Interactions between Hessenberg Varieties, Chromatic Functions, and LLT Polynomials (22w5143)

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1 Overview

The study of Hessenberg varieties and LLT polynomials have been isolated until recently. The workshop 22w5143 brought together researchers in these disparate fields. It was an important opportunity to share recent results, interesting open problems, and new directions.

Hessenberg varieties. Motivated by Hessenberg matrices and algorithms for efficiently calculating eigenvalues in numerical analysis, *Hessenberg varieties* in the flag variety of $GL_n(\mathbb{C})$ were first introduced by De Mari and Shayman [DMS88] and later defined in all Lie types by De Mari, Procesi, and Shayman [DMPS92]. Hessenberg varieties are increasingly important examples of varieties whose geometry and cohomology can be better understood using combinatorial techniques.

Recent results have forged exciting new connections between algebraic combinatorics and the geometry and topology of *regular semisimple Hessenberg varieties*. The *Stanley–Stembridge conjecture* asserts that the chromatic symmetric function of the incomparability graph of a unit interval order is e-positive. Shareshian and Wachs [SW16] conjectured a link between the Stanley–Stembridge conjecture to Hessenberg varieties via the *dot action*, a symmetric group representation on the cohomology ring of a regular semisimple Hessenberg variety defined by Tymoczko [Tym08]. This conjecture was proved in 2015 by Brosnan and Chow [BC18a] and independently by Guay-Paquet [GP16a]. The work referenced above establishes the following research problem which is recognized as a major question in this area: *use the properties of Hessenberg varieties to prove the StanleyStembridge conjecture.*

LLT polynomials. The LLT polynomials were introduced by Lascoux, Leclerc and Thibon [LLT97]. The original motivation was to study certain Fock space representations and plethysm coefficients A few years later, Leclerc and Thibon [LT00] proved that the LLT polynomials are *Schur positive*, by using representation theory.

Bylund and Haiman extended the notion of LLT polynomials by using a new combinatorial model. Now, each LLT polynomial is indexed by a k-tuple of skew shapes, (see [Hag07, p. 92]). Schur positivity for this extended family was later proved by Grojnowski and Haiman, [GH06], but their proof does not give a combinatorial interpretation of the coefficients.

LLT polynomials appear in the study of *diagonal harmonics*. In particular, they play a central role in the study of the *modified Macdonald polynomials*, the *Shuffle conjecture* and the *Delta conjecture*, see [HHL05a, HHL05b, HMZ12, Ser17, HRW18]. Guay-Paquet [GP16a] uses a Hopf algebra approach to show that (unicellular) LLT polynomials are the graded Frobenius series derived from the equivariant cohomology rings of regular semisimple Hessenberg varieties—establishing another connection between Hessenberg varieties and algebraic combinatorics. Meanwhile, Carlsson and Mellit [CM17] give a more convenient model for vertical-strip LLT polynomials, and they prove the *Compositional Shuffle conjecture*, by introducing the *Dyck path algebra*. In Carlsson and Mellit's work, it becomes evident that LLT polynomials and chromatic symmetric functions are closely related. In [AP18], several properties and conjectures regarding chromatic symmetric functions are shown to have an analog in the world of LLT polynomials.

As we see, the family of LLT polynomials can be seen as a building-block in representation-theoretical settings. *Understanding how these expand into Schur polynomials is currently the most central open problem in this area.*

Workshop Topic and Schedule. The work referenced above builds a table of correspondences between Hessenberg varieties, chromatic quasisymmetric functions, and LLT polynomials. As a result, we are able to use geometric information about Hessenberg varieties to prove combinatorial statements and vice versa. The workshop focused on this interplay of ideas and consisted of 42 in-person participants (a mixture of faculty at various career stages) and 40 remote participants (about half of whom were graduate students and postdocs).

The schedule consisted of a mixture of introductory and research presentations, informal discussion, and group work. The workshop included 3 introductory and 7 research talks.

- Erik Carlsson: *Affine Springer fibers* (Introductory talk)
- Laura Escobar: *An introduction to Hessenberg varieties* (Introductory talk)
- Mathieu Guay-Paquet: *Divided Difference Operators for Hessenberg Varieties*
- Jennifer Morse: *Hey Series, How can you help with symmetric functions?*
- Antonio Nigro: *Parabolic Lusztig varieties and chromatic symmetric functions*
- Greta Panova: *Combinatorial identities for CSF of Dyck paths with bounce 2 and 3*
- Bruce Sagan: *Chromatic symmetric functions and sign-reversing involutions*
- Franco Saliola: *Chromatic symmetric functions and LLT polynomials* (Introductory talk)
- Eric Sommers: *Nilpotent Hessenberg varieties and related objects in the setting of general Lie type*
- Foster Tom: *Horizontal-strip LLT polynomials*

Informal discussions served as a jumping-off point for researchers with expertise in particular topics to share the current status of the field. These discussions topics included the following;

- Schur positivity and crystals (facilitated by Per Alexandersson)
- Failed attempts (facilitated by Laura Colmenarejo)
- The dot action and Hessenberg varieties in all Lie types (facilitated by Martha Precup)

The organizers also hosted virtual discussions on Zoom (one on professional development topics and the other on research questions) for graduate students participating remotely.

