On the Convergence of Inversive Distance Circle Packings to the Riemann Mapping

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August 1st, 2023.

Goal: Compute conformal parametrizations of surfaces.

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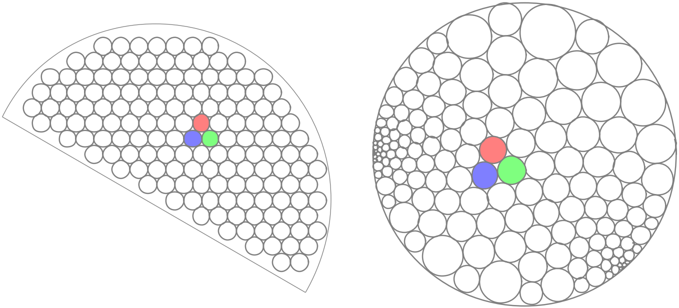


Thurston's idea: Circle packings as discrete conformal maps approximating the Riemann mapping.

Construct discrete conformal maps using circle packings.

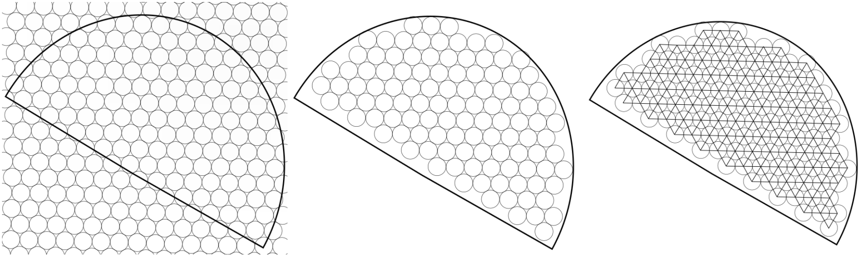
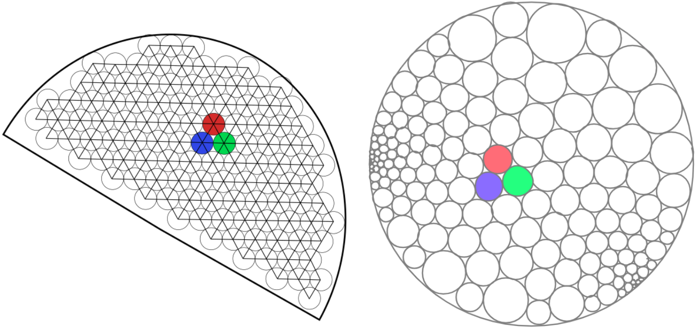


Figure: From a region Ω to the nerve T of a circle packing.

Existence and uniqueness of discrete conformal maps are given by

Theorem (Koebe-Andre'ev-Thurston, 1936, 1971,1978)

Every oriented simplicial triangulation of the 2-disk determined a circle packing of the 2-disk, unique up to Mobius transformations.



Thurston conjectured in 1985 the convergence of discrete conformal maps, confirmed by Rodin-Sullivan in 1987.

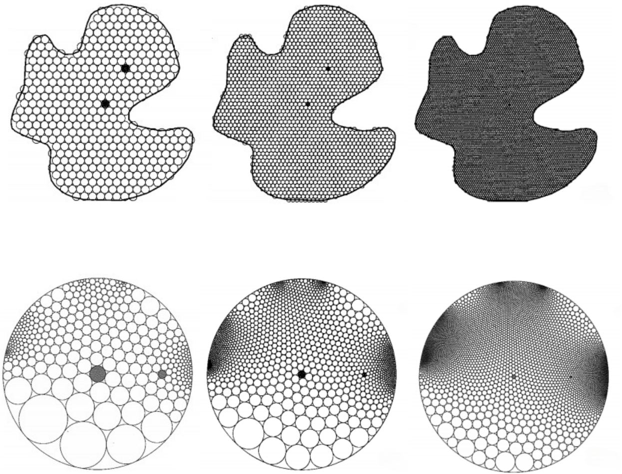


Figure: Approximating the Riemann mapping.[1]

[1]Picture by Stephenson, *Introduction to circle packing*, 2002.

