Stochastic Black-Scholes Equation under Rough Volatility

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Stochastic Modelling of Big Data in Finance, Insurance and Energy Markets

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Outline

Fractional Brownian motion

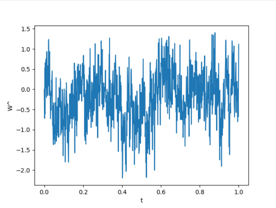
• Denote by W a standard Brownian motion (Wiener process).

• Denote by $\widehat{W}$ a fractional Brownian motion (fBm) of Riemann-Liouville type with Hurst index $0 < H < 1$, i.e.,

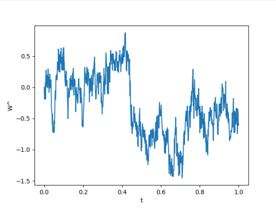
$$\widehat{W}_t := \int_0^t \mathcal{K}(t-s)dW_s, \quad \mathcal{K}(r) := \sqrt{2H} r^{H-1/2}, \quad r > 0. \tag{1}$$

• When $H = \frac{1}{2}$, $W = \widehat{W}$.

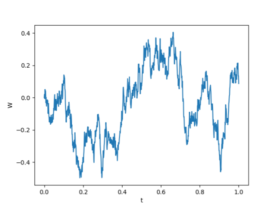
Roughness of the paths



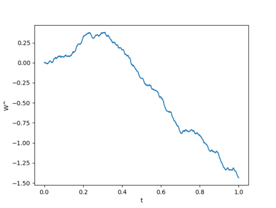
H=0.1



H=0.3



H=0.5



H=0.8

Figure: Rough paths are $(H - \varepsilon)$-Hölder continuous, for any $\varepsilon > 0$.

Volatility is rough

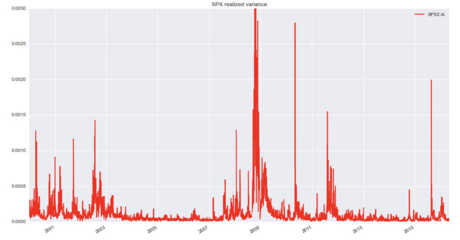
• Consider a general stochastic volatility model given under a risk neutral probability measure as

$$\begin{cases} dS_t = rS_t dt + S_t \sqrt{V_t} \left(\rho \, dW_t + \sqrt{1-\rho^2} \, dB_t \right); \\ S_0 = s_0, \end{cases} \tag{2}$$

where $\rho \in [-1, 1]$ denotes the correlation coefficient and the constant r the interest rate.

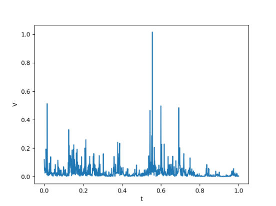
• $(V_t)_{t \geq 0}$ is the *volatility* process. It was set to be constant in classical Black-Scholes models. Later, it was modelled via stochastic differential equations. But, ...

Compare the roughness (intuitively)

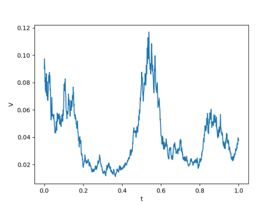


Oxford-Man KRV estimates of SPX realized variance from January 2000 to year 2018 (by J. Gatheral et al.)

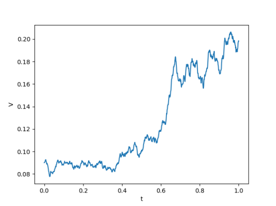
https://tpq.io/p/rough_volatility_with_python.html



A: H=0.12



B: H≈ 0.5



C: H=0.8

Evidence shows roughness of order $H = 0.1$

To mention but a few,

- Alòs, León, & Vives, 2007;
- Bayer, Friz, & Gatheral, 2016;
- Forde & Zhang, 2017;
- Fukasawa, 2011, 2017;
- Gatheral, Jaisson, & Rosenbaum, 2018;
- El Euch, Fukasawa, & Rosenbaum, 2018;
- ...

