Introduction to bootstrap percolation and kinetically constrained models[1]

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Markov Chains with Kinetic Constraints and Applications workshop, Banff

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Bootstrap percolation

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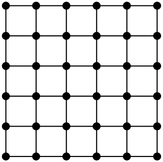
- Geometry: $\mathbb{Z}^2$.



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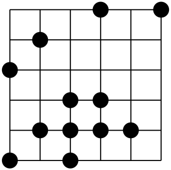

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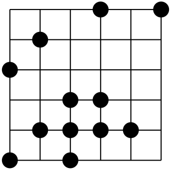

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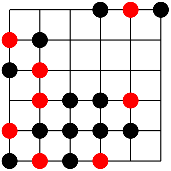

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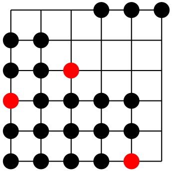

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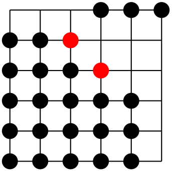

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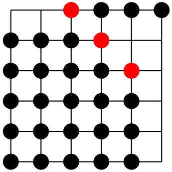

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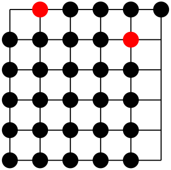

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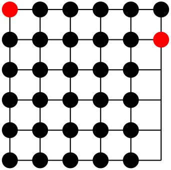

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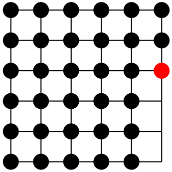

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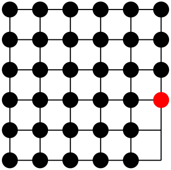

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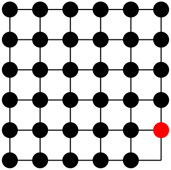

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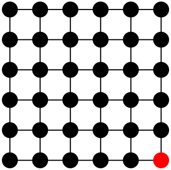

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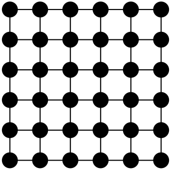

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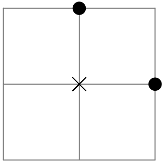

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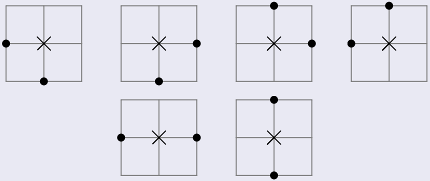

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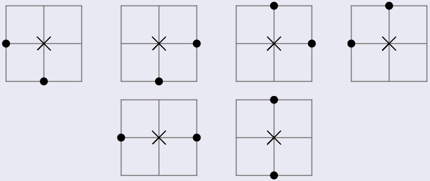

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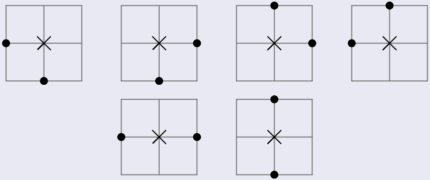
2-neighbour bootstrap percolation



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Examples

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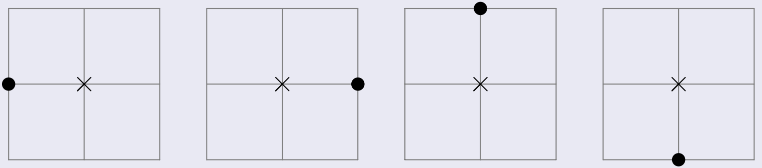

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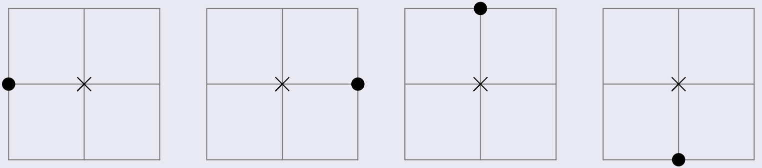

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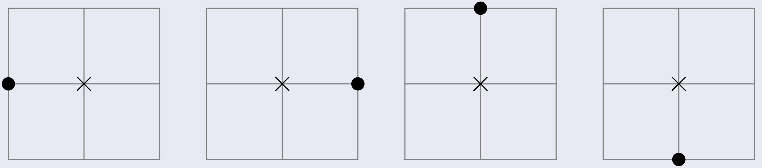

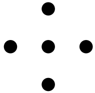

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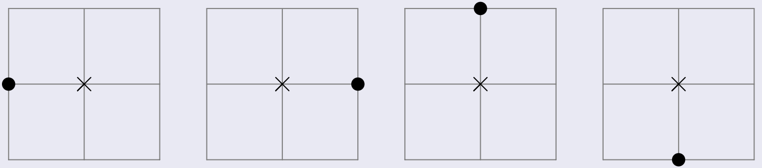