Group discussion played a significant role in the workshop structure. These discussions were fluid, informal, and engaged. The intended goal was to spark ideas and identify researchers interested in similar projects from different mathematical fields or perspectives. This appears to have been met! We refrain from detailed description of the content because of the speculative and open-ended nature of these conversations.

Our survey of workshop participants indicated that they valued the group structure, in some cases more than they anticipated. Although it was not a goal of the workshop that each group continue to meet and solve the problem they discussed, we expected the workshop to motivate long-term collaborations between participants and produce several new publications. Multiple participants noted that the topics discussed had a direct impact on their research and we are aware of at least three different ongoing collaborations formed at the workshop.

Feedback from workshop participants include the following sampling of comments.

- I think the talks were at a reasonable level and my group had an approachable project to which everyone was able to contribute, which was really great. Given how few hours were actually available for group work, we did not make much progress, but I think this was fine. I really appreciated the opportunity to work with my other group members and get to know them better. Most importantly, I learned things!
- The format worked really well in our case. (I was initially very suspicious of the format and in fact tried to get out of being a group organizer, but am glad now they didn't let me).
- It went well. We had a very specific problem to deal with, and the group made progress in some special cases.
- The group discussion was great! We were able to learn very efficiently about some new results straight from the author, with lots of questions and answers and interactive examples from all the participants.
- Despite my critiques, it was still an amazing experience for me. I learned a lot from the few structured talks, as well as talking informally with so many experts. And the memories of the food, the pool, the natural beauty still linger...
- As a postdoc, perhaps the most valuable thing about the workshop was being able to talk informally with the other participants, and I believe I learned quite a bit from those conversations. I really appreciated all the opportunities to discuss over coffee, meals, etc with so many great researchers and my experience of the workshop was incredibly positive. Thank you to the organizers for all your work and excellent planning!

For future workshop organizers, we note that surveyed participants appreciated the open discussions but would have preferred more structure. If we were to do it again, we would instead ask moderators to provide a list of topics to prompt discussions, or perhaps a panel discussion with a list of questions from the moderator.

2 Group discussion and scientific progress

During the week, workshop participants were split into seven groups working on six different projects. Four hours were specifically designated for group discussion, and a number of groups met outside of the other workshop activities for further work and discussion. Participants were welcome to move between groups during discussion. This section briefly describes each project and summarize group activities during the workshop.

Deformations and automorphisms of Hessenberg varieties

Group Leader: Patrick Brosnan; Group Members: Laura Escobar, Jaehyun Hong, Eunjeong Lee, Anton Mellit, Eric Sommers.

Hessenberg varieties are subvarieties of generalized flag varieties associated to a reductive group G and two pieces of data: an element X of the Lie algebra $\mathfrak{g}$ of G and a subset H of the root system Ψ of G. Given this data, one convention is to write $\mathcal{B}(X,H)$ for the associated Hessenberg variety. When X is regular semisimple, $\mathcal{B}(X,H)$ is a smooth subvariety of the variety $\mathcal{B}$ of Borel subgroups of G, which inherits many nice properties from $\mathcal{B}$ itself.

In the data defining a Hessenberg variety, the second datum, H, is obviously combinatorial in nature. In particular, since Ψ itself is finite, there are only finitely many possible Hessenberg data H for a given reductive group G. On the other hand, X can vary continuously in Zariski dense open subset $\mathfrak{g}^{\mathrm{rs}}$ of regular semisimple elements. In particular, if $X, X' \in \mathfrak{g}^{\mathrm{rs}}$, then $\mathcal{B}(X,H)$ and $\mathcal{B}(X',H)$ are deformation equivalent. So, for example, if we work over the complex numbers, $\mathcal{B}(X,H)$ and $\mathcal{B}(X',H)$ are diffeomorphic. This leads to the following question.

Question 1. Suppose X and X' are two regular semisimple elements of $\mathfrak{g}$. Are $\mathcal{B}(X,H)$ and $\mathcal{B}(X',H)$ isomorphic as algebraic varieties?

Note that if $H = \Psi$, then $\mathcal{B}(X, H)$ is just the complete flag variety $\mathcal{B}$. So, in this case, the answer to Question 1 is yes. Similarly, if H is empty or is equal to the set of simple positive roots, the answer is yes.

This question was posed by Patrick Brosnan as a potential group problem at the BIRS Workshop. It was taken up by a group consisting of Brosnan together with Laura Escobar, Jaehyun Hong, Eunjeong Lee, Anton Mellit and Eric Sommers. Very quickly, Mellit came up with a construction that led to a negative answer. We work with the case where H contains every root except the highest root θ. Then essentially the trivial cases are the only cases when the answer to the question is yes.