For the rough volatility literature, you may refer to

https://sites.google.com/site/roughvol/home

Consider a general stochastic volatility model given under a risk neutral probability measure as

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{V_t}\left(\rho\, dW_t + \sqrt{1-\rho^2}\, dB_t\right); \\ S_0 = s_0, \end{cases} \tag{3}$$

where $\rho \in [-1, 1]$ denotes the correlation coefficient and the constant r the interest rate.

Rough Bergomi model (Bayer-Friz-Gatheral-2016)

The stochastic variance is given as

$$V_t = \xi_t \mathcal{E}\left(\eta \widehat{W}_t\right), \tag{4}$$

where ξ_t denotes the *forward variance curve* (a quantity which can be computed from the implied volatility surface), $\mathcal{E}$ denotes the *Wick exponential*, i.e.,
$\mathcal{E}(Z) := \exp\left(Z - \frac{1}{2}\operatorname{var} Z\right)$ for a zero-mean normal random variable Z, and $\eta \geq 0$.
Finally, $\widehat{W}$ denotes a fractional Brownian motion (fBm) of Riemann-Liouville type with Hurst index $0 < H < \frac{1}{2}$, i.e.,

$$\widehat{W}_t := \int_0^t \mathcal{K}(t-s)dW_s, \quad \mathcal{K}(r) := \sqrt{2H} r^{H-1/2}, \quad r > 0. \tag{5}$$

- The process V (or even (S, V)) may not be a Markov process or a semi-martingale.

Rough Heston model (Euch-Rosenbaum-2019)

The stochastic variance satisfies the stochastic Volterra equation

$$V_t = V_0 + \int_0^t \mathcal{K}(t-s)\lambda\left(\theta - V_s\right)ds + \int_0^t \mathcal{K}(t-s)\zeta\sqrt{V_s}dW_s, \tag{6}$$

where the Kernel satisfies

$$\mathcal{K}(r) := r^{\alpha-1}/\Gamma(\alpha), \quad r>0, \quad \frac{1}{2} < \alpha < 1. \tag{7}$$

- The process V (or even (S,V)) may not be a Markov process or a semi-martingale.

Outline

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Setting

- Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space with the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ being generated by two independent Wiener processes W and B.

- $(\mathscr{F}_t^W)_{t\in[0,T]}$ is the filtration generated by the Wiener process W.

Consider a general stochastic volatility model given under a risk neutral probability measure as

$$\begin{cases} dS_t = rS_t dt + S_t \sqrt{V_t}\left(\rho\, dW_t + \sqrt{1-\rho^2}\, dB_t\right); \\ S_0 = s_0, \end{cases} \tag{8}$$

where $\rho \in [-1, 1]$ denotes the correlation coefficient and the constant r the interest rate. We impose the following assumptions on the stochastic variance process V.

Assumption 1

V has continuous trajectories, takes values in $\mathbb{R}_{\geq 0}$, and is adapted to the filtration generated by the Brownian motion W. We further assume that V is integrable, i.e.,

$$E\left[\int_0^T V_s ds\right] < \infty, \quad T > 0.$$

- Both the *rough Bergomi model* and *rough Heston model* satisfy the above assumption.

The fair price of a European option with payoff H, as the smallest initial wealth required to finance an admissible (super-replicating) wealth process, is given by (Cox-Hobson-05)

$$P_t(s) := E\left[e^{-r(T-t)}H(S_T^{t,s})\Big|\mathscr{F}_t\right]. \tag{9}$$

• In the special/classical case when $V_t \equiv \sigma^2$ is a constant, $P_t(s)$ is deterministic and satisfies the so-called Black-Scholes equation:

$$\begin{cases} -\dfrac{\partial P_t(s)}{\partial t} = \dfrac{\sigma^2 s^2}{2} D_{ss}^2 P_t(s) - rsD_s P_t(s) - rP_t(s); \\ \quad P_T(s) = H(s), \end{cases}$$

• Markovianity leads to a *deterministic* value function $P_t(s)$; general models include hidden Markov models to restore the Markovianity by extending the state spaces.

• The process V (or even (S, V)) may not be a Markov process or a semi-martingale; in fact, the adopted rough Bergomi/Heston models are neither.

• It is impossible to characterize the value function $P_t(s)$ with a conventional (deterministic) partial differential equation (PDE).