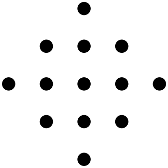

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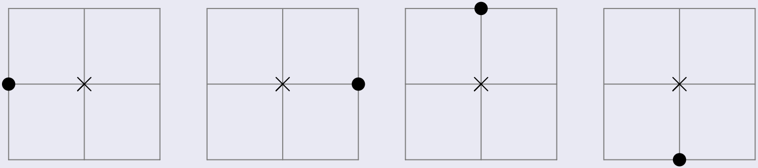

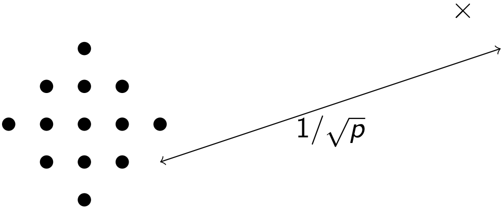

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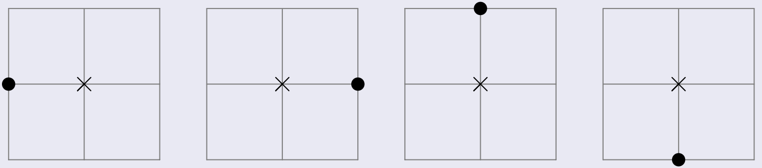

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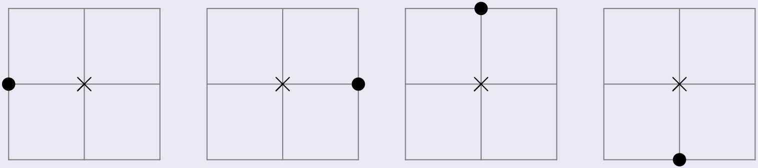

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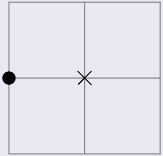

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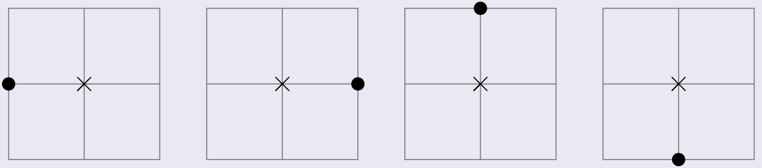

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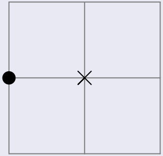

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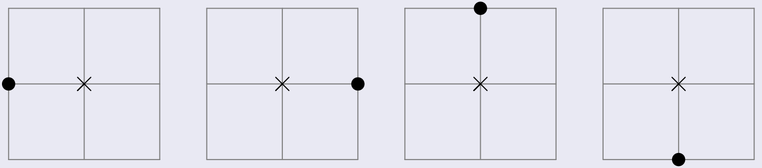

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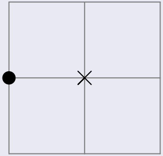

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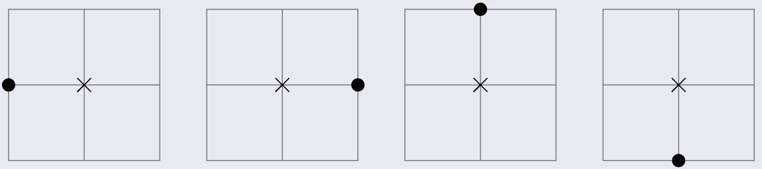

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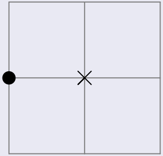

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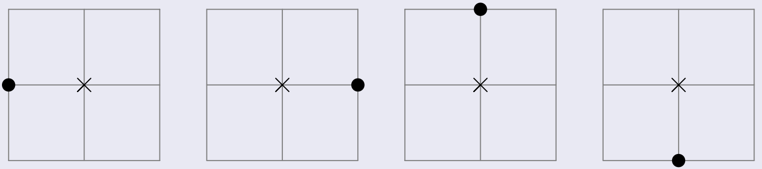

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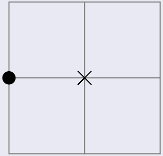

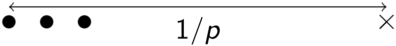

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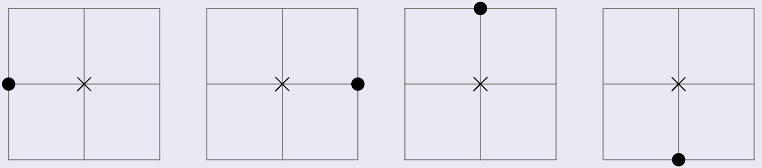

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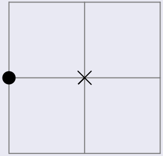