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Thurston conjectured in 1985 the **convergence** of discrete conformal maps, confirmed by

Theorem (Rodin-Sullivan, 1987)

The discrete conformal map of circle packings converges to the Riemann mapping on a simply connected domain.

Idea of the proof of Rodin-Sullivan

Step 1 Construct simplicial homeomorphisms using circle packings to the 2-disk.

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Step 1: Construct simplicial homeomorphisms from the KAT theorem.

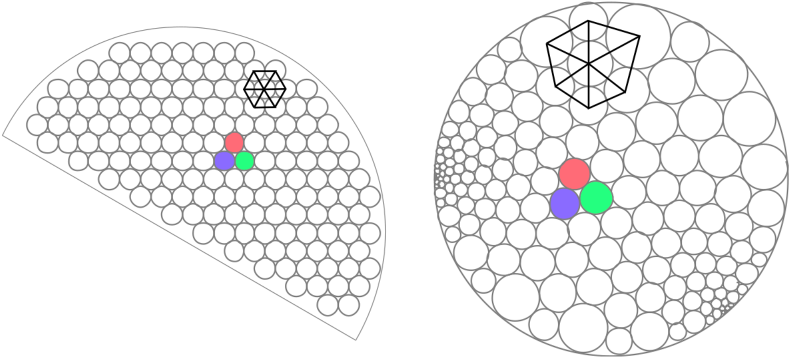
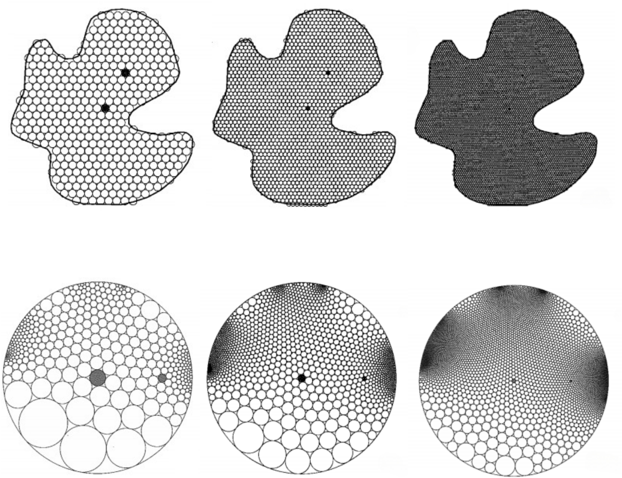
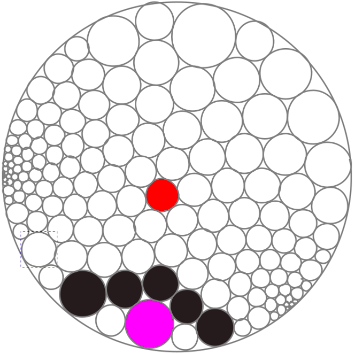


Figure: Discrete conformal maps are piecewise linear.

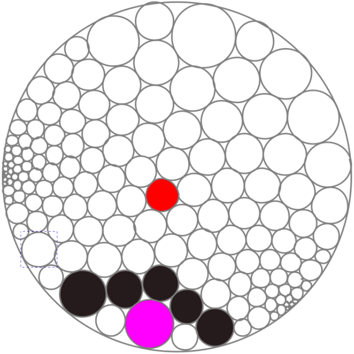


- A sequence of simplicial homeomorphisms $f_n : \Omega_n \to D_n \subset \mathbb{D}^2$,
- Clearly, $\Omega_n \subset \Omega$ and $\Omega_n \to \Omega$,
- It can be shown that $D_n \to \mathbb{D}^2$ using a length-area relation.