The members of the group are in the process of writing up the proofs along with the answers to related questions about automorphisms of the Hessenberg variety $\mathcal{B}(X, H)$.

Operators and algorithms for computing chromatic symmetric functions

Group Leaders: Jim Haglund, Alejandro Morales; Group Members: Farid Aliniaeifard, Logan Crew, Mathieu Guay-Paquet, Megumi Harada, Byung-Hak Hwang, Rosa Orellana, Martha Precup, Franco Saliola, Sophie Spirkl, Michelle Wachs.

Originally I (Haglund) was assigned to be group leader for a project studying the Schur expansion of the unicellular LLT polynomial $\mathrm{LLT}_\pi(X; q)$, where π is a Dyck path and X a set of variables. The main idea was to start with the combinatorial, signed expansion of Alexandersson and Sulzgruber [AS22] of $\mathrm{LLT}_\pi(X; q)$ into elementary symmetric functions, and figure out how to cancel the negative signs, leaving a positive expansion. In recent joint work, Anna Pun, my two PhD students Jennifer Wang and Alex Vetter, and I have figured out how to do this when the shape of the Schur function is an augmented hook. (Most of this work was done by Alex). We are still struggling with how to do any non-augmented hook shape, in fact even the shape 33 is giving us trouble.

On seeing the abstract of the project that Alejandro Morales had proposed, I noticed the two projects had very substantial overlap. At one point I suggested we cancel my project, but the conference organizers decided against it, and in retrospect I am glad they did. They placed both my group and the Morales group in the same room, and we quickly decided to "join forces". One of the main reasons for this is that one of the people in my group, Byung-Hak Hwang, recently posted a preprint on the arXix [Hwa22] in which he gives an exciting refinement of the Shareshian-Wachs conjecture (that the coefficients in the expansion of the chromatic quasisymmetric function corresponding to a Dyck path π, denoted say $\mathrm{SW}_\pi(X; q)$, are nonnegative integers, i.e. the coefficients are in $\mathbb{N}[q]$). Hwang also gives a refinement of an important theorem of Shaeshian-Wachs [SW16], namely that the Schur coefficients of $\mathrm{SW}_\pi(X; q)$ can be described combinatorially in terms of "P-tableaux". The function $\mathrm{SW}_\pi(X; q)$ can be represented by a weighted sum over proper colorings of a certain graph G_π, while $\mathrm{LLT}_\pi(X; q)$ equals the same weighted sum, except over all colorings of G_π. Hwang gave a hour-long presentation to our (combined) group, and we spent most of the first two group meetings discussing this new angle on the problem. (Trying to understand the combinatorics behind the positive expansion of $\mathrm{SW}_\pi(X; q)$ into elementary symmetric functions is one of the main motivating factors behind the conference and continued work on $\mathrm{SW}_\pi(X; q)$.)

The two functions $\mathrm{SW}_\pi(X; q)$ and $\mathrm{LLT}_\pi(X; q)$ are related analytically by a simple plethystic transformation. This allows one to take any formula for one of them in terms of symmetric function operators and easily obtain a corresponding formula for the other. Morales gave a half-hour presentation on how to start with a recent formula for $\mathrm{LLT}_\pi(X; q)$ which occurs in the famous paper of Carlsson and Mellit proving the Shuffle Conjecture [CM17]), and obtain a very efficient formula for computing $\mathrm{SW}_\pi(X; q)$. I also gave a half-hour presentation on how to start with the Schur expansion of $\mathrm{SW}_\pi(X; q)$ in terms of P-tableaux, and get a corresponding formula for the Schur expansion of $\mathrm{LLT}_\pi(X; q)$ involving the special value of Macdonald's q, t-Kostka matrix when $t = q$, i.e. $K_{\lambda,\mu}(q, q)$ in the standard notation. Although there is no known combinatorial interpretation for $K_{\lambda,\mu}(q, q)$ for general λ, μ, perhaps a special study of this special $t = q$ value will lead to one.

Combinatorial formulas for the character values of the dot action

Group Leader: John Shareshian; Group Members: Erik Carlsson, Maria Gillespie, Antonio Nigro, Bruce Sagan, Mark Skandera.

John Shareshian led a group which considered connections between type-A trace generating functions, chromatic symmetric functions, and graded Frobenius characteristics of Hessenberg varieties, and began to investigate the extent to which these extend to connections between their type-B and type-C analogs.