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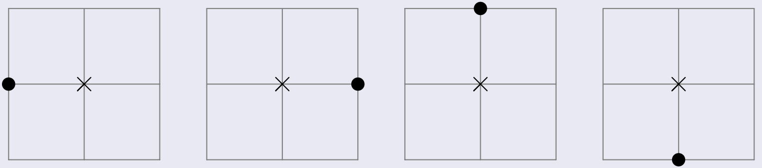

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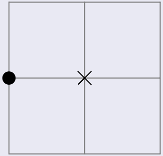

$$\tau \sim \mathcal{G}(p)$$

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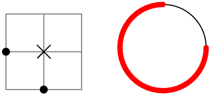

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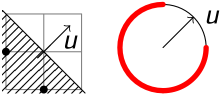

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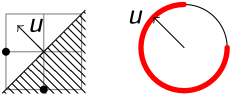

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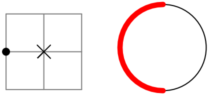

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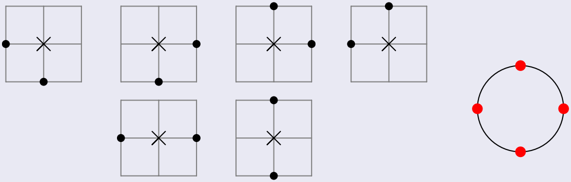
2-neighbour bootstrap percolation



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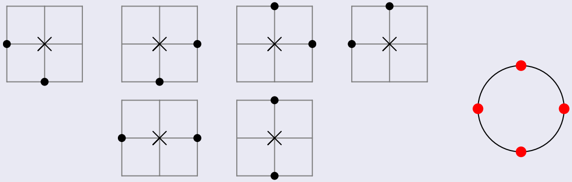

Theorem (van Enter'87)

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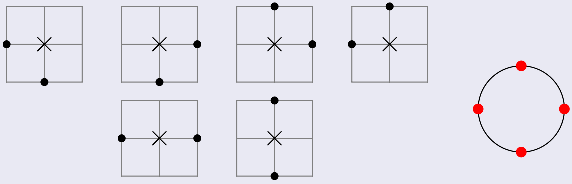

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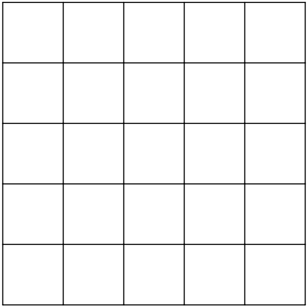

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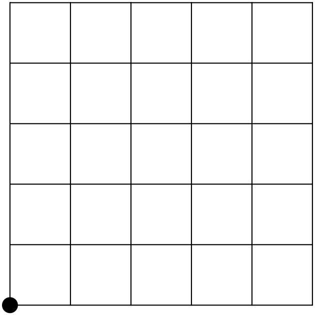

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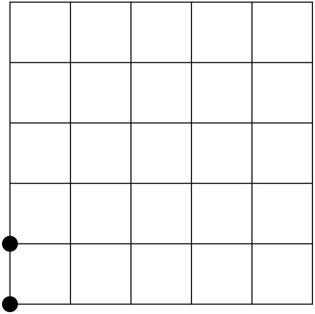

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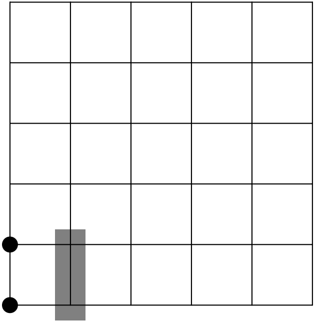

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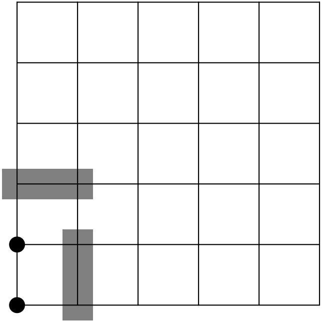

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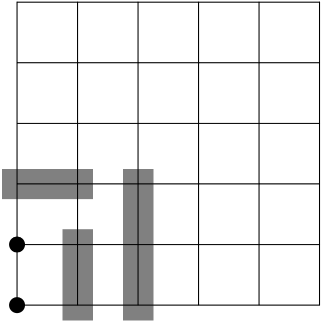

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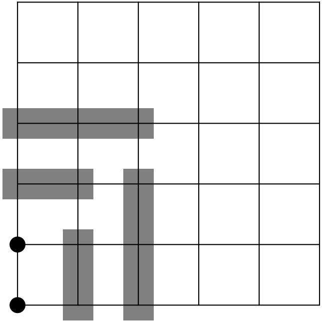