• Indeed, $P_t(s)$ satisfies the following backward stochastic PDE (BSPDE, Yong-Ma-1999-PTRF):

$$\begin{cases} -dP_t(s) = \left[\frac{V_t s^2}{2} D_{ss}^2 P_t(s) + \rho\sqrt{V_t} s D_s \Psi_t(s) - rsD_s P_t(s) - rP_t(s)\right] dt \\ \qquad\qquad -\Psi_t(s)\, dW_t; \\ P_T(s) = H(s), \quad s \in (0, \infty). \end{cases}$$

where both $P_t(s)$ and $\Psi_t(s)$ are unknown random fields.

• Question: Wellposedness and computations ?

Coordinate Transformation

- Taking $X_t = \log(e^{-rt} S_t)$, we may reformulate the above pricing problem, i.e.,

$$u_t(x) := E\left[e^{-r(T-t)} H(e^{X_T^{t,x}+rT}) \middle| \mathscr{F}_t\right], \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{10}$$

subject to

$$\begin{cases} dX_s^{t,x} = \sqrt{V_s}\left(\rho\, dW_s + \sqrt{1-\rho^2}\, dB_s\right) - \frac{V_s}{2}\, ds, \quad 0 \le t \le s \le T; \\ X_t^{t,x} = x. \end{cases} \tag{11}$$

- Obviously, we have $u_t(x) = P_t(e^{x+rt})$ and $u_t(x)$ satisfies BSPDE:

$$\begin{cases} -du_t(x) = \left[\frac{V_t}{2} D^2 u_t(x) + \rho\sqrt{V_t} D\psi_t(x) - \frac{V_t}{2} Du_t(x) - ru_t(x)\right] dt - \psi_t(x)\, dW_s; \\ u_T(x) = H(e^{x+rT}), \quad x \in \mathbb{R}. \end{cases}$$

- Question: Wellposedness and computations ?

Outline

- Consider a general nonlinear BSPDE:

$$\begin{cases} -du_t(x) = \Big[\frac{V_t}{2}D^2u_t(x) + \rho\sqrt{V_t}D\psi_t(x) - \frac{V_t}{2}Du_t(x) \\ \qquad\qquad + F_t(e^x, u_t(x), \sqrt{(1-\rho^2)V_t}Du_t(x), \psi_t(x) + \rho\sqrt{V_t}Du_t(x))\Big]\,dt \\ \qquad\qquad - \psi_t(x)\,dW_s, \quad (t,x) \in [0,T) \times \mathbb{R}; \\ u_T(x) = G(e^x), \quad x \in \mathbb{R}. \end{cases} \tag{12}$$

- The previous stochastic Black-Scholes equation is a particular case when $F_t(x, y, z, \tilde{z}) \equiv -ry$ and $G(e^x) = H(e^{x+rT})$.

- Difficulty lies in the combination of the non-uniform-boundedness of $(V_t)_{t\in[0,T]}$ and the inintegrability of $G(e^x)$ and $F_t(e^x, y, z, \tilde{z})$ w.r.t. x on the whole space $\mathbb{R}$.

Feynman-Kac formula

Let the triple $(Y_s^{t,x}, Z_s^{t,x}, \tilde{Z}_s^{t,x})$ be the L^1-solution to backward SDE (Briand et al.-2003):

$$\begin{cases} -dY_s^{t,x} = F_s(e^{X_s^{t,x}}, Y_s^{t,x}, Z_s^{t,x}, \tilde{Z}_s^{t,x})\, ds - \tilde{Z}_s^{t,x}\, dW_s - Z_s^{t,x}\, dB_s, \quad 0 \le t \le s \le T; \\ \quad Y_T^{t,x} = G(X_T^{t,x}). \end{cases} \tag{13}$$

Theorem

Value function $\Phi_t(x) := Y_t^{t,x}$ is just $\mathscr{F}_t^W$-measurable.

The weak solution (u, ψ) of BSPDE (12) satisfies

$$u_\tau(X_\tau^{t,x}) = Y_\tau^{t,x}, \quad \sqrt{(1-\rho^2)V_\tau}\, Du_\tau(X_\tau^{t,x}) = Z_\tau^{t,x}, \quad \psi_\tau(X_\tau^{t,x}) + \rho\sqrt{V_\tau}\, Du_\tau(X_\tau^{t,x}) = \tilde{Z}_\tau^{t,x},$$

for $0 \le t \le \tau \le T$ and $x \in \mathbb{R}$, where $(Y_\tau^{t,x}, Z_\tau^{t,x}, \tilde{Z}_\tau^{t,x})$ is the unique solution to BSDE (13).

