Quasiconvexity and nonlinear Elasticity

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The classical Calculus of Variations

We are interested in minimizers of

$$\mathscr{F}[u] \equiv \int_\Omega F(\mathrm{D}u)\,\mathrm{d}x, \qquad u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m,$$

where $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ and $m, n \geq 2$.

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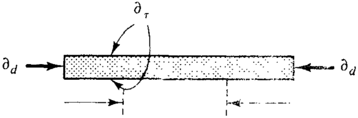
A crucial feature in vectorial problems is that F is often **non-convex**.

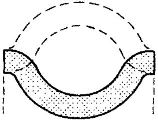
In nonlinear Elasticity, F is the **stored-energy function** of an elastic material with reference configuration Ω.

Neo-Hookean models

Non-uniqueness of solutions $\implies$ F is not convex!



Prescribed displacement



Two solutions in two dimensions; a circle of solutions in three dimensions

Image from the book by Marsden and Hughes

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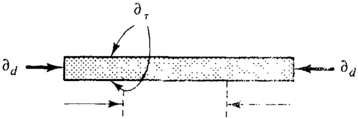

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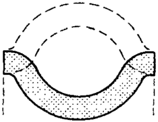

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In a neo–Hookean model, F may take the form

$$F(\mathrm{D}u) = G\left(\frac{|\mathrm{D}u|^n}{\det \mathrm{D}u}\right) + H(\det \mathrm{D}u), \qquad \text{(NH)}$$

known as the **additive isochoric–volumetric** split (Flory 1961).

Quasiconvexity

Natural existence condition for min problems is **quasiconvexity**:

$$|\Omega|F(A) \leq \int_\Omega F(\mathrm{D}u)\,\mathrm{d}x \qquad \text{for all } u \in A + C_c^\infty(\Omega, \mathbb{R}^m),$$

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Assuming that $|F| \leq C(1+|\cdot|^p)$,

$$F \text{ is quasiconvex} \iff \exists \text{ minimizers in } W^{1,p}.$$

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The growth condition fails for (NH):

Open Problem (Ball–Murat 1984, Ball 2002)

Prove existence of minimizers for quasiconvex F satisfying

$$\det A \to 0 \implies |F(A)| \to \infty.$$

Rank-one convexity

A main example is $F = \det$:

$$|\Omega| \det(A) = \int_\Omega \det(\mathrm{D}u)\,\mathrm{d}x \quad \forall u \in A + C_c^\infty(\Omega, \mathbb{R}^n).$$

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In general we have

$$F \text{ is convex } \underset{\not\Longleftarrow}{\Longrightarrow} F \text{ is quasiconvex } \Longrightarrow F \text{ is rank-one convex.}$$

We say that F is **rank-one convex** if, for $\lambda \in (0,1)$,

$$F(\lambda A + (1-\lambda)B) \leq \lambda F(A) + (1-\lambda)F(B),$$

when $\operatorname{rank}(B - A) = 1$. Equiv: Euler–Lagrange system is elliptic.

Morrey's problem

Recall that $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ and the maps are $u\colon \mathbb{R}^n \to \mathbb{R}^m$.

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In particular,

- The case $m = 2, n \geq 2$ is **OPEN**.

Work on this problem by Ball, Šverák, Müller, Dacorogna, Pedregal, Kirchheim, Iwaniec, Astala, Székelyhidi, Faraco...

The main result

Notation: $\mathbb{R}^{2\times2}_{+} \equiv \{A \in \mathbb{R}^{2\times2} : \det A > 0\}$, $K_A \equiv \frac{|A|^2}{\det A}$.

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Theorem (Astala–Faraco–G.–Koski–Kristensen 2023)

Let $F\colon \mathbb{R}^{2\times 2}_+ \to \mathbb{R}$ be as in (NH):

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in the sense that, if g is a diffeomorphism and $q > 1$,

$$\left.\begin{aligned} &u_j = g \text{ on } \partial\Omega \\ &u_j \rightharpoonup u \text{ in } W^{1,2}(\Omega) \\ &\|Ku_j\|_{L^q(\Omega)} \le C \end{aligned}\right\} \implies \liminf_{j\to\infty} \int_\Omega F(\mathrm{D}u_j)\,\mathrm{d}x \ge \int_\Omega F(\mathrm{D}u)\,\mathrm{d}x.$$

Main ingredients

There are four main ingredients in the proof.

Rank-one convexity $\implies$ quasiconvexity:

1) extremal integrands;
2) the Burkholder function;

Quasiconvexity $\implies$ weak lower semicontinuity:

3) Jensen inequalities for principal maps;
4) Stoilow factorization.

