Harmonic dipoles in elasticity

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H. & Rodiac, DCDS 38 (2018)
arXiv:2102.12303
2111.07112

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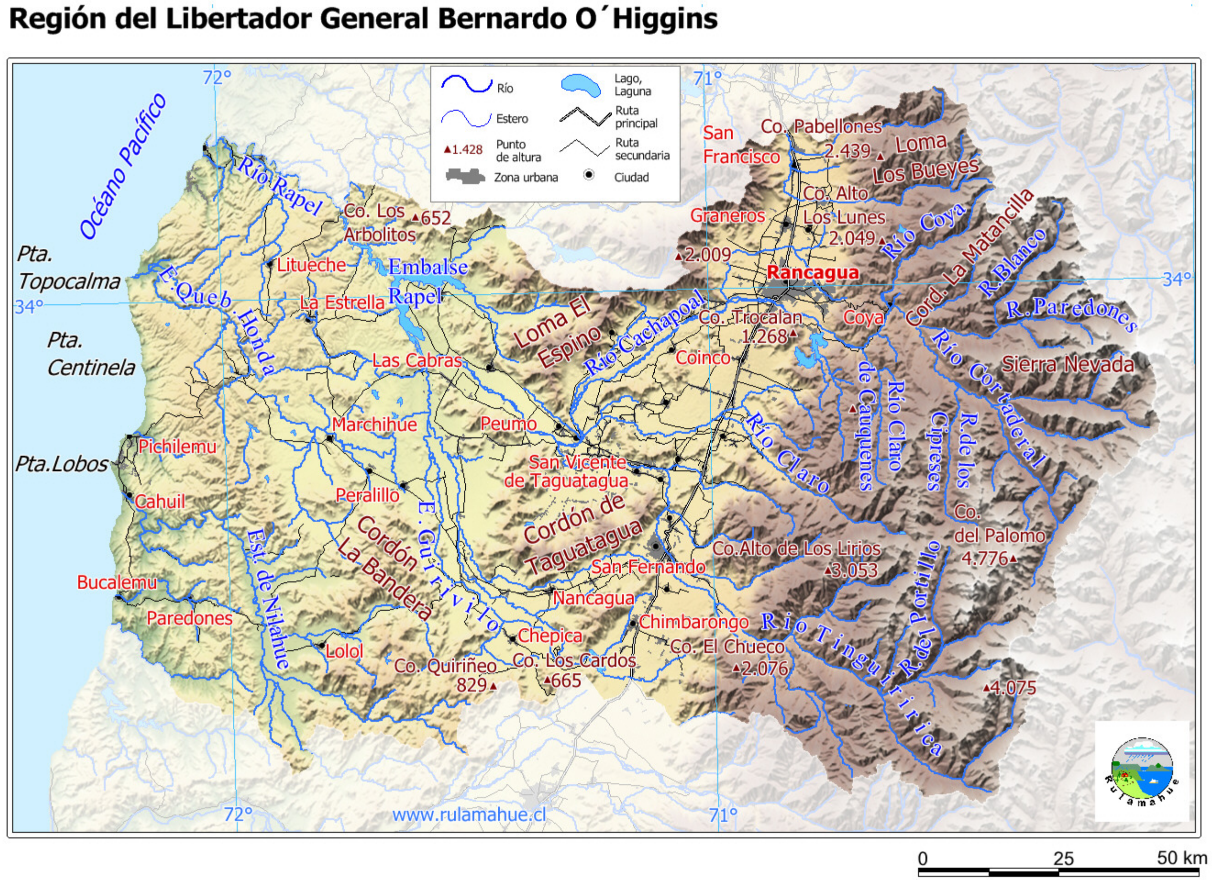
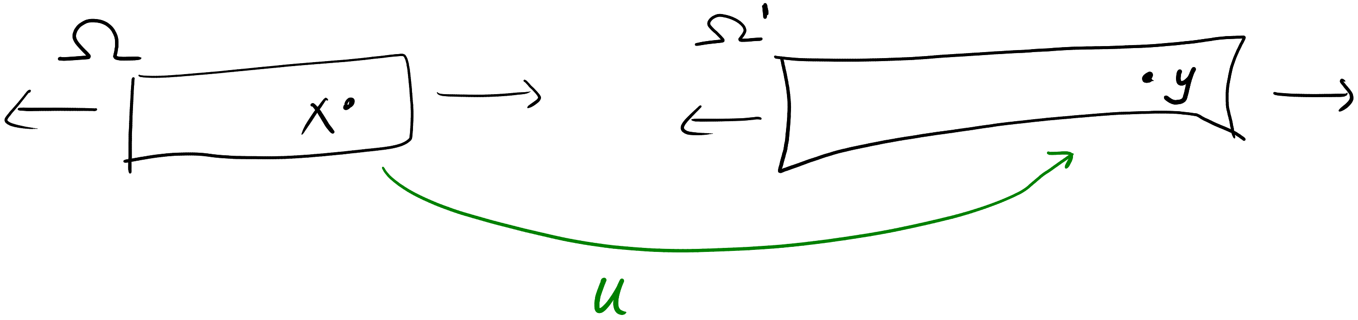
https://www.pinterest.fr/pin/47780446037019908/