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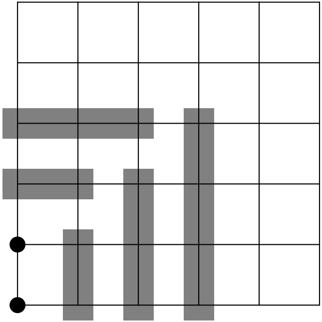

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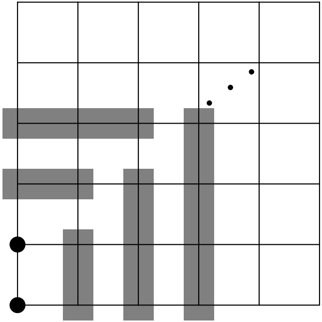

2-neighbour bootstrap percolation

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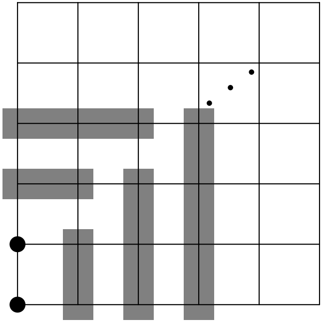

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To be continued...

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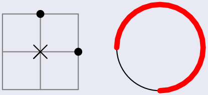

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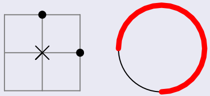

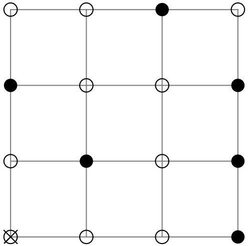

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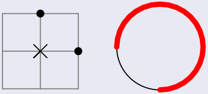

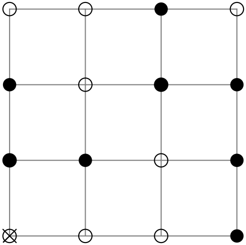

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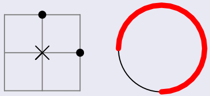

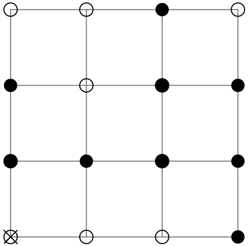

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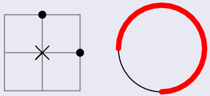

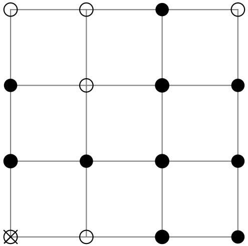

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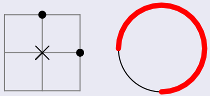

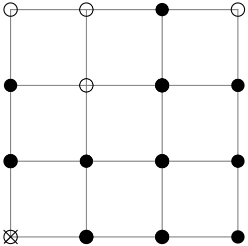

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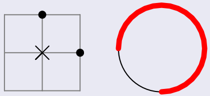

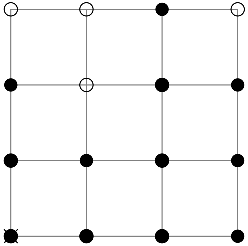

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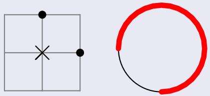

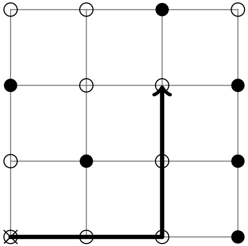

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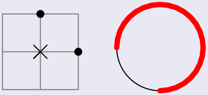

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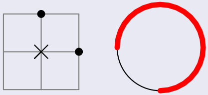

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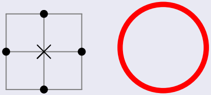

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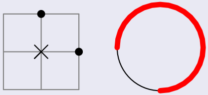

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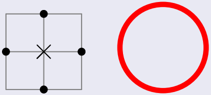

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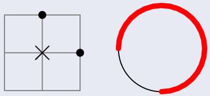

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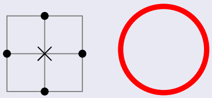

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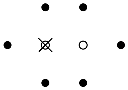

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Theorem (H'22)

For all $\mathcal{U}$ supported in a half-space the conjecture holds.

Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet.$$

- Infection time: $\tau = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.
- Critical probability: $p_c = \inf\{p \in [0, 1] : \pi(\tau = \infty) = 0\}$.

