Laplacians as a Bridge for Discrete Differential Geometry, Numerical Analysis, and Geometric Analysis

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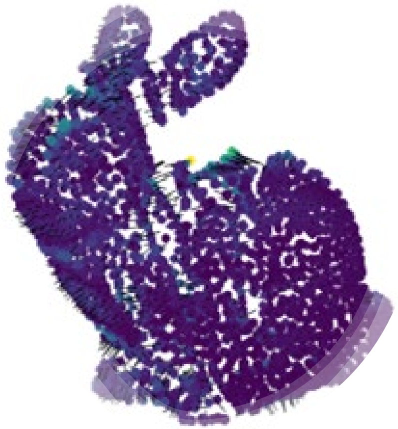
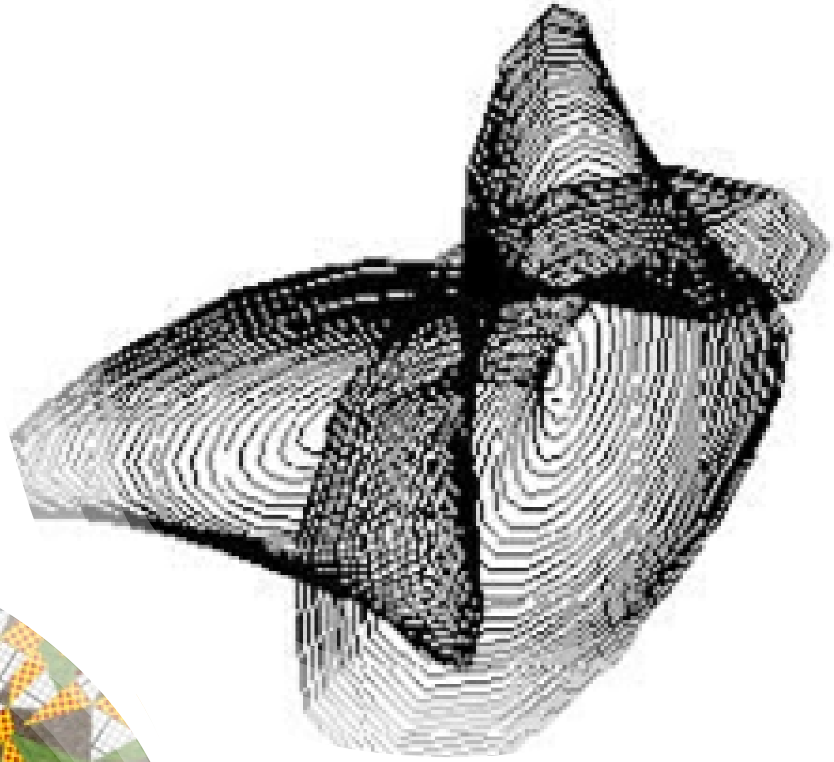
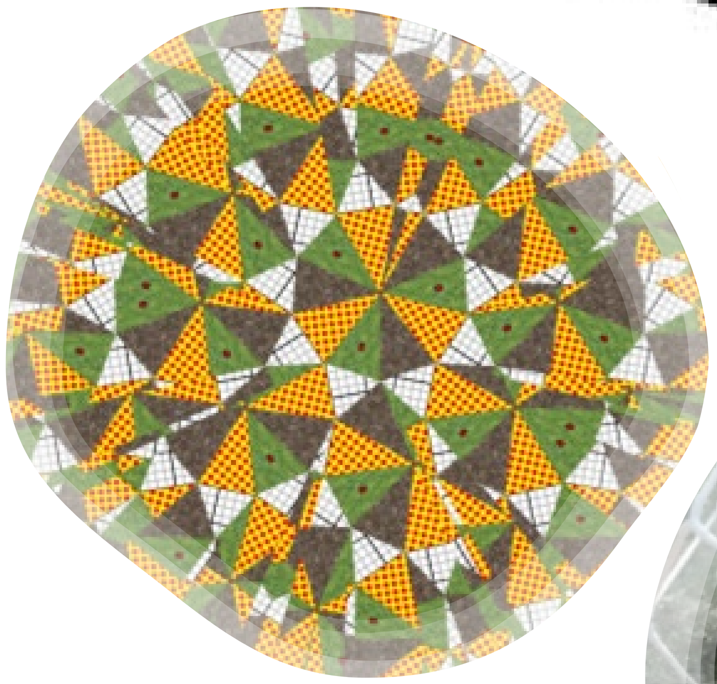
Introductions