https://www.rulamahue.cl/fichas/cl06/cl06.html







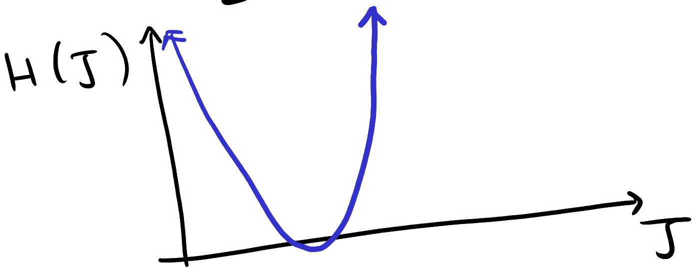
$$\Omega \subseteq \mathbb{R}^3$$

$$y = u(x)$$

$$\min_{u \in \mathcal{A}} \int_\Omega W(Du(x))\,dx$$

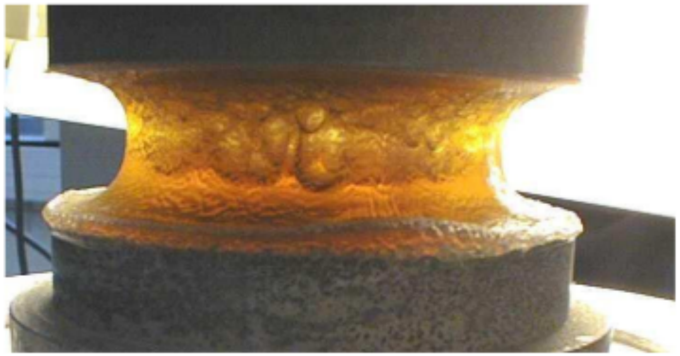
$$\mathcal{A} \subset H^1(\Omega, \mathbb{R}^3)$$

$$W(F) = \frac{|F|^2}{2} + H(\det F)$$

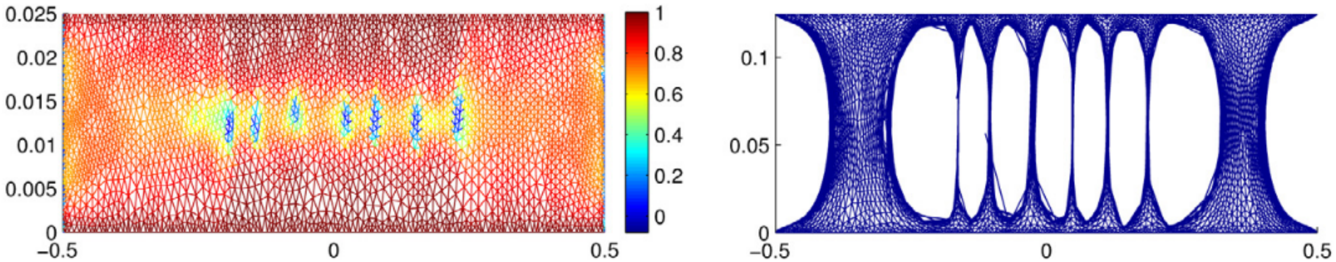


$$W(F) = \mu\left(\frac{|F|^2}{2} - \ln J\right) + \lambda \frac{(J-1)^2}{2}, \quad J = \det F$$

