Singular coherent structures in 2D Euler equation and hydrodynamic limits

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Singular solutions of 2D incompressible Euler equations

$$\partial_t \omega + u \cdot \nabla_x \omega = 0,$$
$$u = \nabla^\perp \Delta^{-1} \omega.$$

- (Generalized) Yudovich solutions $\omega \in L^\infty$: globally well-posed.
- Diperna-Majda solutions $\omega \in L^p$: global existence.
- Weak solutions.

- Q1. What can we say about the behavior of singular solutions?
  - Propagation of certain structures? Singular vortices?
- Q2. Can we derive *singular solutions* as limits?
  - Limits of smooth solutions/ vanishing viscosity limit/etc.
  - *Macroscopic limit* of solutions of Boltzmann equation.

Transport equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla_x \theta = 0, \\ \theta\big|_{t=0} = \theta_0. \end{cases} \tag{Tr}$$

- Associated ODE:

$$\begin{cases} \dfrac{d}{dt}\phi(x,t) = u(\phi(x,t),t), \\ \phi(x,0) = x. \end{cases}$$

Transport equation

- Condition for uniqueness: Osgood.
- $L : (0, m_L) \to \mathbb{R}^+$: modulus of continuity.

$$
\begin{aligned}
|u(x,t) - u(y,t)| &\le \|u\|_L L(|x-y|), \\
\lim_{z\to 0+} \mathcal{M}(z) &= \infty, \\
\mathcal{M}(z) &:= \int_z^{m_L} \frac{dr}{L(r)}.
\end{aligned}
$$

- Osgood's lemma:
$-\mathcal{M}(|\phi(x,t) - \phi(y,t)|) \le -\mathcal{M}(|x-y|) + \int_0^t \|u(s)\|_L ds.$

Transport equation

- u Lipschitz: $L(z) = z$, $\mathcal{M}(z) = \log_+(1/z)$.
- u log-Lipschitz: $L(z) = z \log(1/z)$, $\mathcal{M}(z) = \log \log_+(1/z)$.
- $L(z) = z \log(1/z) \log_2(1/z) \cdots \log_n(1/z)$, $\mathcal{M}(z) = \log_{n+1}(1/z)$.

Transport equation

- For Osgood u, unique integrable solution to (Tr) (Ambrosio and Bernard 2008, Caravenna and Crippa 2021):

$$\theta(x,t) = \theta_0(\phi^{-1}(x,t)). \tag{Flow}$$

- Not much quantitative information about θ.
(EX: Loss of regularity below Lipschitz)
- *Certain singular features propagate by Osgood vector fields.*

Propagation of singular structures

Theorem (Drivas, Elgindi, L. 2022)

Let $L : (0, m_L) \to \mathbb{R}^+$ *be Osgood(i.e.* $\mathcal{M}(0+) = \infty$, $\mathcal{M}(z) = \int_z^{m_L} \frac{dr}{L(r)}$*),* u *div-free with modulus of continuity* L*. Define the seminorm by*

$$[f]_{x,\gamma,L} = \lim_{r\to 0+} \sup_{y: 0<|x-y|<r} \frac{|f(x)-f(y)|}{\mathcal{M}(|x-y|)^\gamma}, \gamma \in \mathbb{R}.$$

Then $\theta = \theta_0(\phi^{-1}(x,t))$ *defined by* (Flow) *preserves the seminorm:*

$$[\theta(t)]_{\phi(x,t),\gamma,L} = [\theta_0]_{x,\gamma,L}.$$

- $\gamma > 0$: singularities, $\gamma < 0$: cusps.
- Chae and Jeong (2020): preservation of logarithmic cusps for Lipschitz drifts.

Propagation of singular structures

- Certain singular structures keep their shape.

Theorem (Drivas, Elgindi, L. 2022)

Let L and $\mathcal{M}$ as before (L Osgood, $\mathcal{M}(z) = \int_z \frac{dr}{L(r)}$.) Let F be a smooth function with at most linear growth at infinity ($\sup_{|z|\geq 1} |F'(z)| < \infty$). If θ_0 has the form

$$\theta_0(x) = F(\mathcal{M}(|x - x_0|)) + b_0, b_0 \in L^\infty$$

near $x = x_0$, then $\theta(x,t)$ given by (Flow) has the form

$$\theta(x,t) = F(\mathcal{M}(|x - \phi(x_0,t)|)) + b, b \in L^\infty$$

near $x = \phi(x_0,t)$.