Summary of type-A results

Let $\mathfrak{S}_n$ be the symmetric group and $H_n(q)$ the corresponding Hecke algebra. To each permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ avoiding the pattern 312 we associate several algebraic and combinatorial objects: a certain element of $H_n(q)$ called the (modified, signless) Kazhdan-Lusztig basis element $\widetilde{C}_w(q)$, the Hessenberg function $m = m(w) = m_1 \cdots m_n$ defined by $m_i = \max\{w_1, \ldots, w_i\}$, a certain Hessenberg variety $\mathcal{H}(m)$, the poset $P = P(w)$ on $\{1, \ldots, n\}$ defined by $i \leq_P j$ if $m_i + 1 \leq j \leq n$, and the incomparability graph $\mathrm{inc}(P)$. We also associate three symmetric functions to w: the (dual) *trace generating function*

$$\omega Y_q(\widetilde{C}_w(q)) := \sum_{\lambda \vdash n} \eta_q^\lambda(\widetilde{C}_w(q)) m_\lambda \tag{1}$$

where $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$ are induced trivial characters of $H_n(q)$ and m_λ are monomial symmetric functions, the *chromatic symmetric function*

$$X_{\mathrm{inc}(P),q} = \sum_{\lambda} \sum_{\substack{\kappa \\ \text{a proper} \\ \text{coloring of} \\ \text{type } \lambda}} q^{\mathrm{asc}_{\mathrm{inc}(P)}(\kappa)} m_\lambda, \tag{2}$$

where each coloring κ takes nonnegative integer values and $\mathrm{asc}_{\mathrm{inc}(P)}$ is a statistic on colorings, and the *graded Frobenius characteristic*

$$\mathrm{ch}_q(\mathrm{H}^*(\mathcal{H}(m))) := \sum_j \mathrm{ch}(\mathrm{H}^{2j}(\mathcal{H}(m))) q^j \tag{3}$$

of a certain *dot action* of $\mathfrak{S}_n$ on the cohomology of $\mathcal{H}(m)$. By [BC18b, CHSS16, GP16b, SW12], we have the equalities $\omega Y_q(\widetilde{C}_w(q)) = \omega X_{\mathrm{inc}(P)} = \mathrm{ch}_q(\mathrm{H}^*(\mathcal{H}(m)))$. By [Lus86] these can be generalized somewhat: if $w \in \mathfrak{S}_n$ is arbitrary and $V(w)$ is the corresponding *Lusztig variety*, then we have $\omega Y_q(\widetilde{C}_w(q)) = \mathrm{ch}_q(\mathrm{H}^*(V(w)))$, where the second symmetric function is defined in terms of a certain action of $\mathfrak{S}_n$ on the cohomology of $V(w)$.

Extension to types B and C

In types B and C there is one Weyl group: the hyperoctahedral group $\mathfrak{B}_n$. Let $H_n^{\mathsf{BC}}(q)$ be its Hecke algebra. Each element $w \in \mathfrak{B}_n$ has *long* and *short* one-line notations given by $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$ and $w_1 \cdots w_n$, where $\overline{i} := -i$. The long one-line notation satisfies $w_{\overline{i}} = -w_i$. Let $\widetilde{C}_w^{\mathsf{BC}}(q)$ be the corresponding Kazhdan-Lusztig basis element of $H_n^{\mathsf{BC}}(q)$. Analogous to the 312-avoiding permutations of $\mathfrak{S}_n$ are certain *type-BC codominant elements w of* $\mathfrak{B}_n$, each of which has a Hessenberg function $m = m(w) = m_{\overline{n}} \cdots m_{\overline{1}} m_1 \cdots m_n$ defined by $m_i = \max\{w_{\overline{n}}, \ldots, w_i\}$. However, the resulting $\frac{1}{n+2}\binom{2n+2}{n+1}$ functions form a proper subset of the $\binom{2n}{n}$ valid Hessenberg functions corresponding to type-B or type-C Hessenberg varieties $\mathcal{H}^{\mathsf{B}}(m)$, $\mathcal{H}^{\mathsf{C}}(m)$. Given BC-codominant $w \in \mathfrak{B}_n$ and its Hessenberg function $m = m(w)$, one may construct an n-element type-BC unit interval order $P = P(w)$, which is a *decorated poset* in the sense that a (possibly empty) subset of its minimal elements are circled: $i <_P j$ if $m_i < j \leq n$ for $i = 1, \ldots, n$, with $|w_i|$ circled if $w_i < 0$.