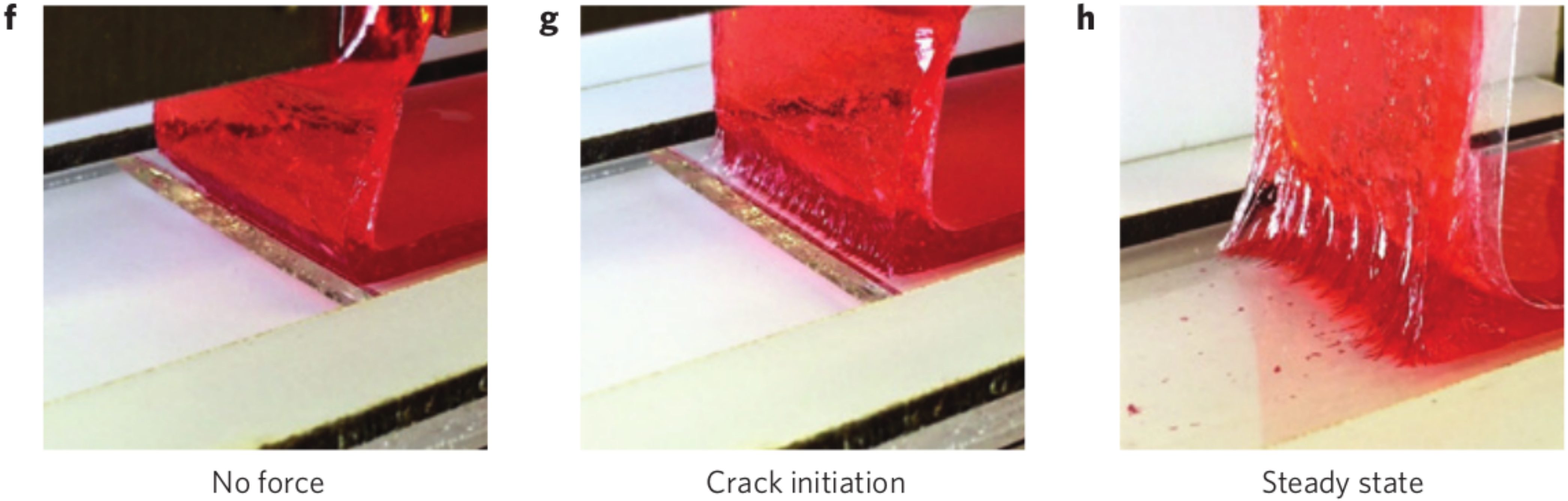
$$\lambda \gg \mu .$$



Robert Nevière
(SNPE Matériaux Energétiques,
Centre de Recherches de Bouchet)