- David Glickenstein
  - University of Arizona
  - Ricci Flow
  - Discrete Conformal Geometry
  - Neural Networks and Machine Learning/AI







Courtesy of Bruce Bayly

More Introductions

Emily Banks	Lee Sidbury	Thomas Doehrman
• Discrete harmonic maps: discrete-to-discrete and point clouds	• Convergence of discrete conformal maps on surfaces	• Geometry of sphere configurations and Laplacian

Laplacians

- Important elliptic differential operator $\triangle$
  - $\frac{d^2}{dx^2}$ on $\mathbb{R}$
  - $\left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^n}\right)^2$ on $\mathbb{R}^n$
  - Div grad on (M, g)
- Key properties:
  - Definiteness
  - Maximum Principle

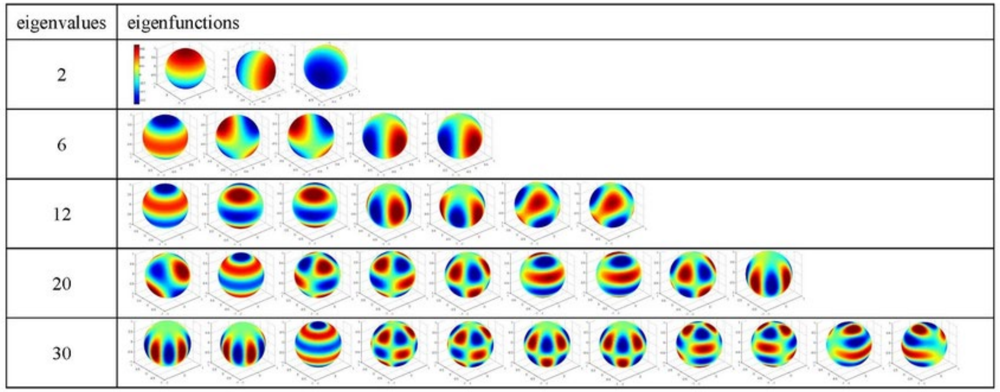
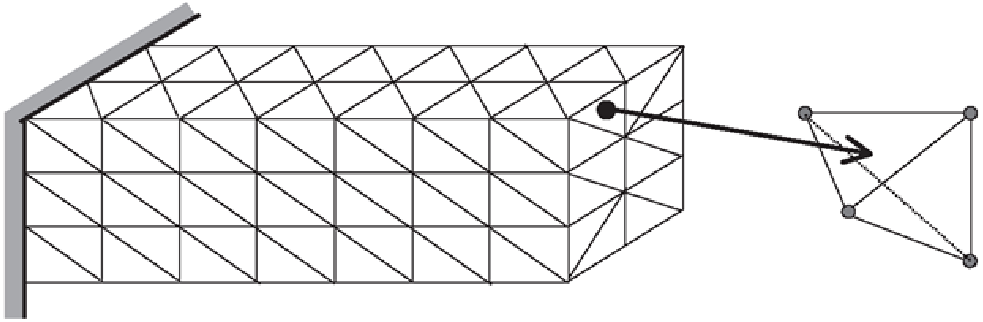
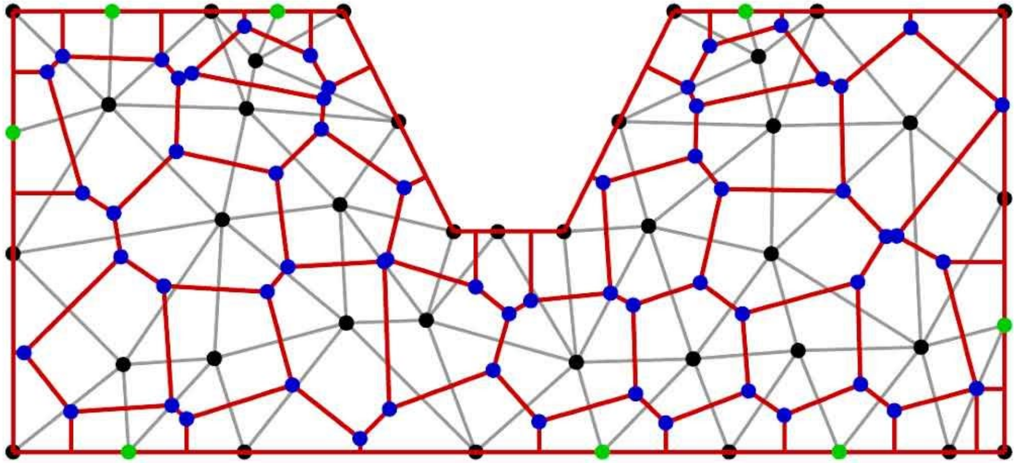