Propagation of singular structures

- What kinds of shape can propagate?
- $\mathcal{M}(|x - x_0|)$, $\sqrt{\mathcal{M}(|x - x_0|)}$, $\log(\mathcal{M}(|x - x_0|))$, etc.
- Pathological shape: $F(z) = \sin(\lambda z)$, $\lambda > 0$ small. $\theta(x, t)$ changes signs like Topologist's sine curve as $x \to \phi(x_0, t)$ $(t \leq T)$.
- Even more singular (i.e. superlinear F)? It seems to be sharp: if F grows faster, $b \notin L^\infty$.

Propagation of singular vortices in 2D Euler equations

- Application: 2D incompressible Euler, singular initial data.
- Singular vortex $\mathcal{M} \to u$ (Biot-Savart).
  BUT, modulus of continuity for u worse than $L = -1/\mathcal{M}'$.
- Cancellation from radial symmetry of $\mathcal{M}$.

Propagation of singular vortices in 2D Euler equations

- $\omega = \mathcal{M}$ in *generalized Yudovich space:* $\|\omega\|_{L^p}$ grows mildly in p.
- $\Theta : [1, \infty) \to \mathbb{R}^+$, $\int_1^\infty \frac{dp}{p\Theta(p)} = \infty$.

$$Y_\Theta := \left\{ f \in \cap_{p\in[1,\infty)} L^p : \|f\|_{Y_\Theta} := \frac{\|f\|_{L^p}}{\Theta(p)} < \infty \right\}.$$

- Modulus of continuity:

$$|u(x,t) - u(y,t)| \lesssim |x-y| \log(1/|x-y|)\Theta(\log(1/|x-y|)).$$

- Existence and uniqueness in Y_Θ (Yudovich 1995, Serfati 1994.)

Propagation of singular vortices in 2D Euler equations

- L: Osgood, $z\log(1/z) \lesssim L(z)$, $\mathcal{M}(z) = \int_z \frac{dr}{L(r)}$.
- $\mathcal{M}(z) = \log\log_+(1/z), \log_3(1/z), \cdots$.
- $\omega = \mathcal{M}(|x - x_0|)$ propagates in 2D Euler equations.

Theorem (Drivas, Elgindi, L. 2022)

Let $\Theta(p) = \log_k(p), k \geq 0$, $L, \mathcal{M}$ *as above,* $b_0 \in Y_\Theta \cap L^1$, $f \in L^1_{loc}(\mathbb{R}; Y_\Theta \cap L^1)$,

$$\omega_0(x) = \mathcal{M}(|x|) + b_0(x).$$

Then there is $b : L^\infty_{loc}(\mathbb{R}; Y_\Theta \cap L^1)$, $\phi_*(t) : \mathbb{R} \to \mathbb{R}^2$ *such that*

$$\omega(x,t) = \mathcal{M}(|x - \phi_*(t)|) + b(x,t).$$

- Meaningful only when $\mathcal{M}$ is more singular than b.

Sketch of the proof.

We find governing equation for ϕ_* and b.
Ansatz: assume b and ϕ_* as above.

$$\omega(x,t) = \omega_s(x,t) + b(x,t), \omega_s(x,t) = \mathcal{M}(|x - \phi_*(t)|).$$

$u_r := -\nabla^\perp(-\Delta)^{-1}b, u_s := -\nabla^\perp(-\Delta)^{-1}\omega_s$: Osgood.
Key observation: ω_s radial, u_s circular, so $u_s \cdot \nabla_x \omega_s = 0$.

$$(\partial_t + u \cdot \nabla_x)\omega_s = (\partial_t + u_r \cdot \nabla_x)\omega_s = (\partial_t + u_r \cdot \nabla_x)(|x - \phi_*(t)|)\mathcal{M}'.$$

$$\frac{d}{dt}\phi_*(t) = u_r(\phi_*(t), t), \phi_*(0) = 0 \Rightarrow (\partial_t + u \cdot \nabla_x)\omega_s = 0.$$

Then equation for b can be written. □

Propagation of singular vortices in 2D Euler equations

- Remark 1. Multiple singular vortices.

$$\omega_0(x) = \sum_{i=1}^{N} \gamma_i \mathcal{M}(|x - x_0^i|) + b_0(x).$$

Evolution of center excludes self-interaction.

$$\frac{d}{dt}\phi_j(t) = -\nabla_x^{\perp}(-\Delta)^{-1}\left[\sum_{i\neq j} \gamma_i \mathcal{M}(|x - \phi_i(t)|) + b(x,t)\right] \circ \phi_j(t),$$

$$\phi_j(0) = x_0^j.$$

- cf. Vortex-wave system (point vortices + perturbation). Point vortices do NOT solve Euler since too singular (Schochet 1996), while the above are actual solutions.
- Remark 2. Is $\log\log_+$ the most singular vortex? (Open).