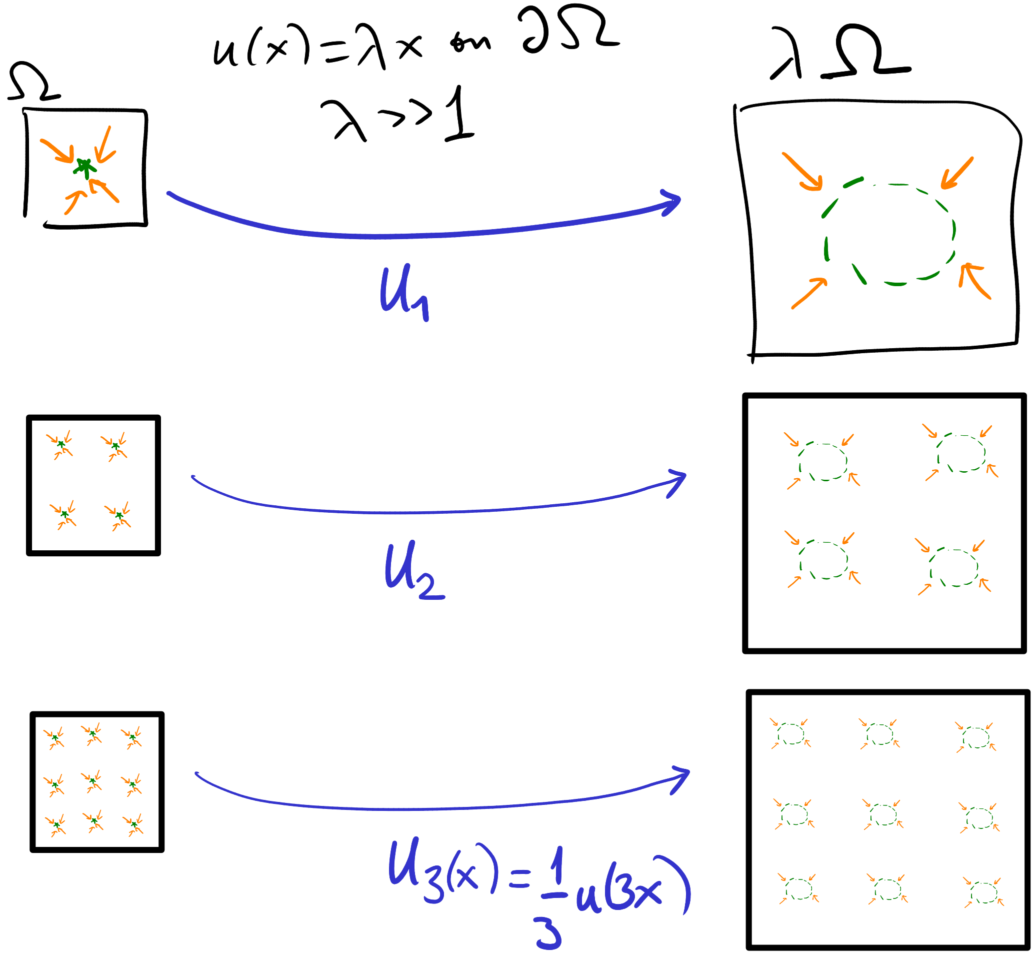
H., Mora-Corral & Xu, CMAME 303 (2016)



H. Yuk, T. Zhang, S. Lin, G. Parada, X. Zhao; Tough bonding of hydrogels to diverse non-porous surfaces. Nature Materials 15 (2016) 190-196

Adhesion energy (swollen gel on glass): 1123 J/m^2

Ball & Murat JFA '84: if cavitation is allowed then $\int_\Omega W(Du(x))\,dx$ is not w. lower semicontinuous in H^1



$$W(Du) = \frac{|Du|^2}{2} + H(\det Du)$$

$$\int_\Omega Du_1 = \int_{\partial\Omega} u_1(x) \otimes \nu(x)$$

$$= \int_{\partial\Omega} u_h(x) \otimes \nu(x)$$

$$= \int_\Omega Du_h$$

$$\therefore \quad u_j \xrightarrow{H^1} u_h$$

$$\int W(Du_h) > \lim_{j\to\infty} \int W(Du_j).$$

Question: can we prove existence of minimizers

of $$E(\underset{\sim}{u}) = \int |D\underset{\sim}{u}|^2 + H(\det D\underset{\sim}{u})\,dx$$

in $$\Big\{ \underset{\sim}{u} \in H^1(\Omega) : \ \underset{\sim}{u} = \underset{\sim}{b} \text{ on } \partial\Omega,$$