Image: Chen-Chi-Wu 2013

Numerical

- (Piecewise linear) Finite element method
- Finite volume method
  - In 2D, FEM is FVM with circumcentric duals (Bank-Rose 1987)
- On a PL surface or manifold (Bobenko-Springborn 2007)
- For a smooth manifold
  - Riemannian barycentric coordinates (e.g., von Deylan-G-Wardetzky 2016)





Variation of discrete conformal structures

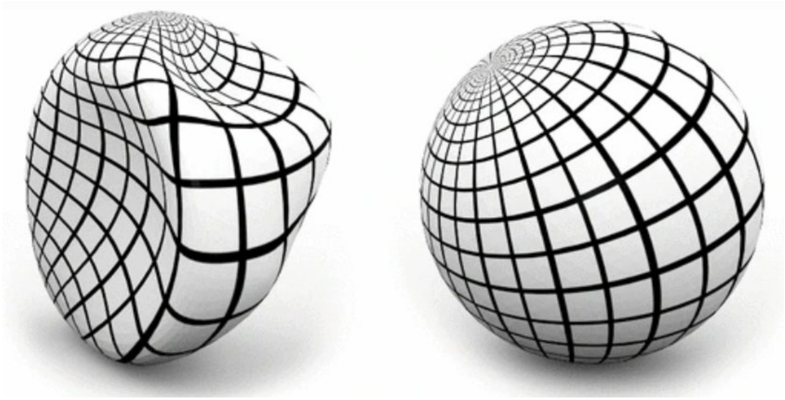


Image: Bobenko-Schelmann-Springborn 2016

- Under a variety of discrete conformal variations in 2D, (Thurston, Z.−X. He 1999, Chow − Luo 2003, G 2011):

$$\frac{d}{dt}K_v = -\triangle \frac{df_v}{dt}$$

- This is similar to the smooth case:

$$\frac{d}{dt}(R\,dA) = -\triangle \frac{df}{dt}\,dA$$

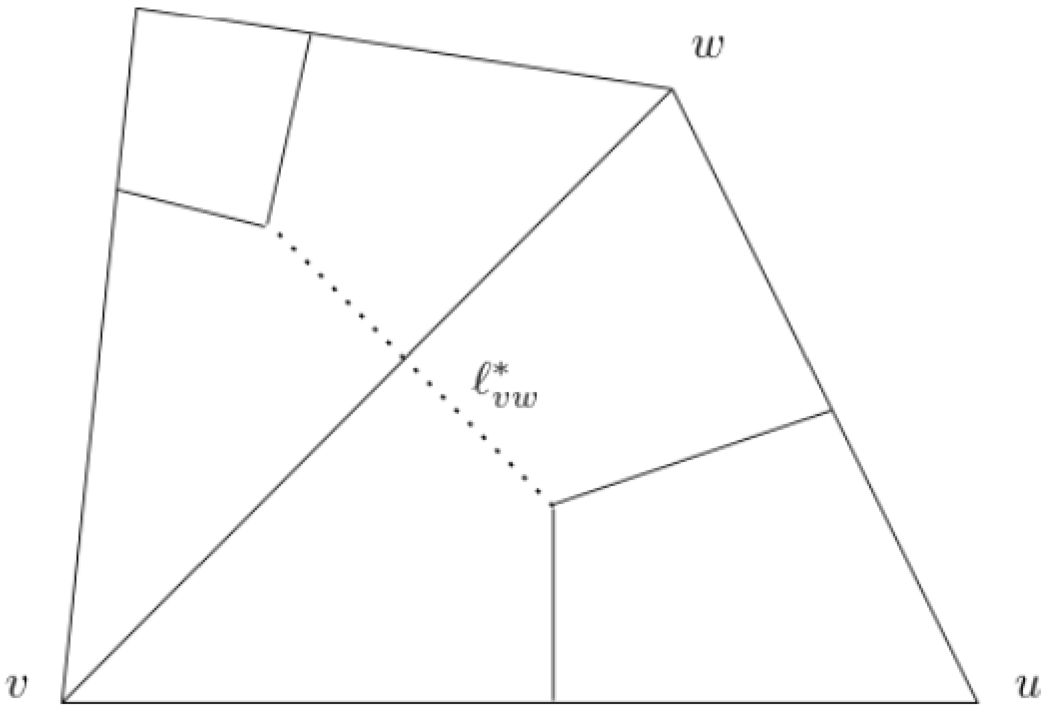
for a conformal variation of metric $g(t) = e^{f(t)}g_0$

