Non-linear equations described by Sato, Segal-Wilson theory

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Integrable non-linear PDE

KdV	$\partial_t u = \frac{1}{4}\partial_x^3 u + \frac{3}{2} u \partial_x u$
Boussinesq	$\partial_t^2 u = -\frac{1}{3}\partial_x^2\left(\partial_x^2 u + 2u^2\right)$
Non-linear Schrödinger	$i\partial_t u = \frac{1}{2}\partial_x^2 u \mp 4\left\|u\right\|^2 u$
Sawada-Kotera	$\partial_t u = \partial_x^5 u + 45u^2\partial_x u - 15\partial_x u \partial_x^2 u - 15u\partial_x^3 u$
Benjamin-Ono	$\partial_t u = -H\left(\partial_x^2 u\right) - u\partial_x u$ (H: Hilbert transf.)
Sine-Gordon	$\partial_t^2 u = \partial_x^2 u - \sin u$
Lund-Regge	$\begin{cases} \partial_t^2 u = \partial_x^2 u + \dfrac{\sin u/2}{2\cos^2 u/2}\left((\partial_t v)^2 - (\partial_x v)^2\right) \\ \partial_t^2 v = \partial_x^2 v + 2\dfrac{\partial_x u \partial_x v - \partial_t u \partial_t v}{\sin u} \end{cases}$

Lax pair

The above integrable systems are known to have the **Lax pair**s:

$$\text{Lax pair:} \begin{cases} L = \partial_x^{\nu} + q_{\nu-2}\partial_x^{\nu-2} + \cdots + q_1\partial_x + q_0 \\ P \text{ differential operators s.t. the order of } [P, L] \leq \nu - 2 \end{cases} .$$

Then, setting

$$[P, L] = f_{\nu-2}\partial_x^{\nu-2} + f_{\nu-3}\partial_x^{\nu-3} + \cdots f_1\partial_x + f_0,$$

with f_j universal polynomials of $\left\{\partial_x^j q_k\right\}$, one has

$$\partial_t q_j = f_j \ \ (0 \leq j \leq \nu - 2) \ \ (\Leftrightarrow \partial_t L = [P, L]) .$$

Example: For the KdV equation $\nu = 2$ $(' = \partial_x)$

$$\begin{cases} L = \partial_x^2 + q \\ P = \partial_x^3 + \dfrac{3}{2}q\partial_x + \dfrac{3}{4}q' \\ [P, L] = \dfrac{1}{4}q''' + \dfrac{3}{2}qq' \end{cases} .$$

Sato, Segal-Wilson theory 1 -KdV-

1. Let $C = \{z \in \mathbb{C};\ |z| = r\}$ and define the **Hardy spaces** $H_\pm$ by
$$H_+ = \left\{\sum_{j\geq 0} a_j z^j\right\},\quad H_- = \left\{\sum_{j<0} a_j z^j\right\}$$
2. $\mathcal{A} = \{\mathbf{a};\ \mathbf{a}(\lambda) = (a_1(\lambda), a_2(\lambda))$ bounded on $C\}$. For $\mathbf{a} \in \mathcal{A}$ define
$$(\mathbf{a}u)(\lambda) = a_1(\lambda)u(\lambda) + a_1(\lambda)u(-\lambda)$$
Then $W_\mathbf{a} = \mathbf{a}H_+ \subset L^2(C)$ satisfies $z^2 W_\mathbf{a} \subset W_\mathbf{a}$.
3. Define projections on $L^2(C)$ by
$$L^2(C) \ni u \to \mathfrak{p}_\pm u(z) = \pm\frac{1}{2\pi i}\int_C \frac{u(\lambda)}{\lambda - z} d\lambda \in H_\pm,\ (z \in D_\pm),$$
and **Toeplitz operator** $T(\mathbf{a})$ on H_+ by
$$T(\mathbf{a})u = \mathfrak{p}_+(\mathbf{a}u)\ \text{ for } u \in H_+.$$
4. $T(\mathbf{a}) : H_+ \to H_+$, $\mathcal{A}^{inv} = \left\{\mathbf{a};\ T(\mathbf{a})^{-1} \text{ exists}\right\}$