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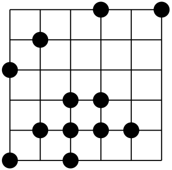

2-neighbour KCM

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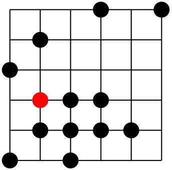

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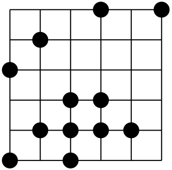

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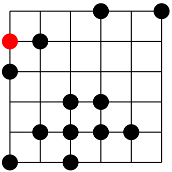

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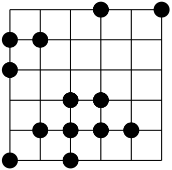

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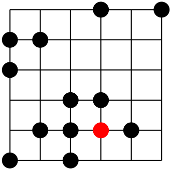

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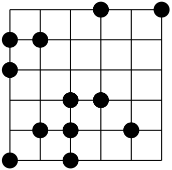

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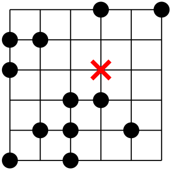

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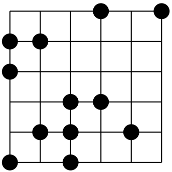

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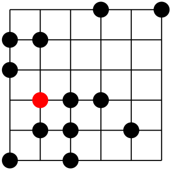

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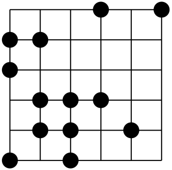

2-neighbour KCM

Theorem (Cancrini–Martinelli–Roberto–Toninelli'08)

For any $\mathcal{U}$ the following are equivalent:

- $\pi(\tau = \infty) = 0$ *in $\mathcal{U}$-bootstrap percolation;*
- $\mathbb{P}_\pi(\tau = \infty) = 0$ *in the $\mathcal{U}$-KCM;*
- 0 *is a simple eigenvalue of the generator of the $\mathcal{U}$-KCM;*
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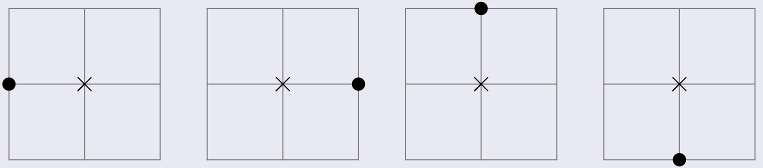
Theorem (CMRT08,H21)

For any $\mathcal{U}$ the following are equivalent:

- *in $\mathcal{U}$-bootstrap percolation τ has an exponential moment;*
- *in $\mathcal{U}$-KCM τ has an exponential moment;*
- $T_{\mathrm{rel}} < \infty$ *for the $\mathcal{U}$-KCM.*

1-neighbour KCM



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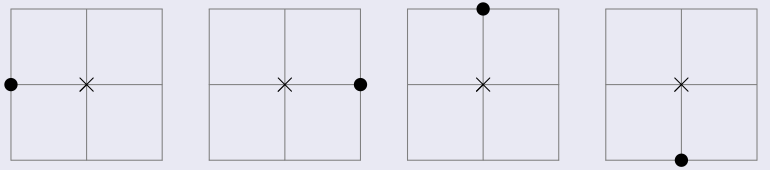

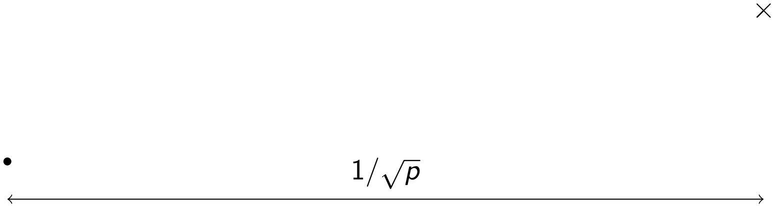

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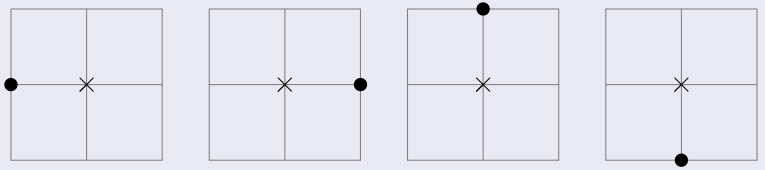

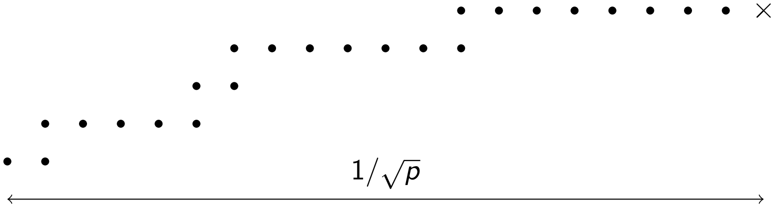

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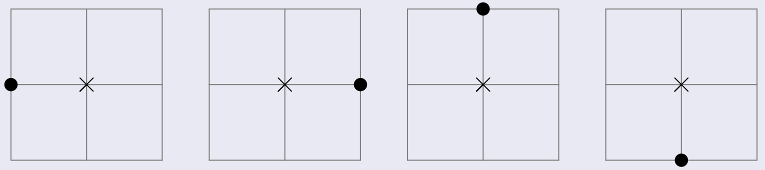