- There are similar (but not quite as nice) formulas in dimension 3 (G 2011)

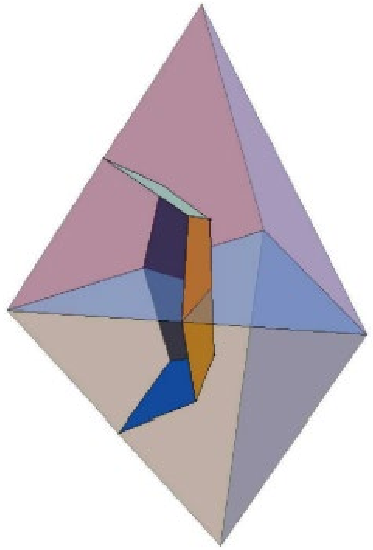
Graph Laplacian

- In each case, the Laplacian is a graph Laplacian with weights coming from the dual structure, generalizing “classical” finite volume Laplacian:

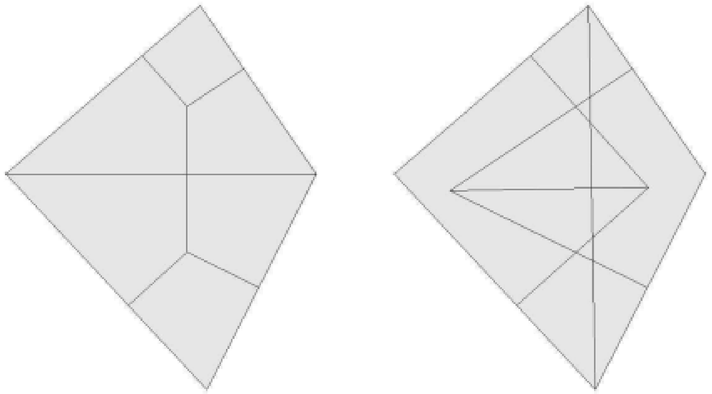
$$\triangle f_v = \sum_{w \sim v} \frac{l_{vw}^*}{l_{vw}} (f_w - f_v)$$