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Proposition (Voss–Martin–Ghiba–Neff 2021)

For F as in the theorem,

$$F \text{ is rank-one convex} \implies F = G + c\mathscr{W}$$

where $c \geq 0$, G is polyconvex and

$$\mathscr{W}(A) \equiv K_A - \log K_A + \log \det A.$$

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Recall G is polyconvex if $G = g(A, \det A)$ and $g\colon \mathbb{R}^5 \to \mathbb{R}$ is convex.
$\mathscr{W}$ is **not polyconvex**, since

$$\lim_{t\to 0} \mathscr{W}(t\,\mathrm{Id}) = \lim_{t\to 0} 1 + \log(t^2) = -\infty.$$

But it suffices to prove the theorem for $\mathscr{W}$.

2: The Burkholder function

$\mathscr{W}$ is closely connected to the **Burkholder function** (1984)

$$B_p(A) = \Big((\tfrac{p}{2} - 1)|A|^2 - \tfrac{p}{2} \det A \Big) |A|^{p-2}, \qquad p \geq 2.$$

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Conjecture (Iwaniec 1990s)

The Burkholder function is quasiconvex.

This conj has huge implications in harmonic and complex analysis.

2: The Burkholder function (continued)

Theorem (G.–Kristensen 2022, AFGKK 2023)

If $u \in A + C_c^\infty(\Omega, \mathbb{R}^2)$ and $B_p(\mathrm{D}u)$ **doesn't change sign**, then

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Earlier results by Astala–Iwaniec–Prause–Saksman 2012–2015.

Our proof combines their complex interpolation argument with an extremality argument using gradient Young Measures, cf. G. 2018.

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Consider the involution

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Its characteristic property is that, if $v = u^{-1}$ is a diffeo,

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$\widehat{\cdot}$ preserves poly-, quasi- and rank-one convexity, but not convexity. One can calculate

$$\mathscr{F}(A) \equiv \lim_{p \searrow 2} 2\frac{B_p(A) + \det A}{p-2} = |A|^2 - (1 + \log|A|^2)\det A,$$

$$\mathscr{W}(A) = \widehat{\mathscr{F}}(A) + 1.$$

2: The Burkholder function (continued)

Corollary

If $u \in A + C_c^\infty(\Omega, \mathbb{R}^2)$ is a smooth diffeo then

$$|\Omega| \mathscr{F}(A) \leq \int_\Omega \mathscr{F}(\mathrm{D}u)\,\mathrm{d}x,$$

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These are **sharp versions** of Müller 1990, Koskela–Onninen 2008. For instance, locally,

$$u \in W^{1,2}, \det \mathrm{D}u \geq 0 \quad \implies \quad \det \mathrm{D}u \in L \log L.$$

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Here we show:

$$\int_\Omega \det \mathrm{D}u \, \log |\mathrm{D}u|^2 \, \mathrm{d}x \leq \int_\Omega |\mathrm{D}u|^2 \, \mathrm{d}x - |\Omega|(\mathscr{F}(A) + 1).$$

Quasiconvexity $\implies$ weak lsc

3: Jensen inequalities for principal maps

A map $u\colon \mathbb{C} \to \mathbb{C}$ is **principal** if

$$u(z) = z + \sum_{j=1}^{\infty} \frac{b_j}{z^j} \qquad \text{when } |z| > 1.$$

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Theorem (AFGKK 2023)

Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{C})$ be a principal map with $K_u \in L^1(\mathbb{D})$. Then

$$\mathscr{W}\left(\fint_{\mathbb{D}} \mathrm{D}u \,\mathrm{d}x\right) \leq \fint_{\mathbb{D}} \mathscr{W}(\mathrm{D}u)\,\mathrm{d}x.$$

This is a Jensen inequality **without linear boundary conditions!**

3: Jensen inequalities for principal maps (continued)

Recall: $\mathscr{W}(A) = K_A - \log K_A + \log\det A$. If $b_1 = 0$, want to show:

$$\mathscr{W}(\mathrm{Id}) \leq \fint_{\mathbb{D}} \mathscr{W}(\mathrm{D}u)\,\mathrm{d}x.$$

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Since $u = \mathsf{Id}$ at ∞, by quasiconvexity we have

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The main point is that, when u is holomorphic, ψ is **harmonic**:

$$\psi \equiv \mathscr{W}(\mathrm{D}u) - \mathscr{W}(\mathrm{Id}) = 2\log|u'|.$$

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Applying the mean value at ∞, we get

$$0 = \psi(\infty) = \int_{\mathbb{C}\setminus\mathbb{D}} \psi\,\mathrm{d}x = \int_{\mathbb{C}\setminus\mathbb{D}} [\mathscr{W}(\mathrm{D}u) - \mathscr{W}(\mathsf{Id})]\,\mathrm{d}x$$

i.e. $\mathbb{C}\setminus\mathbb{D}$ is a **null quadrature domain** (Sakai 1981).