Propagation of possible nonuniqueness

- 2D Euler with $\omega_0 \in L^p, 1 \leq p < \infty$.
- Diperna and Majda(1987): global existence.
- Vishik(2018): non-uniqueness with forcing.
- Let $\omega_1(t), \omega_2(t)$ be two solutions from $\omega_0 \in L^p$.
  How different are they?
- Non-uniqueness "propagates" with speed $\|u\|_{L^\infty}$ for $p > 2$.

Propagation of possible nonuniqueness

Theorem (Drivas, Elgindi, L. 2022)

1. *Let $u_1, u_2 \in C([0,T); W^{1,p})$ be two distinct weak solutions to 2D velocity-Euler with $u_1(0) = u_2(0)$. Then $u_1 - u_2$ cannot be smooth.*
2. *Let $\omega_0 \in L^1 \cap L^p$, smooth away from origin. Let ω_0^ϵ be regularized data, which are uniformly smooth away from $B_1(0)$, and let ω^ϵ be corresponding solution.*
   Let ω_* be a subsequential limit of ω^ϵ, $\epsilon \to 0$. Then ω_* is a weak solution to 2D Euler equation, which is smooth outside of $B_{1+Ct}(0)$ where $C = \sup_\epsilon \|u^\epsilon\|_{L^\infty}$.

Singular Euler solutions as limits

- Singular solutions: limit of regular solutions.
  - Limit of regular Euler solutions (e.g. Crippa, De Lellis 2008)
  - Vanishing viscosity limit (e.g. Constantin, Drivas, Elgindi 2020)
  - *Macroscopic* limits of smaller scale description of fluids?

Singular Euler solutions as limits of Boltzmann

- Hilbert's sixth problem (1900): developing limiting processes between physical models of different scales.
- Ruling out small scale fluctuations by averaging.
- If fluids are not regular, the limiting process becomes nontrivial.

Kinetic description: Boltzmann equation

- $\partial_t F + v \cdot \nabla_x F = Q(F, F)$.
- (Hard-sphere) Collision $Q(F, F)(v)$

$$Q(F, G)(v) = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot \sigma| (F_{v'} G_{v'_*} - F_v G_{v_*}) \mathrm{d}v_* \mathrm{d}\sigma.$$

$(v', v'_*) \to (v, v_*)$ after collision, σ: collision cross-section.

- (local) Maxwellian: R density, U velocity, Θ temperature.

$$M_{R,U,\Theta}(v) = \frac{R}{(2\pi\Theta)^{\frac{3}{2}}} \exp\left(-\frac{|v - U|^2}{2\Theta} \right).$$

Non-dimensionalization

- Non-dimensionalize, take the limit.
- Two non-dimensional numbers
  - $\mathrm{St} := \frac{\text{macroscopic length}}{\text{microscopic length}}$
  - $\mathrm{Kn} := \frac{\text{mean free path length}}{\text{macroscopic length}}$: frequency of collision.
- Non-dimensionalized Boltzmann equation:

$$\mathrm{St}\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} Q(F, F).$$

- $\mathrm{Ma} := \frac{\text{(macroscopic) velocity scale}}{\text{(microscopic) velocity scale}} = \mathrm{St}.$
- $\frac{1}{\mathrm{Re}} = \frac{\mathrm{Kn}}{\mathrm{Ma}}$ (Von Karman).