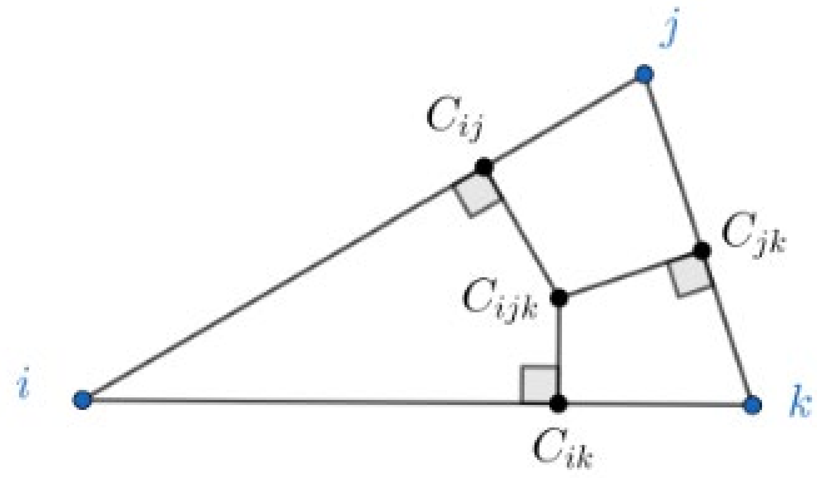
Properties of discrete Laplacians



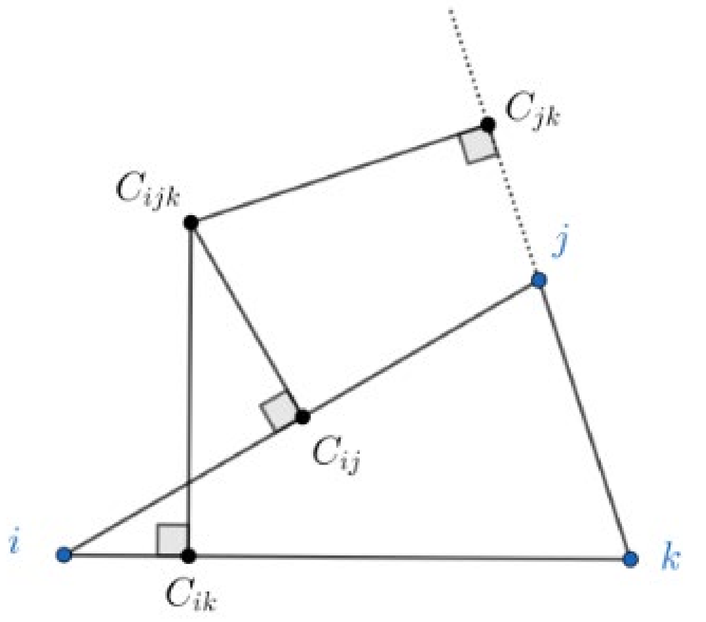
- Maximum principle
  - Related to positivity of the coefficients
  - Related to weighted Delaunay in 2D
  - No free lunch theorem of Wardetzky-Mathur-Kalberer-Grinspun 2007
    - Only admit a Laplacian that is symmetric, local, positive coefficients, zero on linear functions is to be weighted Delaunay.
- Definiteness
  - Maximum principle is sufficient
  - In 2D Weighted Delaunay flips reduce (positive) eigenvalues (Rippa 1990, Bobenko-Springborn 2007, G 2005)
  - Simplex-by-simplex argument…



Simplex-by-simplex definiteness



- Proof of definiteness simplex-by-simplex
  - Suppose the Laplacian is definite on each simplex
  - Laplacian over the whole complex is a sum of definite operators
  - Now check the kernel
- Well-centered assumption (e.g., in Discrete Exterior Calculus)
- 2D finite elements/circumcentric dual
- 3D Sphere Packing (Cooper-Rivin 1996, G 2005)