It would be nice to define type-BC analogs of the three symmetric functions (1) – (3) with the analog of (2) providing combinatorial interpretations of type-BC trace evaluations and of a $\mathfrak{B}_n$-action on the cohomology of the type-B or type-C Hessenberg varieties. A natural type-BC analog of (1) uses type-BC induced trivial characters [AK94] and type-BC monomial symmetric functions [Mac79],

$$\omega Y_q^{\mathsf{BC}}(\widetilde{C}_w^{\mathsf{BC}}(q)) := \sum_{(\lambda,\mu)\vdash n} (\eta_q\eta_q)^{\lambda,\mu}(\widetilde{C}_w^{\mathsf{BC}}(q))(mm)_{\lambda,\mu}. \tag{4}$$

Alternatively, one may replace $(mm)_{\lambda,\mu}$ with the plethystic variant $m_\lambda[x+y]m_\mu[x-y]$ defined in [BRW96]. A natural type-BC analog of (2) uses colorings of vertices of $\mathrm{inc}(P)$ by nonzero integers

$$X^{\mathsf{BC}}_{\mathrm{inc}(P),q} := \sum_{\lambda} \sum_{\substack{\kappa \\ \text{a proper} \\ \text{coloring of} \\ \text{type } (\lambda,\mu)}} q^{\mathrm{asc}_{\mathrm{inc}(P)}(\kappa)}(mm)_{\lambda,\mu}, \tag{5}$$

where we allow $\kappa(v) > 0$ only for vertices of $\mathrm{inc}(P)$ corresponding to elements of P which are not circled. A natural analog of (3) uses actions of $\mathfrak{B}_n$ on the cohomology of type-B or type-C Hessenberg varieties

$$\mathrm{ch}_q(\mathrm{H}^*(\mathcal{H}^{\mathsf{B}}(m))) := \sum_j \mathrm{ch}(\mathrm{H}^{2j}(\mathcal{H}^{\mathsf{B}}(m)))q^j \tag{6}$$

$$\mathrm{ch}_q(\mathrm{H}^*(\mathcal{H}^{\mathsf{C}}(m))) := \sum_j \mathrm{ch}(\mathrm{H}^{2j}(\mathcal{H}^{\mathsf{C}}(m)))q^j. \tag{7}$$

By [Ska21] the equalities $\omega Y_q^{\mathsf{BC}}(\widetilde{C}_w^{\mathsf{BC}}(q)) = \omega X^{\mathsf{BC}}_{\mathrm{inc}(P),q}$ hold when $q = 1$. The research team plans to study the symmetric functions (6) and (7) and to relate these to (4) and (5).

Cell closures for Hessenberg varieties

Group Leader: Julianna Tymoczko; Group Members: Per Alexandersson, Laura Colmenarejo, Sean Griffin, Greta Panova, Meesue Yoo.

Let $GL_n(\mathbb{C})$ be the group of $n \times n$ invertible matrices with complex entries and $B \subseteq GL_n(\mathbb{C})$ be the subgroup of upper-triangular matrices. The *flag variety* is the quotient $GL_n(\mathbb{C})/B$ which can be described equivalently as nested subspaces

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n$$

where each V_i is an i-dimensional subspace. The correspondence between the cosets gB and nested subspaces $V_\bullet$ is obtained by taking the span of the first i columns of g as the subspace V_i for each i. Using a variation of Gaussian elimination on the columns, we can also represent each flag uniquely as a permutation matrix w plus a matrix N_w that is nonzero only in entries that are both above and to the left of ones in the permutation w. This process chooses a specific representative of each *Schubert cell* $C_w = BwB/B$.

Hessenberg varieties are parametrized by two objects: an $n \times n$ matrix X and a *Hessenberg function* $h : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ such that h is

- **nondecreasing:** $h(i) \geq h(i-1)$ for all $i \geq 2$ and
- **upper-triangular:** $h(i) \geq i$ for all i.

The Hessenberg variety is defined as

$$\mathcal{H}ess(X, h) = \{V_\bullet \mid XV_i \subseteq V_{h(i)} \text{ for all } i\}.$$

We know a huge amount about the geometry and topology of the closures of Schubert cells including that their boundary is a union of other Schubert cells, the partial order determined by this closure—called the *Bruhat order*—is also characterized combinatorially by whether one permutation appears as a subword of another, whether the closure of a Schubert cell is singular is determined by whether the corresponding partition contains certain patterns, and what kind of singularity is also determined by permutation patterns, and so on.

In this project, our goal was to generalize these kinds of results in classical Schubert calculus to Hessenberg varieties in the case when the Hessenberg function is $h(i) = \min(i+1, n)$, and X is a nilpotent matrix.

Cells for X nilpotent and $h(i) = \min(i+1, n)$. Hessenberg varieties have a well-known paving by affines that can be expressed as an intersection with (carefully chosen) Schubert cells [Tym06]. However, very little is known about the closures or singularities of the pieces of these pavings.

Observation 2. *We refer to the pieces of the affine paving as Hessenberg Schubert cells. During the group sessions, we found a bijection between Hessenberg Schubert cells in the nilpotent case and rook walks in a multidimensional chess board. More concretely, if the Jordan blocks of X have sizes $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ then the cells correspond to rook walks in a $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_\ell$ board. For instance, if we have two Jordan blocks of the same size, it corresponds to the rook walks enumerated in A051708 in the OEIS.*

For 2-row Springer fibers, Goldwasser, Sun, and the fourth author [GST] have draft results showing that the cells can be described using noncrossing matchings, such that the cell closures correspond to certain unnestings of the matchings. With other coauthors, they have partially extended this to 3-row Springer fibers.