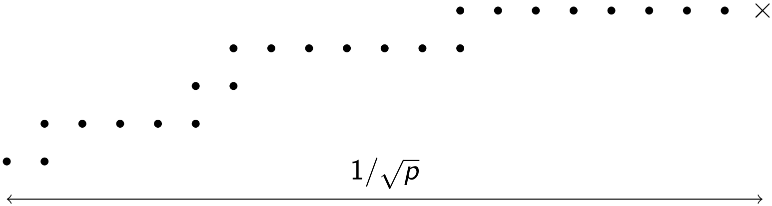

$$\tau \leqslant \exp(1/\sqrt{p})$$

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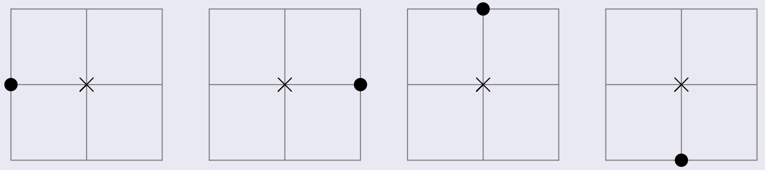

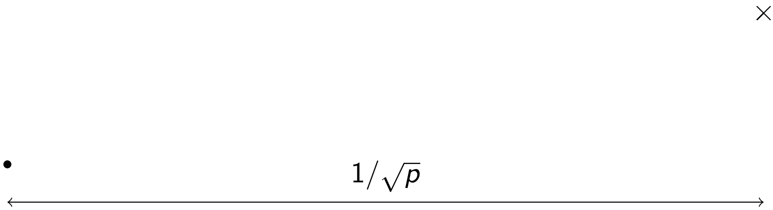

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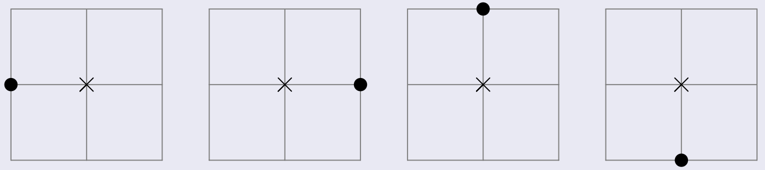

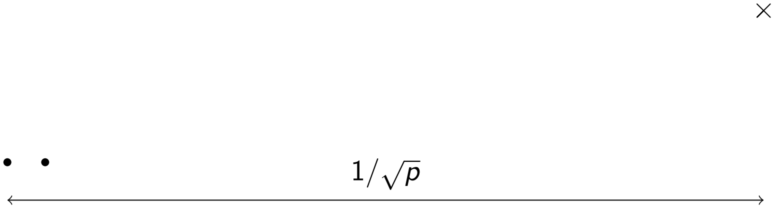

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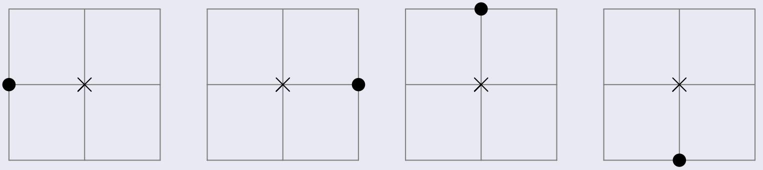

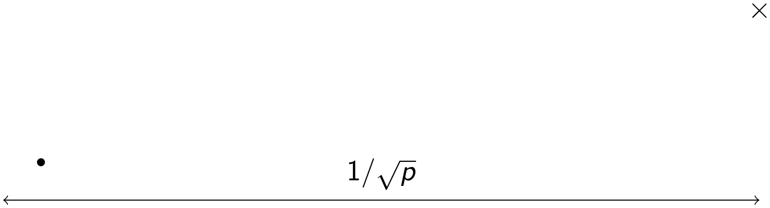

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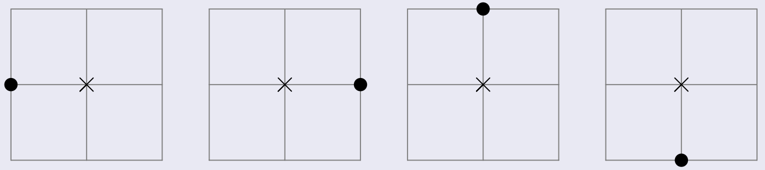