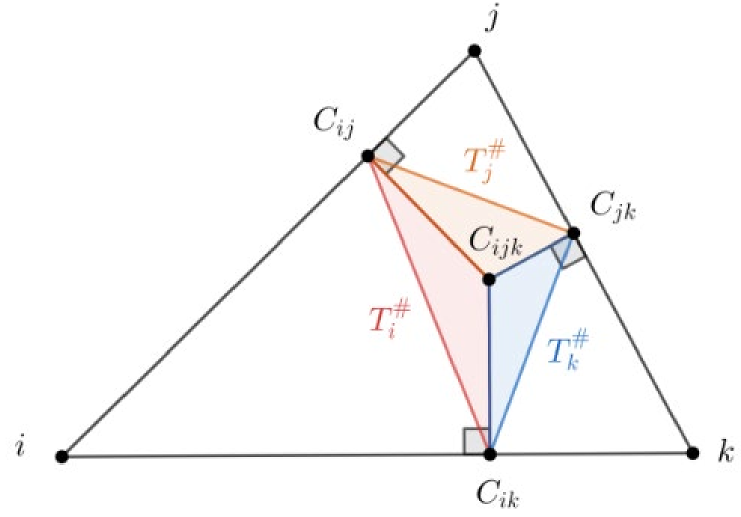


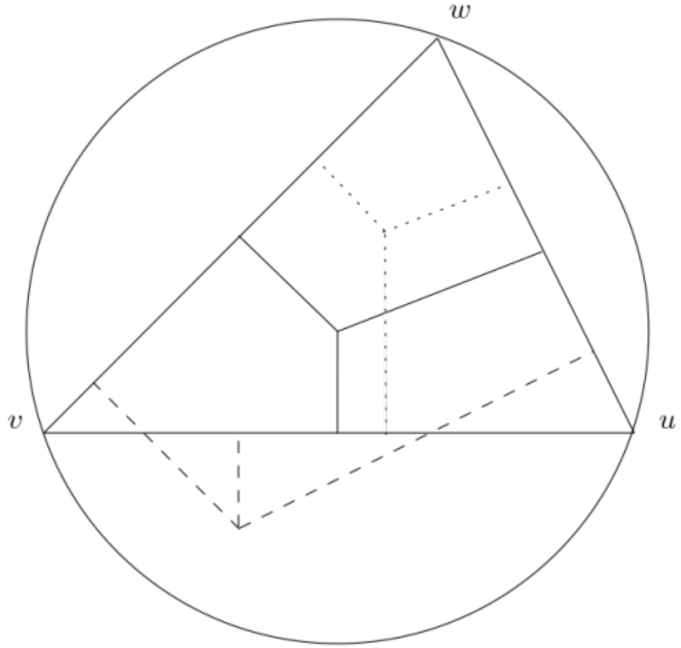
2D Kirchhoff determinant

- In each simplex, we can consider the Kirchhoff determinant to determine when a simplex has a degenerate Laplacian.
- In 2D, the Kirchhoff determinant K(T) is:

$$K(T) = \frac{h_{ij}h_{ik}}{l_{ij}l_{ik}} + \frac{h_{ij}h_{jk}}{l_{ij}l_{jk}} + \frac{h_{jk}h_{ik}}{l_{jk}l_{ik}} = \frac{A(T_{Pedal})}{A(T)}$$

- Simson’s Theorem says this is zero iff the center is on the circumcircle, so positive if the center is inside





General case (Doehrman-G 2022, Doehrman 2023)

- *Let K(T) denote the Kirchhoff determinant of the Laplacian of simplex T of dimension N.*
- *There exist simplices $T^{\#}$ and $T^{\flat}$ associated to a simplex T such that*

$$\mathrm{K(T)} = \frac{(-1)^N N^N}{(N!)^2} \frac{\mathrm{Vol}(T^{\#})^{N-1}}{\mathrm{Vol}(T)} = \frac{(-1)^N N^N}{(N!)^2} \frac{\mathrm{Vol}(T)^{N-1}}{\mathrm{Vol}(T^{\flat})}$$