For any chain of circles separating a boundary circle C and center,

$$diam(C)^2 \leq (2\sum_{i=1}^{n} r_i)^2 \leq 4n\sum r_i^2 = 4n \cdot Area.$$



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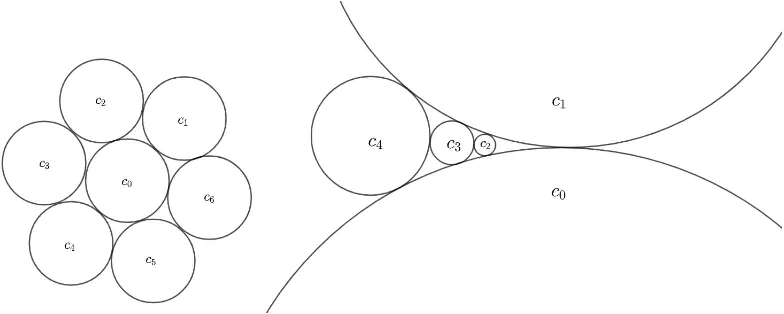
For many disjoint chains with n_j circles,

$$diam(C)^2 \sum_j \frac{1}{n_j} \leq 4 \cdot TotalArea \leq 4$$

Step 2: f_n are K-quasiconformal maps from the ring lemma.

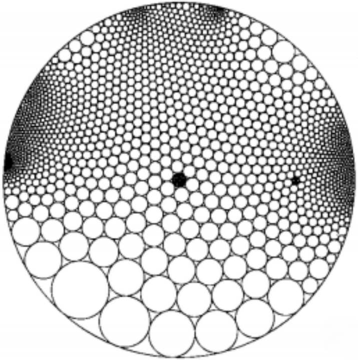


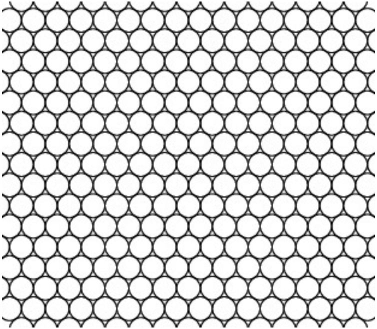
Lemma (Ring lemma)

There exists a lower bound on the ratio of two adjacent radii.

This implies that the angles of triangles can not be too small.

Step 3: the limit f is an 1-quasiconformal map.





Theorem (Rodin-Sullivan, 1987)

A circle packing of a simply connected domain in the plane with (infinite) hexagonal pattern is the regular hexagonal packing.

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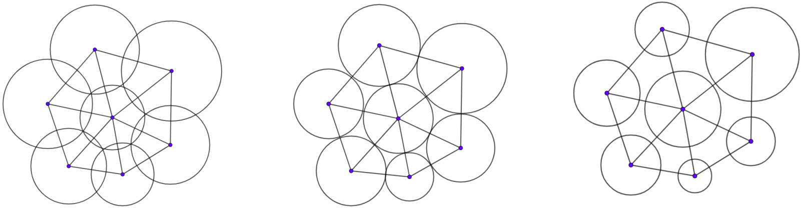
We will adapt this framework to prove the convergence of other discrete conformal maps.

Generalization of circle packings: *inversive distance circle packings*.



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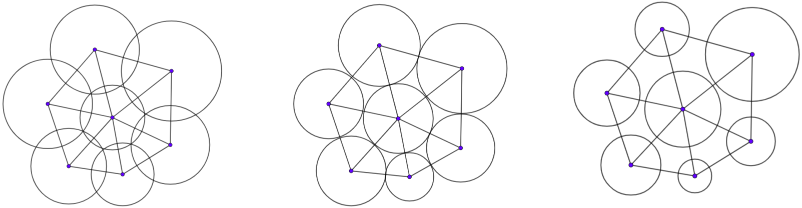

(a) $0 < \Theta < 1$, (b) $\Theta = 1$, (c) $\Theta > 1$.