Our BIRS project group investigated the following question in the particular case when X is nilpotent and $h(i) = \min(i+1, n)$.

Question 3. Can we characterize the cell closures for Hessenberg varieties for particular X and h? What about the intersections of the components? What combinatorial parametrization of the cells are most useful?

Let X be the nilpotent matrix defined by $Xe_i = e_{i-1}$ for $i \notin \{1, 1+\lambda_1, \ldots, 1+\lambda_1+\cdots+\lambda_{\ell(\lambda)-1}\}$ and $Xe_i = 0$ otherwise. We made several observations following some computational experiments. We record them here:

- The intersection of Schubert cell C_w with our Hessenberg variety $\mathcal{H}ess(X, h)$ is non-empty if and only for each $i \notin \{1, 1+\lambda_1, \ldots, 1+\lambda_1+\cdots+\lambda_{\ell(\lambda)}\}$

$$\text{if } w^{-1}(i) < w^{-1}(i-1) \text{ then } w^{-1}(i-1) = w^{-1}(i) + 1.$$

- In particular, if w is a shuffle of

$$\{1, \ldots, \lambda_1\},\ \{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}, \ldots, \{1+\lambda_1+\cdots+\lambda_{\ell(\lambda)-1}, \ldots, \lambda_1+\cdots+\lambda_\ell\},$$

  then C_w intersects the Hessenberg variety.

Irreducible components for the 2-row rectangular nilpotent case. A variety is **equidimensional** if all of its irreducible components are the same dimension, which must equal the dimension of the variety. It is well known [Spa76] that Springer fibers are equidimensional and their irreducible components are in bijection with standard Young tableaux.

Unlike with Springer fibers, Hessenberg varieties are not always equidimensional. A complete list of equidimensional Hessenberg varieties or condition for Hessenberg varieties to be equidimensional is not currently known.

Using a Sagemath computation, we found Hessenberg varieties that are not equidimensional when $h(i) = i+1$ for $i < n$ and X is nilpotent. Moreover, we checked the following properties of $\mathcal{H}ess(X, h)$.

Observation 4. *Let X be a nilpotent matrix whose Jordan type is rectangular of shape (k, k) where $n = 2k$, and let $h(i) = \min(i+1, n)$.*

1. *For $k = 2$, the irreducible components of $\mathcal{H}ess(X, h)$ are dimensions 4 and 3. For $k = 3$, the irreducible components are dimensions $7, 6, 6, 6$. These are surprising results since they are among the simplest Hessenberg varieties that are not Springer fibers (which are known to be equidimensional).*

2. *Nonetheless for $k = 2$, both irreducible components are Cohen-Macaulay.*

The observations above lead us to the following questions.

Question 5. Are the irreducible components of $\mathcal{H}ess(X, h)$ always codimension at most 1 in the case of X nilpotent of Jordan type (k, k) and Hessenberg function $h(i) = \min(i + 1, n)$?

Question 6. In the case above, are all irreducible components of $\mathcal{H}ess(X, h)$ Cohen-Macaulay?

To elaborate on the case $k = 2$, both irreducible components $\mathcal{H}ess(X, h)$ intersect the big open subspace of the flag variety associated with the longest permutation:

$$U_{w_0} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 \\ x_{41} & x_{42} & x_{43} & 1 \end{pmatrix} \right\}.$$

On this open subset, the 4-dimensional component satisfies the equations $x_{21}x_{32} - x_{31} + x_{42} = 0$ and $x_{21}x_{42} - x_{41} = 0$ while the 3-dimensional component satisfies three equations $x_{21} = x_{41} = x_{43} = 0$.

Combinatorics of Theta operators

Group Leader: Michele D'Adderio; Group Members: JiSun Huh, Philippe Nadeau, Anne Schilling, George Seelinger, Andy Wilson.

Background and open problems. The Theta operators Θ_f (where f is any symmetric function) were introduced in [DIVW21] and they already proved to be an important ingredient in the theory of plethystic operators and Macdonald polynomials, e.g. by providing a closed conjectural formula for the Frobenius characteristic of $\mathfrak{S}_n$ diagonal coinvariants with two sets of commuting variables and two sets of anticommuting variables [DIVW21], and by enabling a proof of the compositional Delta conjecture [DM22].

A closer study of these operators led to the following **conjectural** formula [DIL+22]:

$$\Delta_{e_1}\Theta_{e_1^{n-1}}e_1\Big|_{t=1} = \sum_{T\in \mathsf{RTT}_0(1^n)} q^{\mathsf{inv}(T)}x^T \tag{8}$$

where $\mathsf{RTT}_0(1^n)$ are certain *rooted tiered trees*. This is in fact a particular case of a more general conjecture: see [DIL+22]*Conjecture 6.4.

While proving the full (8) might be an ambitious problem, the following consequence might be more tractable (certainly very intriguing).