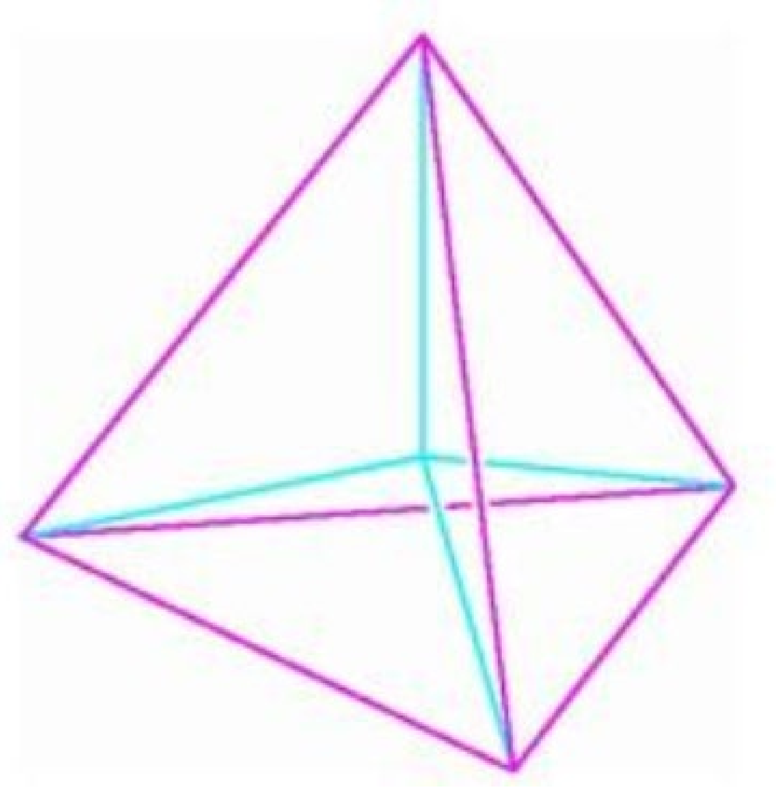
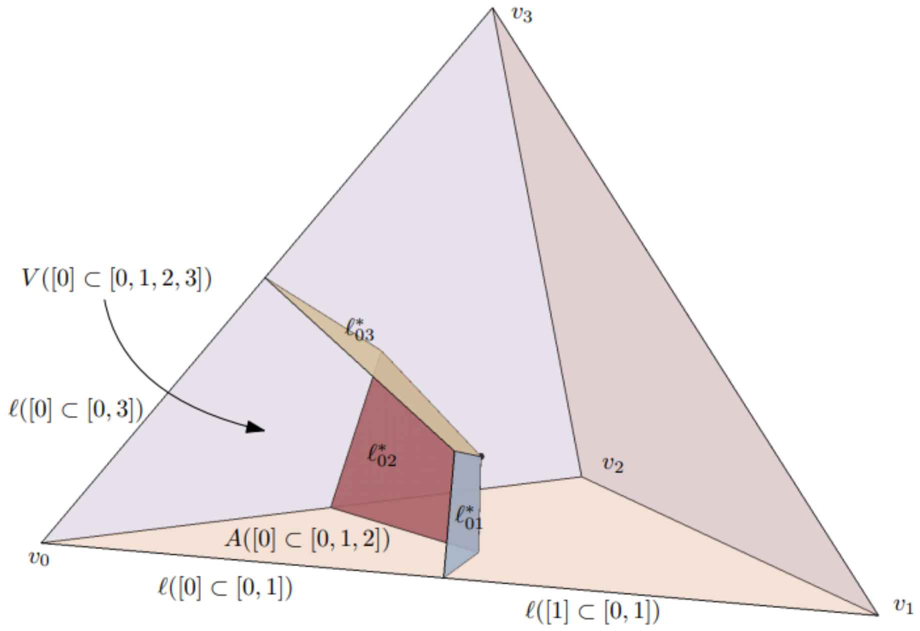
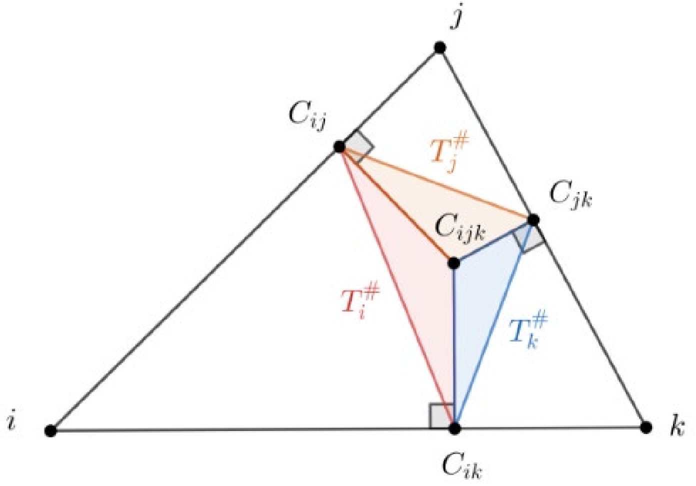


Image: Banchoff

$T^{\#}$ and $T^{\flat}$





- $n_{ij} = \frac{l^*_{ij}}{l_{ij}}(v_j - v_i)$
- $n_i = -\sum_{j\sim i} n_{ij}$
- $T^{\#}$ is a simplex determined by the n_i
- $T^{\flat}$ is a simplex with the directions determined by the n_i such that the plane for n_i goes through vertex v_i