Given a triangulation T of $\mathbb{D}^2$ with a weight $\Theta : E(T) \to \mathbb{R}_{>0}$, define a discrete metric $l : E(T) \to \mathbb{R}_{>0}$ on $(\mathbb{D}^2, T)$ as

$$l_{ij} = \sqrt{r_i^2 + 2\Theta r_i r_j + r_j^2}.$$

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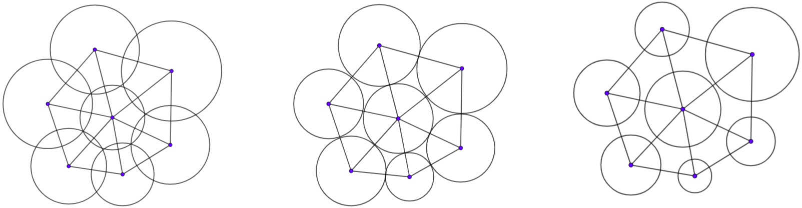

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Inversive distance circle packings are more flexible.

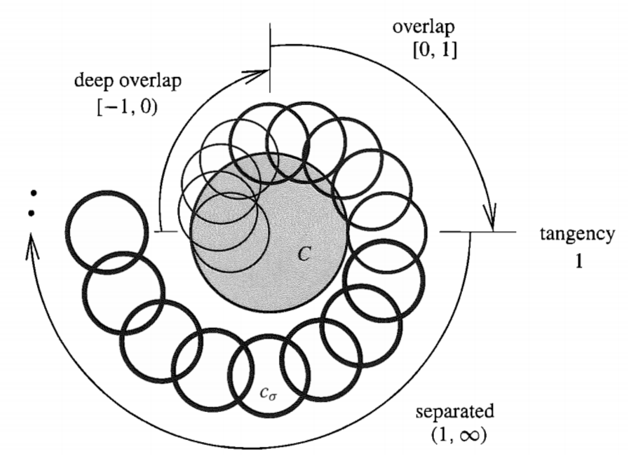


Figure: The inversive distance Θ between two circles.[2]

$$\Theta = \frac{l_{ij}^2 - r_i^2 - r_j^2}{2 r_i r_j}$$

[2]Picture by Stephenson, *Introduction to circle packing*, 2002.

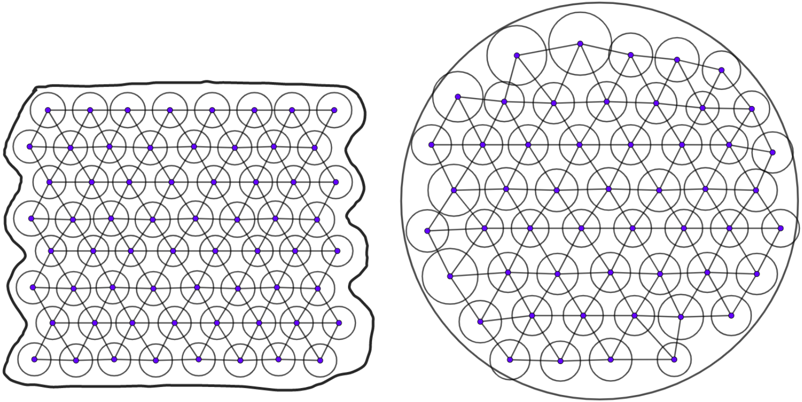
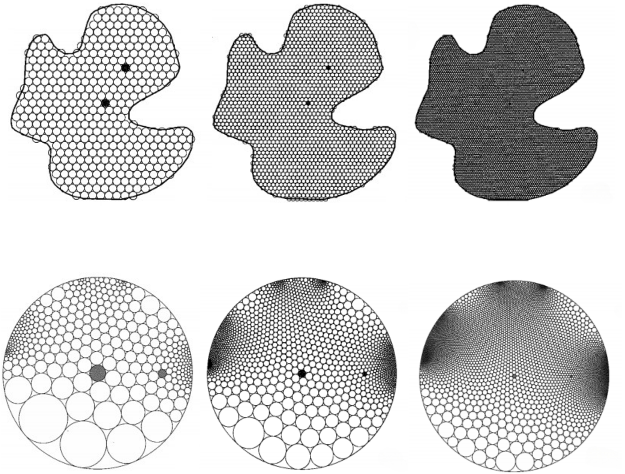
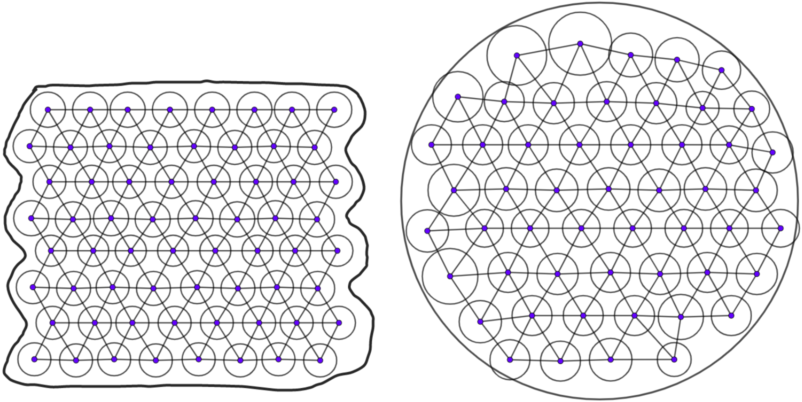


