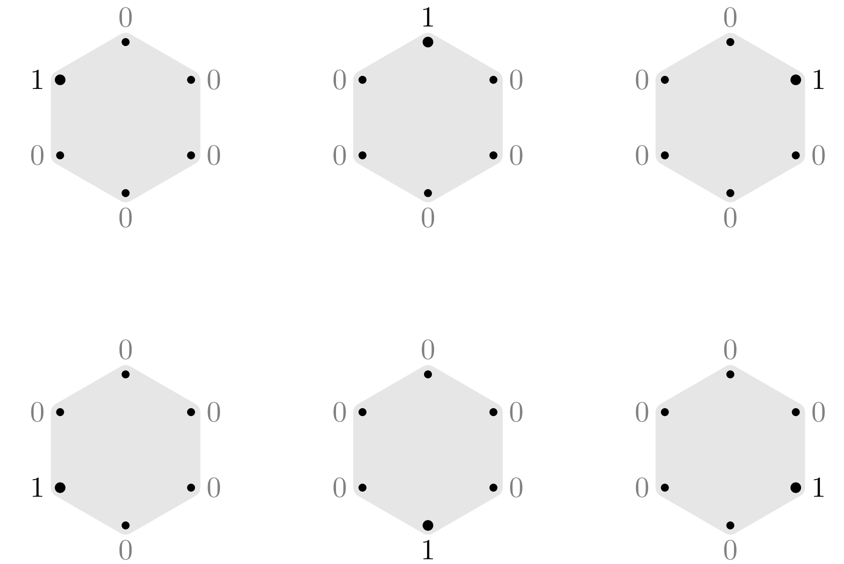
The elements $t_1, t_2, t_3, r_1, r_2, r_3 \in H$, which satisfy all three divisibility conditions, are:



(A.7)

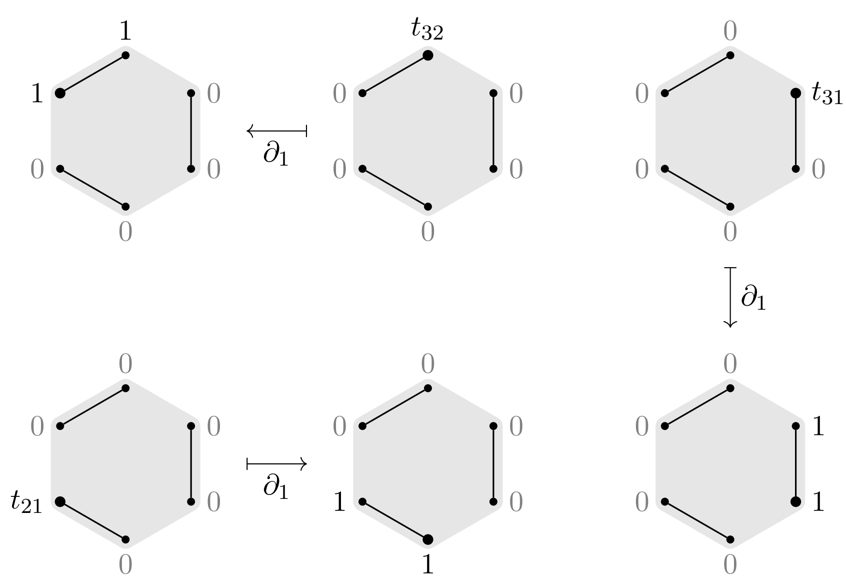
For each Hessenberg function $h : \{1, 2, 3\} \to \{1, 2, 3\}$, there is a corresponding set of divisibility conditions $C = \{(i \leftrightarrow k) \mid i < k \leq h(i)\} \subseteq S_3$, and a corresponding subring $H_C \subseteq H$. Below we give a $\mathbb{C}[t_1, t_2, t_3]$-linear basis for each of these H_C. These bases are well-known examples of *flow-up* bases. We use the abbreviation $t_{ik} = (t_i - t_k)$, and indicate when ∂_1 or ∂_2 takes one basis element to another.

For $h = (1, 2, 3)$, we have $C = \varnothing$, and $H_C = H$ has the basis:



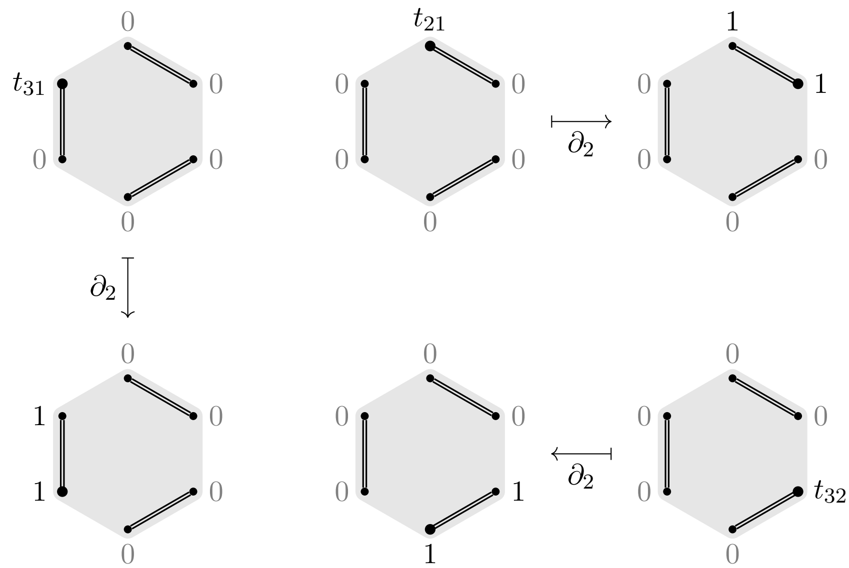
(A.8)

For $h = (2, 2, 3)$, we have $C = \{s_1\}$, and H_C has the basis:



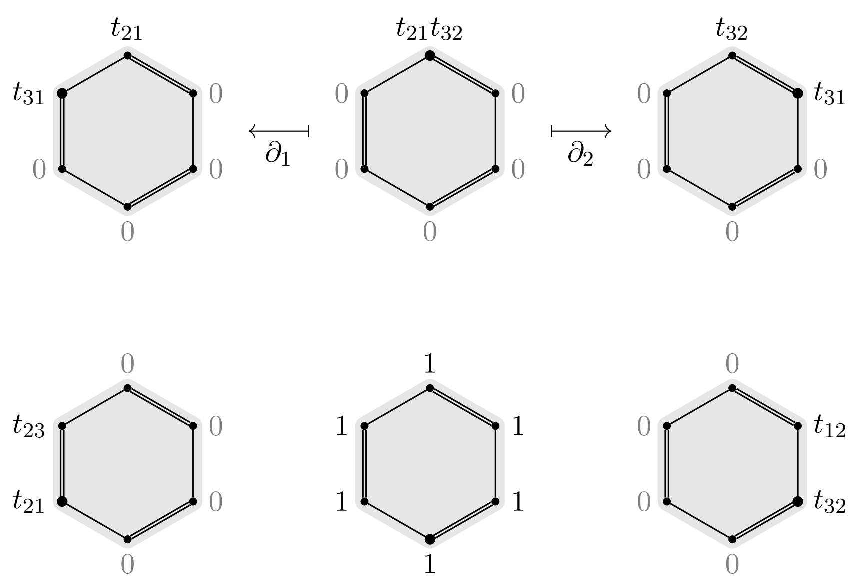
(A.9)

For $h = (1, 3, 3)$, we have $C = \{s_2\}$, and H_C has the basis:



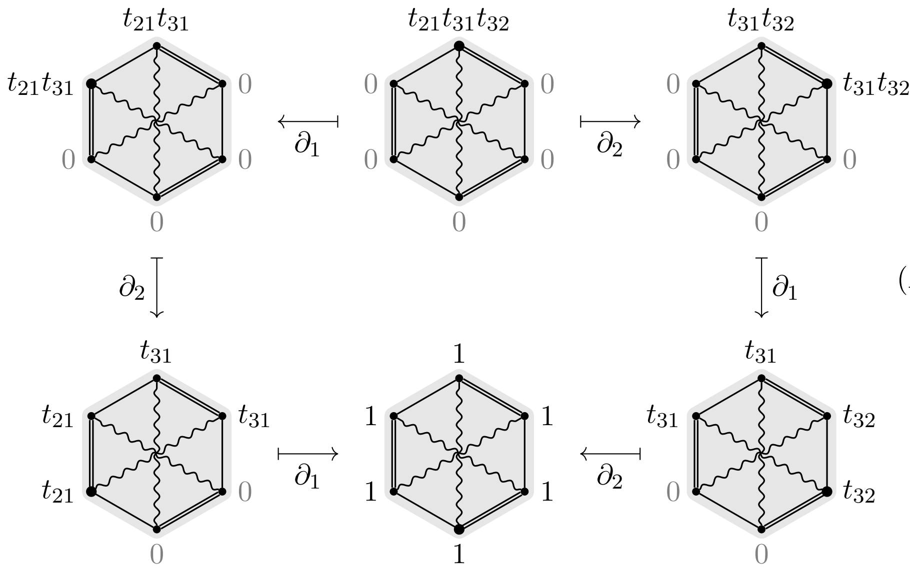
(A.10)

For $h = (2, 3, 3)$, we have $C = \{s_1, s_2\}$, and H_C has the basis:



(A.11)

For $h = (3, 3, 3)$, we have $C = \{s_1, s_2, w_0\}$, and H_C has the basis:



(A.12)

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