Discrete conformal geometry of polyhedral surfaces

Feng Luo (Rutgers), joint work w/ D. Gu, J. Sun, T. Wu (1)

§1. Riemann surfaces

S — connected surface :

g — a Riemannian metric :

Two g, $\tilde g$ conformal if they define the same angles

$\Leftrightarrow$ $\boxed{\tilde g = e^u g}$, $u: S \to \mathbb{R}$.

A Riemann surface (R S) is $(S, [g])$

= surface + notion of angle.

Conformal map = angle preserving map

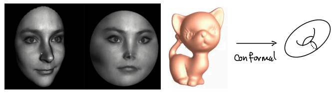
Eg1. ∀ smooth Σ =  or  $\subset \mathbb{R}^3$ is a R.S.

Riemann Mapping: ∀ Ω = (blob) $\subsetneq \mathbb{C}$ conformal to $\mathbb{D}$ = ○

Unif. Thm (V1) Simply connected Riemann surf is conformal to $\mathbb{C}, \mathbb{D}, S^2$.

(2)

Eg 2,3



conformal

Unif Thm (V2). ∀ Riemannian metric g on surface Σ, $\exists\, u: \Sigma \to \mathbb{R}$ s.t. $e^u g$ is complete of constant curvature 0, −1, or 1.

Structure preserving discretization

discrete Riemann surfaces w/ intrinsic structure

1. discrete (e.g. polyhedral surf), computable
2. discrete uniformization theorem
3. Convergence of discrete R.S. and maps

Guides: f conformal $\Leftrightarrow$ $Df: \mathbb{R}^2 \to \mathbb{R}^2$ conformal

$\Leftrightarrow$ $Df = \lambda \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\Leftarrow$ circle preserving

$\Leftrightarrow$ cross ration preserving $\frac{aa'}{bb'}$

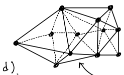
Circle preserving $\Rightarrow$ Thurston's circle packing dis. R.S

cross ration preserving $\Rightarrow$ vertex scaling dis. R. S

(3)

RM. D. Glickenstein's generalization, 1-parameter family of dis. R.S.

Basic ingredient:

polyhedral surfaces:  gluing Δ's

P.L. surfaces (S, V, d), edge length l

P.L. metrics (S, V, T, l)

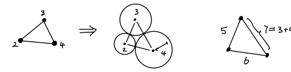
§2. Circle Packing P.L. metrics

(S, T, V) triangulated surface :

$r: V \to \mathbb{R}_{>0}$ radius function

$\Rightarrow$ polyhedral metric on (S, T, V, l), $l: E \to \mathbb{R}_{>0}$

$\boxed{l(uv) = r(u) + r(v)}$

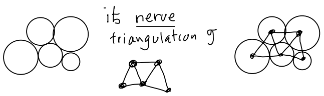
Eg4. 

discrete curvature of a PL metric

$\boxed{K_d(v) = 2\pi - \sum_{i=1}^{m} a_i}$:

(4)

its nerve triangulation T

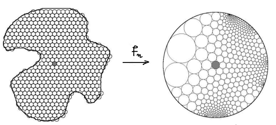


Köbe-Andreev-Thurston (dis. Riemann mapping thm)

∀ triangulated disk :  ∃! circle packing of $\mathbb{D}$ 

whose nerve is isomorphic to the triangulation.

Thurston's Convergence Conjecture and Rodin-Sullivan thm :



K. Stephenson

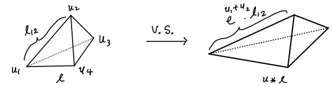
§3. Vertex Scaling distization of R.S.

Triangulated surface (Σ, V, T): , $l: E \to \mathbb{R}_{>0}$ + $u: V \to \mathbb{R}$

Def. (vertex scaling) $u*l$ of l : $\boxed{(u*l)(v_1v_2) = e^{u(v_1)+u(v_2)} \cdot l(v_1v_2)}$

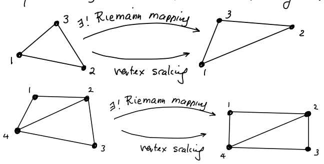
edge: v_1 — l — v_2 $\xrightarrow{V.S.}$ v_1 — v_2

(5)

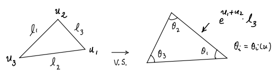
Eg 5. 

RM Cross ratio invariant: $\frac{aa'}{bb'} = \frac{xx'}{yy'}$

Eg 6. Any two triangles are conformal + differ by a vertex scaling



Prop. (L) (variational principle).