Figure: Discrete conformal maps using inversive distance circle packings.

Can we repeat this process for inversive distance circle packings?





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No KAT theorem for inversive distance circle packings!

Two issues: triangle inequalities and convexity.

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Recipe from the work of Luo-Sun-Wu in 2022,

- Construct simplicial homeomorphisms using **special** triangulations.
- Show the rigidity of the infinite hexagonal **weighted Delaunay** inversive distance circle packings.

Proposition

Suppose (P, T, l) *is a flat polyhedral disk with an equilateral triangulation* T *such that exactly three boundary vertices* p, q, r *have curvature* $\frac{2\pi}{3}$*, and the metric* l *is an inversive distance circle packing metric induced by a constant label* u *and a constant weight* $\Theta \to (1, +\infty)$*. Then for* **sufficiently large** n*, there is an inversive distance circle packing* $\tilde{u} : V_{(n)} \to \mathbb{R}$ *for* **the n-th standard subdivision** $(P, T_{(n)}, l_{(n)})$ *such that*

- $K_i(\tilde{u}) = 0$ *for all* $v_i \in V_{(n)} - \{p, q, r\}$*,*
- $K_i(\tilde{u}) = \frac{2\pi}{3}$ *for all* $v_i \in \{p, q, r\}$*,*

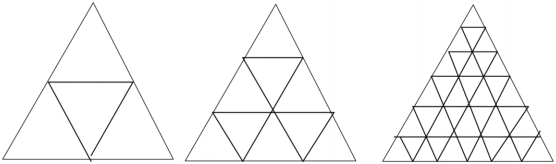


Figure: The n-th standard subdivision of one triangle.[3]

[3] Picture by Luo-Sun-Wu, 2022.

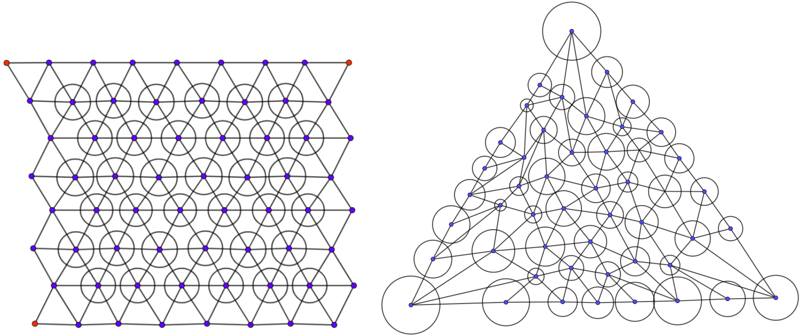


Figure: Discrete conformal maps using inversive distance circle packings.