Problem 7. Show that the right hand side of (8) is a symmetric function.

Remark 8. 1) Notice that in [DIL+22] a similar formula is proved for $\Theta_{e_1^{n-1}}e_1\Big|_{t=1}$, whose coefficients in the monomial expansion are known to be Kac polynomials of dandelion quivers.

2) In similar situations (e.g. in the Delta conjecture) the symmetry is proved by showing that the given formula is a positive sum of known symmetric functions (e.g. LLT in the Delta case). Notice also that the whole symmetric function seems to be Schur positive, even before the specialization $t = 1$, leaving the natural (and possibly related) open problem:

Problem 9. Find a statistic tstat on rooted tiered trees such that

$$\Delta_{e_1}\Theta_{e_1^{n-1}}e_1 = \sum_{T\in \mathsf{RTT}_0(1^n)} q^{\mathsf{inv}(T)}t^{\mathsf{tstat}(T)}x^T.$$

3) In the remarkable [IR22], the following formula is **proved**:

$$\Delta_{e_1}\Theta_{e_1^{n-1}}e_1\Big|_{t=1} = \sum_{p\in \mathrm{PF}^{\varnothing}_{(1^n)}} q^{\mathsf{area}(p)}e_{\eta(p)}. \tag{9}$$

Again, this is in fact a special case of a much more general result: see [IR22, Theorem 1.1]. It should be also noticed that before the specialization $t = 1$ we do not have e-positivity in general, though it seems to hold in the above case for $t = t + 1$.

Despite being just a combination of (8) and (9), it is worth stating the following:

Problem 10. Prove directly/combinatorially the equality of the right hand side of (8) and the right hand side of (9).

Summary of group discussion. During the discussion group in Banff, first of all we talked about the general setting of the problems, with the relevant background. Then we focused our attention to Problem 10. More specifically, we looked at two special cases: the specialization at $q = 0$, and the scalar product with e_n.

For the specialization at $q = 0$ we first looked at what the right hand side of (9) gives in this case, and we tried to find a recursion that would prove the expansion in the elementary symmetric functions. Then we looked at what the right hand side of (8) gives, and we tried to find an expansion in terms of Gessel fundamental quasisymmetric functions. In this case it seems already problematic to characterize the trees with zero inversions. In both cases our progress was only partial.

For the scalar product with e_n, the identity translates into an identity between generating functions of rooted labelled forests and parking functions. During the discussion group we studied a known result in the literature providing a bijective proof of this special case, with the idea of trying to extend this known bijection to get more informations about our problem. Also in this last case our progress was only partial, mainly for lack of time.

LLT polynomials, generalized Catalan combinatorics, and multiparameter extensions

Group Leader: François Bergeron; Group Members: Timothy Chow, Samantha Dahlberg, Jennifer Morse, Foster Tom, Alexander Woo.

The plan of our group was to explore the algebraic and geometric interplay between vertical strip LLT-polynomials and generalized Catalan combinatorics. On the combinatorial side, the objects considered are sets of partitions contained in a fixed "triangular" partition, as well as their decorated versions (a.k.a. parking functions). These correspond to the "under any line" paths defined in [BHM+21], occurring in conjunction with the exploration of Schiffmanns algebra of operators and extensions. We also intend to study multiparameter versions of the resulting enumerations, that involve LLT-polynomials.

During this week, the group first reviewed the problem, including definitions, key examples, and other foundational material to establish a common vocabulary and understanding. Then the group discussed various speculative approaches that at this point remain imprecise. The bulk of the rest of our work was an exploration of whether we could generalize the formulas of Negut [Neg14] so that they hold for many parameters. The group continues to explore this idea without definitive conclusions (yet).

References

[AK94] Susumu Ariki and Kazuhiko Koike. A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations. *Adv. Math.*, 106(2):216–243, 1994.

[AP18] Per Alexandersson and Greta Panova. LLT polynomials, chromatic quasisymmetric functions and graphs with cycles. *Discrete Mathematics*, 341(12):3453–3482, December 2018.

[AS22] Per Alexandersson and Robin Sulzgruber. A combinatorial expansion of vertical-strip LLT polynomials in the basis of elementary symmetric functions. *Adv. Math.*, 400:Paper No. 108256, 58, 2022.

[BC18a] Patrick Brosnan and Timothy Y. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Adv.Math.*, 329:955–1001, April 2018.

[BC18b] Patrick Brosnan and Timothy Y. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Adv. Math.*, 329:955–1001, 2018.

[BHM+21] Jonah Blasiak, Mark Haiman, Jennifer Morse, Anna Pun, and George Seelinger. A shuffle theorem for paths under any line. *ArXiv e-prints*, 2021.

[BRW96] D. Beck, J. Remmel, and T. Whitehead. The combinatorics of transition matrices between the bases of the symmetric functions and the B_n analogues. *Discrete Math.*, 153:3–27, 1996.