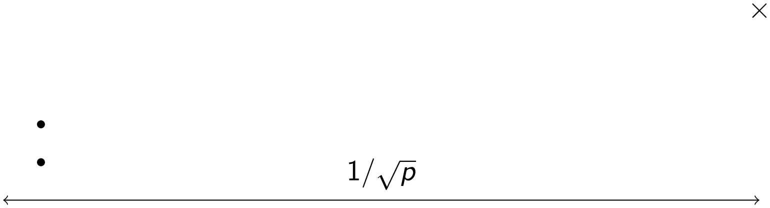

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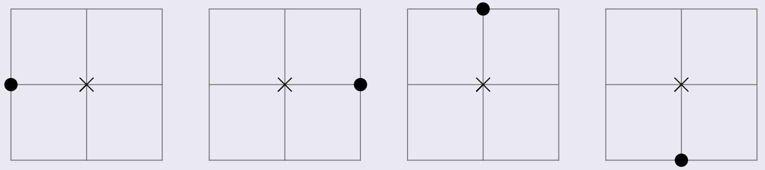

$$1/\sqrt{p}$$

$$\tau \leqslant (1/\sqrt{p})^2/p = 1/p^2$$

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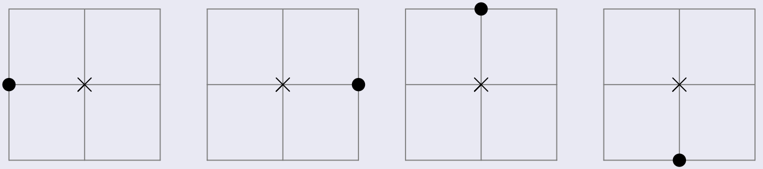

Theorem (CMRT08,Shapira'20)

For the 1-neighbour KCM we have

$$T_{\text{rel}} = \begin{cases} \Theta(p^{-3}) & d = 1, \\ p^{-2+o(1)} & d = 2, \\ \Theta(p^{-2}) & d \geqslant 3. \end{cases}$$

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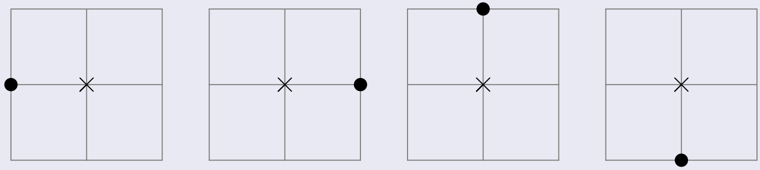

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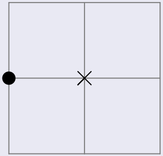

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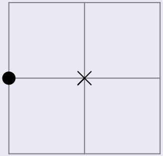

Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

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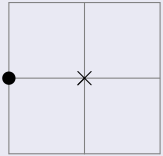

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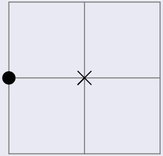

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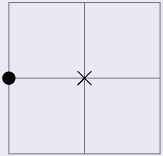

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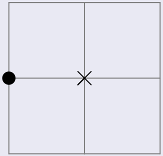

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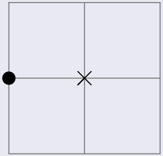

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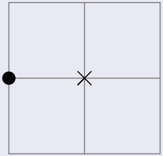

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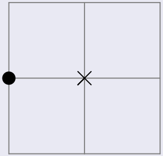

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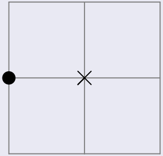

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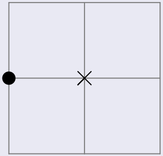

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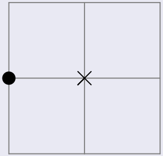

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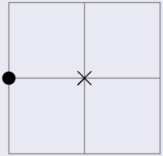

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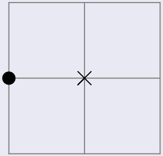

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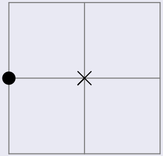

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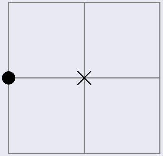

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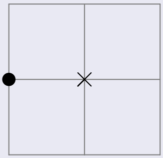

Lemma (Mauch–Jackle'99, Sollich–Evans'99, Chung–Diaconis–Graham'01)

Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1}-1$.

Theorem (Aldous–Diaconis'02, CMRT08)

$$T_{\mathrm{rel}} = \exp\left(\frac{\log^2(1/p)}{2\log 2 + o(1)}\right).$$

Lemma (Two-block Poincaré inequality of CMRT)

Let X_1 and X_2 be two independent RV valued in the finite sets $\mathbb{X}_1, \mathbb{X}_2$. Let $\mathcal{H} \subset \mathbb{X}_1$ with $p := \mathbb{P}(X_1 \in \mathcal{H}) > 0$. Then for any $f : \mathbb{X}_1, \mathbb{X}_2 \to \mathbb{R}$

$$\mathrm{Var}(f) \leqslant \frac{1}{1-\sqrt{1-p}}\mathbb{E}[\mathrm{Var}(f|X_2) + \mathbb{1}_{X_1 \in \mathcal{H}} \mathrm{Var}(f|X_1)]$$

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Idea: $e^{-1/T_{\mathrm{rel}}} = \lim_{t\to\infty}(d_{\mathrm{TV}}(\mu_t, \pi))^{1/t}$.