No KAT theorem for inversive distance circle packings! **But it exists after sufficient subdivisions.**

Theorem (Chen-L.-Xu-Zhang, 2022, arXiv:2211.07464)

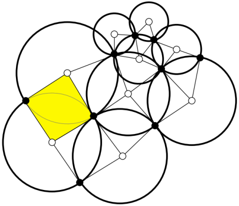
Let Ω be a Jordan domain in the complex plane with three distinct boundary points p, q, r specified. Let f be the Riemann mapping from the equilateral triangle $\triangle ABC$ to $\overline{\Omega}$ such that $f(A) = p$, $f(B) = q$, $f(C) = r$. Then there exists a sequence of weighted triangulated polygonal disks $(\Omega_n, \mathcal{T}_n, \eta_n, (p_n, q_n, r_n))$ with inversive distance circle packing metrics l_n, where $\mathcal{T}_n$ is a triangulation of Ω_n, $\eta_n : E_n \to (1, +\infty)$ is a weight defined on $E_n = E(\mathcal{T}_n)$ and p_n, q_n, r_n are three distinct boundary vertices of $\mathcal{T}_n$, such that

(a) $\Omega = \cup_{n=1}^{\infty} \Omega_n$ *with* $\Omega_n \subset \Omega_{n+1}$, *and* $\lim_n p_n = p$, $\lim_n q_n = q$, $\lim_n r_n = r$.

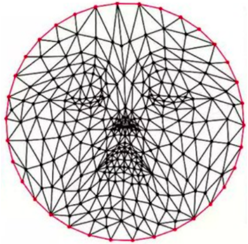
(b) *discrete conformal maps f_n from $\triangle ABC$ to $(\Omega_n, \mathcal{T}_n, \eta_n, l_n)$ with $f_n(A) = p_n$, $f_n(B) = q_n$, $f_n(C) = r_n$ exist.*

(c) *discrete conformal maps f_n converge uniformly to the Riemann mapping f.*

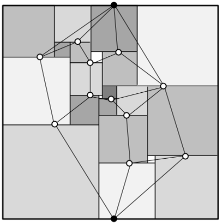
Other discrete conformal maps and their convergence



(a) Circle Patterns, *Bücking*, 2018.



(b) Tutte embedding, *Dym – Slutsky – Lipman*, 2019.



(c) Square tiling, *Georgakopoulos – Panagiotis*, 2020.

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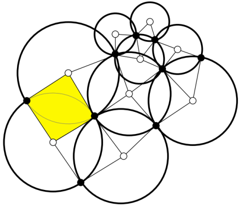

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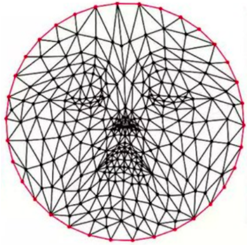

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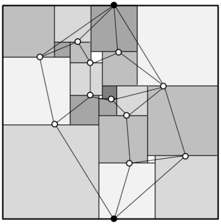

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Other convergence scheme:
convergence of discrete conformal factors to the smooth factor on general surfaces.

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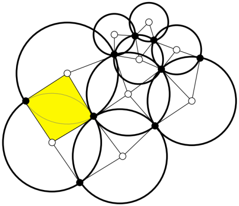

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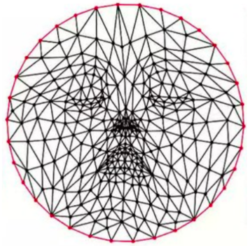

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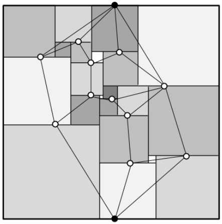

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Thank you!