Sato, Segal-Wilson theory 2 -KdV-

1. $\Gamma = \left\{ g = e^h ;\ h \text{ anal. in a nhb of } D_+ \right\} \Rightarrow g\mathcal{A} \subseteq \mathcal{A}$ but $g\mathcal{A}^{inv} \not\subseteq \mathcal{A}^{inv}$
2. For $e_{t,x}(z) = e^{xz+tz^3} \in \Gamma$, $\mathbf{a} \in \mathcal{A}$ assume $e_{t,x}^{-1}\mathbf{a} \in \mathcal{A}^{inv}$ for $\forall x, t \in \mathbb{R}$, namely
$$\mathfrak{p}_+ : e_{t,x}^{-1} W_{\mathbf{a}} \left(= W_{e_{t,x}^{-1}\mathbf{a}} \right) \to H_+ \text{ is 1 to 1}$$
3. $f(t,x,z) = \left(\mathbf{a} T \left(e_{t,x}^{-1}\mathbf{a} \right)^{-1} 1 \right)(z) \in W_{\mathbf{a}}$ satisfies
$$\begin{cases} f = e^{xz+tz^3} \left(1 + \sum_{j \geq 1} r_j(t,x) z^{-j} \right) \in e_{t,x}\left(1 + H_-\right) \\ f'' = e^{xz+tz^3} \left(z^2 + r_1 z + r_2 + 2r_1' + O\left(z^{-1}\right)\right) \end{cases}$$
$$\Rightarrow \quad e_{t,x}^{-1}\left(f'' - z^2 f - 2r_1' f \right) = O\left(z^{-1}\right) \in H_-$$
4. Since $f'' - z^2 f - 2r_1' f \in W_{\mathbf{a}}$, applying $\mathfrak{p}_+$ one has
$$f'' - z^2 f - 2r_1' f = 0 \Rightarrow -\partial_x^2 f + 2r_1' f = -z^2 f \quad \text{(Schrödinger)}$$
5. Comparing the coefficients of z^{-j}, one has
$$2r_{j+1}' + r_j'' - 2r_1' r_j = 0 \Longrightarrow r_j \text{ is determined by } \left\{ r_1^{(k)} \right\}_{1 \leq k \leq j}$$

Sato, Segal-Wilson theory 3 -KdV-

1. Take derivatives ∂_t, ∂_x in $f = e^{xz+tz^3}\left(1 + r_1 z^{-1} + r_2 z^{-2} + \cdots\right)$

$$\begin{cases} \partial_t f = e_{t,x}\left(z^3 + r_1 z^2 + r_2 z + r_3 + O\left(z^{-1}\right)\right) \\ f' = e_{t,x}\left(z + r_1 + O\left(z^{-1}\right)\right) \\ f''' = e_{t,x}\left(z^3 + r_1 z^2 + r_2 z + r_3 + 3r_1' z + 3r_2' + 3r_1'' + O\left(z^{-1}\right)\right) \end{cases}$$

$$\Rightarrow \partial_t f \;- f''' + 3r_1' f' \;- 3\left(r_2' + r_1'' - r_1 r_1'\right) f \quad \in e_{t,x}^{-1} H_-$$

$$\Rightarrow \partial_t f \;- f''' + 3r_1' f' \;- 3\left(r_2' + r_1'' - r_1 r_1'\right) f \;= 0.$$

Computing the coefficient of z^{-1} gives

$$\partial_t r_1 - \left(3r_3' + 3r_2'' + r_1'''\right) + 3r_1'\left(r_2 + r_1'\right) - 3\left(r_2' + r_1'' - r_1 r_1'\right) r_1 = 0$$

$$\Longrightarrow \quad \partial_t r_1 = \frac{1}{4} r_1''' - \frac{3}{2}\left(r_1'\right)^2$$

2. $u = -2a_1' \Longrightarrow \partial_t u = \frac{1}{4}u''' + \frac{3}{2}uu'$ (KdV)
3. $e_{t,x}(z) = e^{xz+tz^n}$(odd n) gives the **KdV hierarchy** $\{\mathrm{KdV}_n\}_{\text{odd } n}$
4. $u(t,x) = -2\partial_x^2 \log \tau_{\mathbf{a}}\left(e_{t,x}\right)$ with $\tau_{\mathbf{a}}(g) = \det\left(g^{-1} T(g\mathbf{a})\, T(\mathbf{a})^{-1}\right)$.

Sato, Segal-Wilson theory 4 -Boussinesq-

1. $(\nu = 3)$ The underlying operator: $L = \partial_x^3 + q_1\partial_x + q_0$
2. $\mathcal{A} = \{\mathbf{a};\ \mathbf{a}(\lambda) = (a_1(\lambda), a_2(\lambda), a_3(\lambda))$ bdd on $C\}$. $\omega = e^{2\pi i/3}$

$$\begin{cases} (\mathbf{a}u)(\lambda) = a_1(\lambda)u(\lambda) + a_2(\lambda)u(\omega\lambda) + a_3(\lambda)u(\omega^2\lambda) \\ W_{\mathbf{a}} = \mathbf{a}H_+ \Longrightarrow z^3 W_{\mathbf{a}} \subset W_{\mathbf{a}} \end{cases}$$