Then $\left[\frac{\partial \theta_i}{\partial u_j}\right]_{3\times 3}$ is symmetric + negative semi-definite.

$\Rightarrow$ ∃ locally concave function $W(u)$ s.t. $\nabla W = (\theta_1, \theta_2, \theta_3)$.

(6)

RM. Colin de Verdierer for CP, Glickenstein for all generalized cases.

Bobenko-Pinkall-Springborn Rigidity Thm.

(1). W can be extended to a concave function on $\mathbb{R}^3$.

(2). If l, $u*l$ are two PL metrics on cpt (S, V, T) with the same discrete curvature, then u = const.

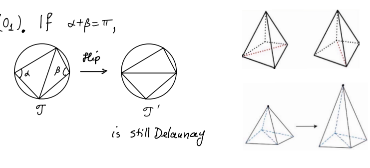
RM. X. Xu's generalization to all cases.

Discrete conformal requires: (S, V, T, l) P.L. surf :

Delaunay triangulation : ∀ edge e

$\alpha + \beta \le \pi$

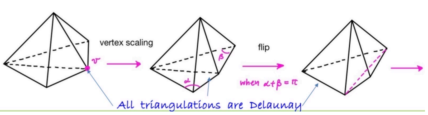
(O_1). If $\alpha + \beta = \pi$,



(O_2). Two Delaunay triangulated $(S, V, T, l) \overset{V.S}{\sim} (S, V, T, u*l)$.

(7)

Def. Two P.L metrics (S, V, d) and (S, V, d') are discrete conformal if ∃ a sequence of (O_1) + (O_2) moves from (S, V, d) to (S, V, d')

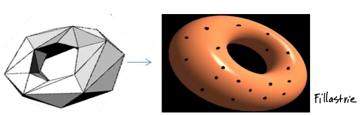


Thm (Gu-L-Sun-Wu) ∀ P.L metric d on a cpt (S, V) and ∀ $K^*: V \to (-\infty, 2\pi)$ w/ $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$, ∃! up to scaling a P.L metric d^* on (S, V) s.t.

(1) $d^* \overset{d.c.}{\sim} d$,

(2) $K_{d^*} = K^*$.

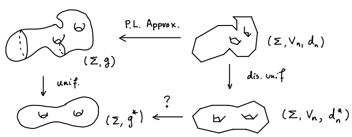
For $K^* = 2\pi\chi(S)/|V|$, d^* is the discrete unif metric.

Eg 7. 

Fillastrie

(8)

Convergence Question.



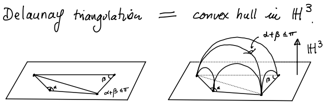
Gu-L-Wu: Convergence holds for the torus Σ = ◎.

Thm (Wu-Zhu) Convergence holds for all closed surfaces of genus > 1 in hyperbolic background setting if the approximation P.L surfaces satisfy: , $\alpha_n \ge \varepsilon$, $\alpha_n + \beta_n \le \pi - \varepsilon$, for some $\varepsilon > 0$.

§4. Relationship with the Weyl Problem

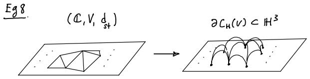
∃ relationship between dis. conf. and convex surfaces in $\mathbb{H}^3$.

Key: Delaunay triangulation = convex hull in $\mathbb{H}^3$



Delaunay condition   hyperbolic convex hull

Eg8 $(\mathbb{C}, V, d_{st})$ → $\partial C_H(V) \subset \mathbb{H}^3$ Convex hyperbolic surface in $\mathbb{H}^3$.



(9)

Unif. Thm. ∀ non-cpt s.c. Riemann surf is conformal to $\mathbb{C}$ or $\mathbb{D}$.

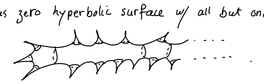
Dis. Unif. Problem ∀ non-cpt simply connected P.L surf is discrete conformal to $(\mathbb{C}, V)$ = or $(\mathbb{D}, V)$ = ,

V is discrete w/ $\partial V = \{\infty\}$ or ∂D, unique up to Möbius transf.

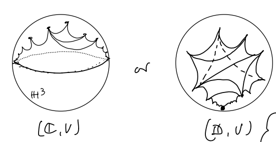
⇕

Weyl Problem in $\mathbb{H}^3$

S = genus zero hyperbolic surface w/ all but one ending being cusps



Then S is isometric to a unique convex surface in $\mathbb{H}^3$ whose boundary components in $\partial\mathbb{H}^3$ are points and circles.



$(\mathbb{C}, V)$ or $(\mathbb{D}, V)$

Thank you!