Remark

For hedging, $Z = \sqrt{(1-\rho^2)V}\, Du$ is delta, and $\tilde{Z} = \psi + \rho\sqrt{V}\, Du$ corresponds to portfolios in the forward variance curve $((E[V_{t+u}|\mathscr{F}_t])_{u\in[0,T-t]}$ using liquid variance swaps or European options) in rough Heston models; see El Euch-Rosenbaum-2018.

Omar El Euch, and Mathieu Rosenbaum.
Perfect hedging in rough Heston models
Ann. Appl. Probab., 28(6), 3813–3856, 2018.

Theorem: existence and uniqueness of weak solution

Suppose further that there is an infinitely differentiable function ζ such that $\zeta(x) > 0$ for all $x \in \mathbb{R}$ and

$$G(e^{\cdot + X_T^{0,0}})\zeta(\cdot) \in L^2(\Omega, \mathscr{F}_T; L^2(\mathbb{R})), \quad \zeta(\cdot) F_\cdot(e^{\cdot + X_\cdot^{0,0}}, 0, 0, 0) \in L^2(\Omega \times [0,T]; L^2(\mathbb{R})). \tag{14}$$

Then BSPDE (12) admits a unique weak solution (u, ψ).

Outline

1. Rough Volatility Models

2. Stochastic Black-Scholes Equation
   - Pricing European options
   - Wellposedness of BSPDEs and Feynman-Kac formula

3. Examples
   - European put

Outline

- Consider the rough Bergomi model with parameters: $H = 0.07$, $\eta = 1.9, \rho = -0.9$, $r = 0.05$, $T = 1$, $X_0 = \ln(100)$. For simplicity, we choose the forward variance curve to be $\xi(t) \equiv 0.09$, independent of time.
- We have the associated FBSDE:

$$\begin{cases} dX_s^{0,x} = \sqrt{V_s}\left(\rho\, dW_s + \sqrt{1-\rho^2}\, dB_s\right) - \dfrac{V_s}{2}\, ds, \quad 0 \le s \le T; \\ X_0^{0,x} = x; \\ \quad V_s = \xi_s\, \mathcal{E}(\eta\, \widehat{W}_s) \quad \text{with} \quad \widehat{W}_s = \displaystyle\int_0^s \sqrt{2H}(s-r)^{H-1/2}\, dW_r, \quad s \in [0,T]; \\ dY_s^{0,x} = rY_s^{0,x}\, ds + Z_s^{0,x}\, dW_s + \overline{Z}_s^{0,x}\, dB_s, \quad s \in [0,T]; \\ Y_T^{0,x} = (K - e^{X_T^{0,x} + rT})^+. \end{cases}$$

- $N = 20$ in the Euler Scheme and set a single hidden layer whose number of neurons is equal to half of the total number of neurons in the input and output layers.
- activation function: Sigmoid; Optimizer: Adam

	Reference value	$RSD = \frac{standard\ deviation}{average\ value}$	Estimated value	RSD
$K = 90$	4.9550	0.0259	4.9535	0.0228
$K = 100$	7.8284	0.0135	7.8061	0.0201
$K = 110$	12.1844	0.0100	12.1940	0.0143
$K = 120$	18.1631	0.0077	18.1699	0.0055

Table: Prices of European put options at t=0 under the different strike prices K.

- Reference values are calculated by Monte Carlo method.

- Christian Bayer, Jinniao Qiu, and Yao Yao.
  Pricing Options Under Rough Volatility with Backward SPDEs
  SIFIN, 13(1), 179–212, 2022.

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  Ann. Appl. Probab., 28(6), 3813–3856, 2018.

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  Proceedings of the National Academy of Sciences, 115(34):8505–8510, 2018.

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  Some machine learning schemes for high-dimensional nonlinear pdes.
  Mathematics of Computation, 89(324), 1547–1580, 2020.

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  Deep PPDEs for rough local stochastic volatility.
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Thank You !