Hydrodynamic limit

- More collisions $\mathrm{Kn} \to 0$: averages representative of the distribution (hydrodynamic regime).
- $\mathrm{Ma} << 1$: macroscopic velocity $<<$ particle velocity - incompressible regime.
- $\mathrm{Ma} = \mathrm{Kn} \to 0$: incompresible Navier-Stokes.
- $\mathrm{Kn} << \mathrm{Ma} \to 0$: *incompressible Euler.*

Hydrodynamic limit

- $\varepsilon = \mathrm{St} = \mathrm{Ma} \to 0, \kappa = \kappa(\varepsilon) = \frac{1}{\mathrm{Re}} \to 0$ for

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon \kappa} Q(F^\varepsilon, F^\varepsilon),$$

- Goal: $\frac{1}{\varepsilon} \int_{\mathbb{R}^3} v F^\varepsilon(x, t, v) \mathrm{d}v \to u(x, t)$.
- $x \in \mathbb{T}^2$ (symmetric in z direction).

Hydrodynamic limits toward Euler equation

- Hilbert expansion: perturbative method.
- Singular limit ($\kappa \to 0$): use the local Maxwellian $\mu := M_{1,\varepsilon \boldsymbol{u},1}$
- $F^{\varepsilon} = \mu + \varepsilon f_R \sqrt{\mu} + \text{(correctors)}$.
- We ask $\lim_{\varepsilon \to 0} f_R = 0$: $\frac{1}{\varepsilon} \int v F^{\varepsilon} = u + \int v f_R \sqrt{\mu} + \cdots$.
- Stability estimate of f_R.

Hydrodynamic limits toward Euler equation

- Regularity requirements for u:
  - Relative entropy (Saint-Raymond 2003): $\nabla_x u \in L^1_t L^\infty_x$ needed, $\frac{1}{\varepsilon}\int vF^\varepsilon \to u$ weakly.
  - L^2 stability of f_R: $u \in L^2_t H^k_x$ needed, $\frac{1}{\varepsilon}\int vF^\varepsilon \to u$ strongly in L^2.
  - H^k stability of f_R: higher regularity for u needed, stronger convergence.

Issues

1. Not enough regularity: $\nabla_x u \notin L^\infty$.
2. Singular structures only observable in stronger topology (e.g. interfaces in vortex patch)
3. Viscosity effect blurs singular structures.
4. Large perturbation(general data): $f_R = o(1)$, but as large as possible.

Issues

- Issues 3 and 4: Incompressibility - size ε^{-1}, Euler equation - size ε^0, viscosity term - size κ.
  - Need to suppress up to size κ: (i) put viscosity term in Euler (κ-NS), or (ii) further corrector expansions (but $\kappa = \varepsilon$: too singular).
  - $f_R = o(\kappa)$ optimal: comparable to viscosity effect.
- Issues 1 and 2: approximation of u bt u^β (Euler solution with initial data $u_0^\beta = u_0 \star \phi_\beta$.)
  - $\phi_\beta \to_{\beta\to 0} \delta_0$: $\beta(\varepsilon) \to 0$.
  - Perturbation around $\mu^\beta = M_{1,\varepsilon u^\beta,1}$, stability $u^\beta \to u$ in $W^{1,p}, p < \infty$.
  - $\frac{1}{\varepsilon}\int F^\varepsilon v dv = u^\beta + o(1) \to u$.
  - u^β smooth, β can be adjusted: stability estimate for f_R in $H^2_x L^2_v$.

Issues

- Issues 2 and 4: using strong topology gives a better scaling.
  - f_R equation: partially coercive, but two problems (more than L^2 required).
  - (i) perturbation around local Maxwellian - higher moment.
  - (ii) nonlinearity $Q(f_R\mu^\beta, f_R\mu^\beta)$ - integral with rapidly decaying multiplier: only lacks integrability in x.
  - $H^2_x L^2_v$ and interpolation $L^\infty \subset H^2$ treats (ii). (i): small prefactor.
  - Scaling: $f_R \sim o(\kappa), \partial_x f_R \sim o(\sqrt{\kappa}), \partial_x^2 f_R \sim o(1)$.
- Issues 2 and 3: new expansion designed.
  - Scales of various terms tractable as only one is (mostly) used.

Main theorem

Theorem (Kim, L. 2022)

For a singular solution u of 2D Euler equation ($\omega \in L^p, \|\omega\|_{L^p} = \Theta(p)$), there exists a sequence of Boltzmann solutions

$$F^\varepsilon = \mu_\beta + O(\kappa\varepsilon)$$

such that $\frac{1}{\varepsilon}\int vF^\varepsilon \mathrm{d}v = u^\beta + O(\kappa) \to u$ in $W^{1,p}$. Moreover, u^β solves Euler equation as well.

- EX: u vortex patch $\to$ u^β smooth Euler, a patch with β-thick layer.