[CHSS16] Samuel Clearman, Matthew Hyatt, Brittany Shelton, and Mark Skandera. Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements. *Electron. J. Combin.*, 23(2), 2016.

[CM17] Erik Carlsson and Anton Mellit. A proof of the shuffle conjecture. *Journal of the American Mathematical Society*, 31(3):661–697, November 2017.

[DIL+22] Michele D'Adderio, Alessandro Iraci, Yvan LeBorgne, Marino Romero, and Anna Vanden Wyngaerd. Tiered trees and theta operators. *ArXiv e-prints*, 2022.

[DIVW21] Michele D'Adderio, Alessandro Iraci, and Anna Vanden Wyngaerd. Theta operators, refined delta conjectures, and coinvariants. *Adv. Math.*, 376:Paper No. 107447, 59, 2021.

[DM22] Michele D'Adderio and Anton Mellit. A proof of the compositional delta conjecture. *Adv. Math.*, 402:Paper No. 108342, 17, 2022.

[DMPS92] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg varieties. *Trans. Amer. Math. Soc.*, 332(2):529–534, 1992.

[DMS88] Filippo De Mari and Mark A. Shayman. Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix. *Acta Appl. Math.*, 12(3):213–235, 1988.

[GH06] Ian Grojnowski and Mark Haiman. Affine Hecke algebras and positivity of LLT and Macdonald polynomials. *ArXiv e-prints*, 2006.

[GP16a] Mathieu Guay-Paquet. A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra. *ArXiv e-prints*, 2016.

[GP16b] Matthieu Guay-Paquet. A second proof of the shareshian–wachs conjecture, by way of a new hopf algebra. *ArXiv e-prints*, 2016.

[GST] Talia Goldwasser, Garcia Sun, and Julianna Tymoczko. Cell closures for the (n, n) Springer fiber. *In preparation*.

[Hag07] James Haglund. *The q, t-Catalan numbers and the space of diagonal harmonics (University lecture series)*. American Mathematical Society, 2007.

[HHL05a] James Haglund, Mark D. Haiman, and Nicholas A. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18(03):735–762, July 2005.

[HHL05b] James Haglund, Mark D. Haiman, Nicholas A. Loehr, Jeffrey B. Remmel, and Alexei Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke Mathematical Journal*, 126(2):195–232, February 2005.

[HMZ12] James Haglund, Jennifer Morse, and Mike Zabrocki. A compositional shuffle conjecture specifying touch points of the Dyck path. *Canad. J. Math.*, 64(4):822–844, August 2012.

[HRW18] James Haglund, Jeffrey B. Remmel, and Andrew T. Wilson. The delta conjecture. *Transactions of the American Mathematical Society*, 370(6):4029–4057, February 2018.

[Hwa22] Byung-Hak Hwang. Chromatic quasisymmetric functions and noncommutative p-symmetric functions. *ArXiv e-prints*, 2022.

[IR22] Alessandro Iraci and Marino Romero. Delta and theta operator expansions. *ArXiv e-prints*, 2022.

[LLT97] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras and unipotent varieties. *J. Math. Phys*, 38:1041–1068, 1997.

[LT00] Bernard Leclerc and Jean-Yves Thibon. Littlewood–Richardson coefficients and Kazhdan–Lusztig polynomials. In *Combinatorial Methods in Representation Theory*, volume 28, pages 155–220. Mathematical Society of Japan, 2000.

[Lus86] George Lusztig. Character sheaves. V. *Adv. Math.*, 61(2):103–155, 1986.

[Mac79] I.G. Macdonald. *Symmetric Fuctions and Hall Polynomials*. Oxford University Press, Oxford, 1979.

[Neg14] Andrei Negut. The shuffle algebra revisited. *Int. Math. Res. Not. IMRN*, (22):6242–6275, 2014.

[Ser17] Emily Sergel. A proof of the square paths conjecture. *Journal of Combinatorial Theory, Series A*, 152:363–379, November 2017.

[Ska21] Mark Skandera. Graph coloring and hyperoctahedral group character evaluations. In preparation, 2021.

[Spa76] N. Spaltenstein. The fixed point set of a unipotent transformation on the flag manifold. *Nederl. Akad. Wetensch. Proc. Ser. A* **79**=*Indag. Math.*, 38(5):452–456, 1976.

[SW12] John Shareshian and Michelle Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In A Bjorner, F Cohen, C De Concini, C Procesi, and M Salvetti, editors, *Configuration Spaces*, pages 433–460, Pisa, 2012. Edizione Della Normale.

[SW16] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, 295:497–551, 2016.

[Tym06] Julianna S. Tymoczko. Linear conditions imposed on flag varieties. *Amer. J. Math.*, 128(6):1587–1604, 2006.

[Tym08] Julianna S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 365–384. Amer. Math. Soc., Providence, RI, 2008.