Probabilistic proof

Two chains couple as soon as we update X_1 so that $\mathcal{H}$ occurs and then X_2. There are $\lfloor N/2 \rfloor$ attempted updates at X_2 preceded by an update at X_1, where $N \sim \mathcal{P}(t)$. Each succeeds with probability p, so

$$d_{\mathrm{TV}}(\mu_t, \pi) \leqslant \mathbb{P}(\text{not coupled at time } t) \leqslant \mathbb{E}\left[(1-p)^{\lfloor N/2 \rfloor}\right]$$

$$\approx \mathbb{E}\left[\left(\sqrt{1-p}\right)^N\right] = \exp\left(-t\left(1-\sqrt{1-p}\right)\right).$$

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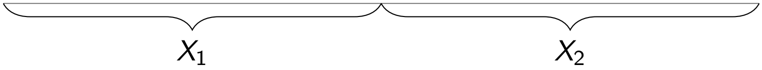

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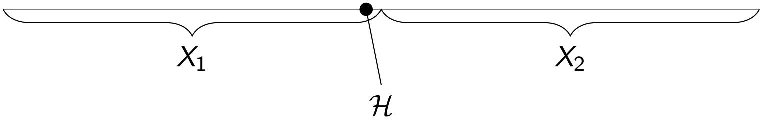

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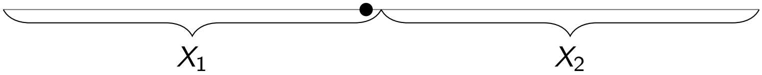

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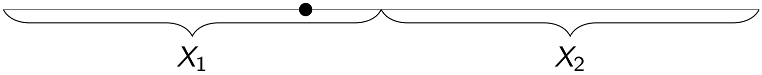

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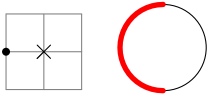

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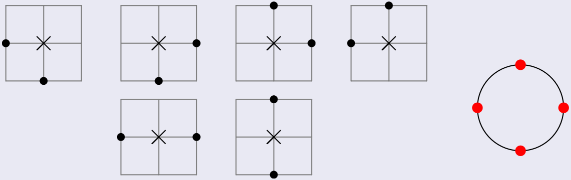
Theorem (Martinelli–Toninelli'19, Martinelli–Morris–Toninelli'19, Marêché'20, Marêché–Martinelli–Toninelli'20)

For a supercritical KCM we have

- $T_{\mathrm{rel}} = p^{-\Theta(1)}$ *if* $\mathcal{U}$ *is unrooted;*
- $T_{\mathrm{rel}} = \exp(\Theta(\log^2(1/p)))$ *if* $\mathcal{U}$ *is rooted.*

2-neighbour KCM



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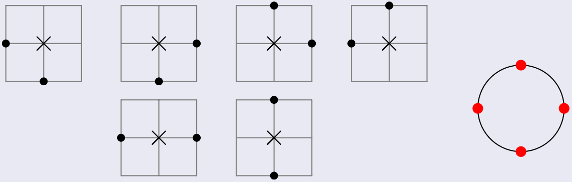

Theorem (H–Martinelli–Toninelli'20+)

For the 2-neighbour model

$$\tau = \exp\left(\frac{\pi^2 + o(1)}{9p}\right) = \left(\tau^{\mathrm{BP}}\right)^{2+o(1)}.$$

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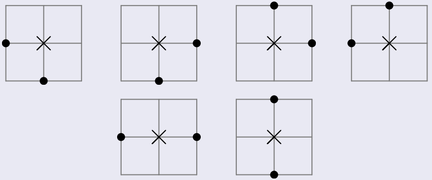

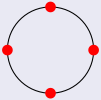

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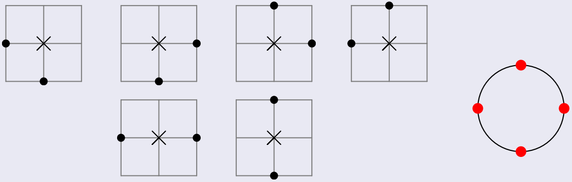

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Bootstrap percolation universality statements for supercritical and subcritical families extend to higher dimensions modulo adapting the definition as needed. For every critical family there exists an integer $1 \leqslant r \leqslant d-1$ such that

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For KCM the analogous universality result (with a rooted/unrooted distinction for supercritical families) is not known. More precisely, the upper bounds for supercritical and critical families are still missing.

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- Find the asymptotics of τ starting from a product measure different from the invariant one. Even for the 1-neighbour case this is open when p is not close to 1.
- Study universality: beyond $\mathbb{Z}^d$, in inhomogeneous settings, for conservative KCM, for plaquette models, ...

Thank you.

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