3. $f = \mathbf{a}\left(T\left(e_{t,x}^{-1}\mathbf{a}\right)^{-1}1\right)(z) = e_{t,x}\left(1 + \sum_{r\geq 1} r_j z^{-j}\right)$ $(e_{t,x} = e^{xz+tz^2})$

$$\begin{cases} Lf = z^3 f \quad \text{with } q_0 = 3\left(r_1' r_1 - r_1'' - r_2'\right), q_1 = -3r_1' \\ \partial_t f - \partial_x^2 f + 2r_1' f = 0 \end{cases}$$

4. The coefficients $\{r_1, r_2\}$ satisfies

$$\begin{cases} \partial_t r_1 = 2r_2' + r_1'' - 2r_1' r_1 \\ \partial_t r_2 = 2r_1 r_2' - r_2'' - \frac{2}{3}r_1''' + 2\left(r_1'\right)^2 + 2\left(r_1'' - r_1' r_1\right) r_1 \end{cases}$$

$$\partial_t^2 r_1 = \left(-\frac{1}{3}r_1''' + 2\left(r_1'\right)^2\right)' \Rightarrow \partial_t^2 q_1 = -\frac{1}{3}\left(q_1'' + 2q_1^2\right)'' \ \textbf{(Boussinesq)}$$

5. e^{xz+tz^n}(even integer n) generates the **Boussinesq hierarchy**

1. $\mathbf{H}_{\pm} = H_{\pm} \times H_{\pm} \subset \mathbf{L}^2(C)$, $\mathcal{A} = \left\{ A(\lambda) = (a_{ij}(\lambda))_{1 \le i,j \le 2} \,;\, a_{ij} \text{ bdd on } C \right\}$

$$W_A = A\mathbf{H}_+ \Longrightarrow zW_A \subset W_A$$

2. Group: $\mathbf{\Gamma} = \left\{ G(g) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} ; \, g = e^h \in \Gamma \right\}$
3. $e_{t,x}(z) = e^{i\left(xz + tz^2\right)}$, I =the identity 2×2 matrix

$$F = AT\left(G(e_{t,x})^{-1} A\right)^{-1} I = G(e_{t,x}) \left(I + \sum_{j \ge 1} R_j z^{-j} \right)$$

4. **Dirac op.**:$\mathcal{L} = i \begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix} + \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}$ with

$R_1 = \frac{1}{2} \begin{pmatrix} * & q_2 \\ q_1 & * \end{pmatrix}$

$$\mathcal{L}\, {}^t F = z\, {}^t F$$

Sato, Segal-Wilson theory 6 -Non linear Schrödinger-

1. $R_1 = \frac{1}{2}\begin{pmatrix} * & q_2 \\ q_1 & * \end{pmatrix}$ satisfies $q_1 = 2r_2, q_2 = 2r_1$

$$\begin{cases} i\partial_t q_1 = \frac{1}{2}\partial_x^2 q_1 - q_2 q_1^2 \\ i\partial_t q_2 = -\frac{1}{2}\partial_x^2 q_2 + q_1 q_2^2 \end{cases}$$

2. For $A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \Longrightarrow q_2 = \overline{q_1} \Longrightarrow$ **de focussing NLS**

$$\mathcal{L} = \mathcal{L}^* \quad \text{(self-adjoint)}$$

3. For $A = \begin{pmatrix} a & b \\ \overline{b} & -\overline{a} \end{pmatrix} \Longrightarrow q_2 = -\overline{q_1} \Longrightarrow$ **focussing NLS**

$$J\mathcal{L}J = \mathcal{L}^* \quad (J\text{-self-adjoint}) \ \left(J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C\right)$$

Open problems

1. KdV: $L = -\partial_x^2 + q \Longrightarrow$Weyl functions $m_\pm$

$$m(z) = \begin{cases} -m_+\left(-z^2\right) & \text{if} \quad \operatorname{Re} z > 0 \\ m_-\left(-z^2\right) & \text{if} \quad \operatorname{Re} z < 0 \end{cases}$$

2. $\mathcal{A}^{inv} \ni A = (1, m(z)/z)$ enables us to extend the theory to unbdd. C and more general initial data including smooth ergodic ones.
3. de focussing NLS: Set $\phi = \dfrac{i-m}{i+m}$ with the Weyl function m.

$$A = \begin{pmatrix} 1 & \phi \\ \overline{\phi} & 1 \end{pmatrix} \in \mathcal{A}^{inv} \Longrightarrow \text{Extend to general initial data ?}$$

4. Boussinesq eq., focussing NLS the underlying L is non-self adjoint
5. Costruct the theory for the other integrable systems.

Bibliography

[1] M. Sato: *Soliton Equations as Dynamical Systems on an Infinite Dimensional Grassmann Manifolds*, Suriken Koukyuroku **439** (1981), 30-46. (http://www.kurims.kyoto-u.ac.jp/en/publi-01.html)

[2] G.Segal-G.Wilson:Loop groups and equations of KdV type, Publ. IHES, **61** (1985),5-65.

Thanks

Thank you for attention !