Proposition (AFGKK 2023)

Let g be a diffeo, $q > 1$. For any sequence u_j we have

$$\left.\begin{array}{l} u_j = g \text{ on } \partial\Omega \\ u_j \rightharpoonup u \text{ in } W^{1,2}(\Omega) \\ \|Ku_j\|_{L^q(\Omega)} \leq C \end{array}\right\} \implies \liminf_{j\to\infty} \int_\Omega \mathscr{W}(\mathrm{D}u_j)\,\mathrm{d}x \geq \int_\Omega \mathscr{W}(\mathrm{D}u)\,\mathrm{d}x.$$

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By YM machinery wlog can take $\mathrm{D}u_j \rightharpoonup \mathrm{Id}$. Want to replace u_j with a better sequence (cf. Astala–Faraco 2002).

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$$\left.\begin{array}{l} u_j = g \text{ on } \partial\Omega \\ u_j \rightharpoonup u \text{ in } W^{1,2}(\Omega) \\ \|Ku_j\|_{L^q(\Omega)} \leq C \end{array}\right\} \implies \liminf_{j\to\infty} \int_\Omega \mathscr{W}(\mathrm{D}u_j)\,\mathrm{d}x \geq \int_\Omega \mathscr{W}(\mathrm{D}u)\,\mathrm{d}x.$$

By YM machinery wlog can take $\mathrm{D}u_j \rightharpoonup \mathrm{Id}$. Want to replace u_j with a better sequence (cf. Astala–Faraco 2002).

Iwaniec–Šverák 1992: $\exists$ holomorphic h_j, principal maps f_j with

$$u_j = h_j \circ f_j,$$

with $h_j(z) \to z$ in C^∞_{loc}.

4: Stoilow Factorization

Proposition (AFGKK 2023)

Let g be a diffeo, $q > 1$. For any sequence u_j we have

$$\left.\begin{array}{l} u_j = g \text{ on } \partial\Omega \\ u_j \rightharpoonup u \text{ in } W^{1,2}(\Omega) \\ \|Ku_j\|_{L^q(\Omega)} \leq C \end{array}\right\} \implies \liminf_{j\to\infty} \int_\Omega \mathscr{W}(\mathrm{D}u_j)\,\mathrm{d}x \geq \int_\Omega \mathscr{W}(\mathrm{D}u)\,\mathrm{d}x.$$

By YM machinery wlog can take $\mathrm{D}u_j \rightharpoonup \mathrm{Id}$. Want to replace u_j with a better sequence (cf. Astala–Faraco 2002).

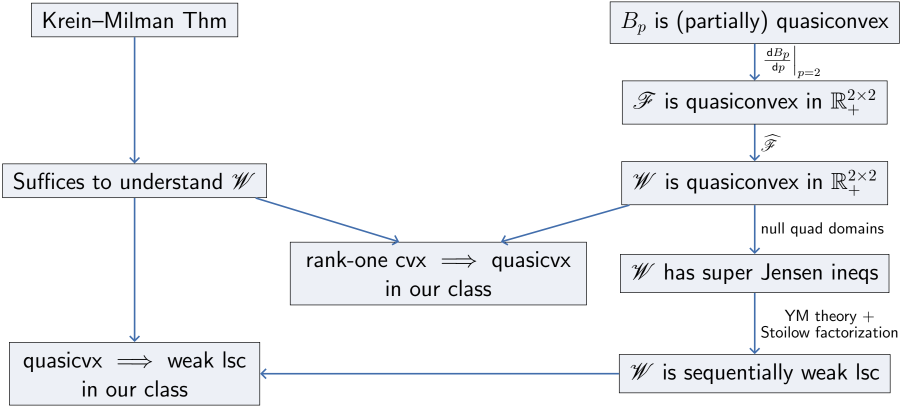
Iwaniec–Šverák 1992: $\exists$ holomorphic h_j, principal maps f_j with

$$u_j = h_j \circ f_j,$$

with $h_j(z) \to z$ in C^∞_{loc}. Then apply Jensen's ineq for principal maps:

$$\liminf_{j\to\infty} ⨍_{\mathbb{D}} \mathscr{W}(\mathrm{D}u_j)\,\mathrm{d}x = \liminf_{j\to\infty} ⨍_{\mathbb{D}} \mathscr{W}(\mathrm{D}f_j)\,\mathrm{d}x \geq \mathscr{W}(\mathrm{Id}).$$

Proof outline



Outlook

Further directions: regularity

We have see that, when combined,

- Jensen inequalities for principal maps
- the Stoilow factorization

yield existence theorems without growth conditions.

Question

Can these tools be used to prove **regularity** results?

Even in the simple polyconvex example

$$F(A) = |A|^2\left(1 + \frac{1}{(\det A)^2}\right)$$

almost nothing is known about regularity of minimizers, but see Bauman–Owen–Phillips 1991, Iwaniec–Kovalev–Onninen 2013.