$\underset{\sim}{u}$ is one-to-one a.e.,

$\det D\underset{\sim}{u} > 0$ a.e.,

$$E(\underset{\sim}{u}) \le E(\underset{\sim}{b}), \quad \operatorname{Div}\big((\operatorname{adj} D\underset{\sim}{u})\,\underset{\sim}{g} \circ \underset{\sim}{u}\big) = (\operatorname{div} \underset{\sim}{g}) \circ \underset{\sim}{u} \det D\underset{\sim}{u} \text{ for all } \underset{\sim}{g} \in C^\infty(\mathbb{R}^3;\mathbb{R}^3) \Big\}$$

$$\underset{\sim}{g}(\underset{\sim}{y}) = \frac{1}{3}\underset{\sim}{y}$$

$$\det D\underset{\sim}{u} = \frac{1}{3}\operatorname{Div}\big((\operatorname{adj} D\underset{\sim}{u})\,\underset{\sim}{u}\big)$$

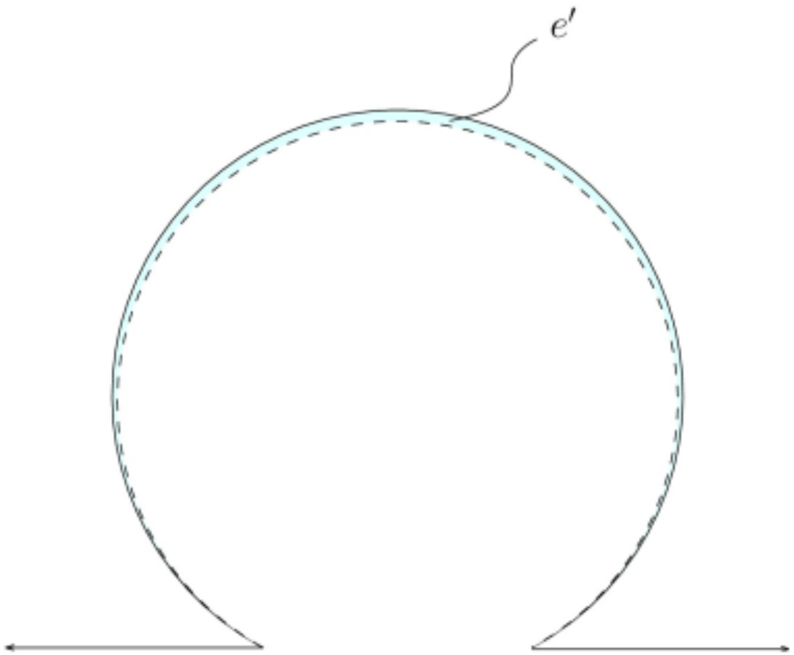
S. Conti & C. De Lellis '03:

Step 1.

Before the deformation:

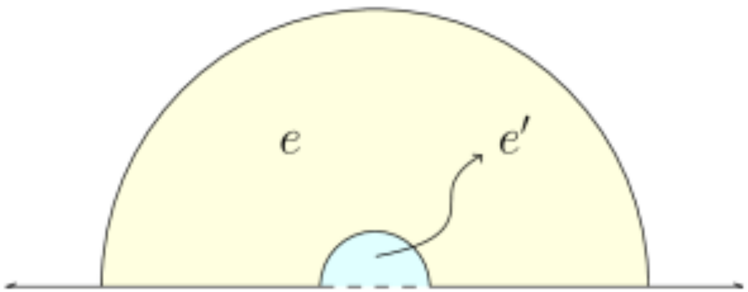
Region e' is mapped onto:

And then onto:

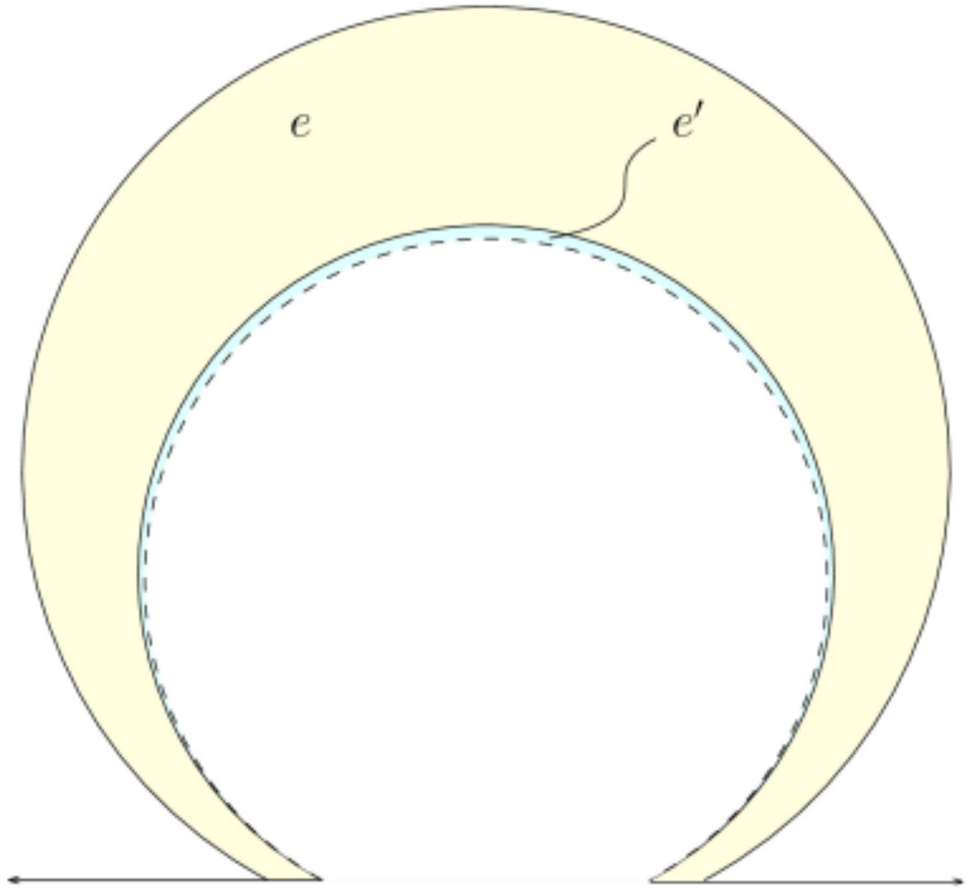


Step 2.

Before the deformation:

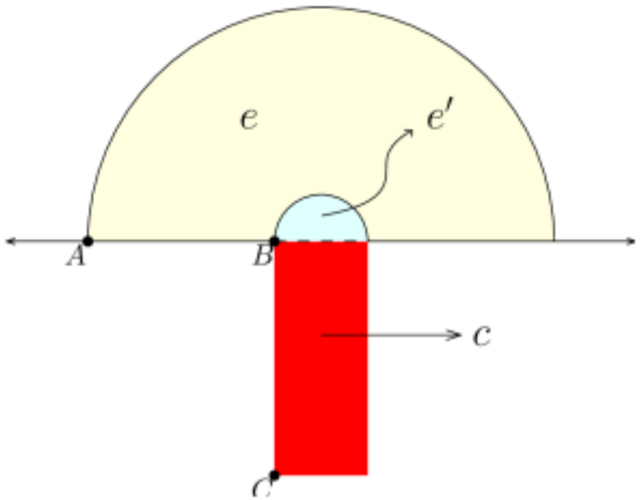


After the deformation:

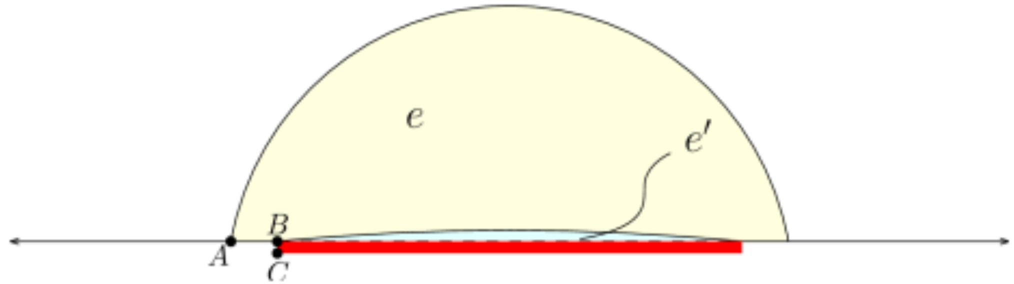


Step 3.

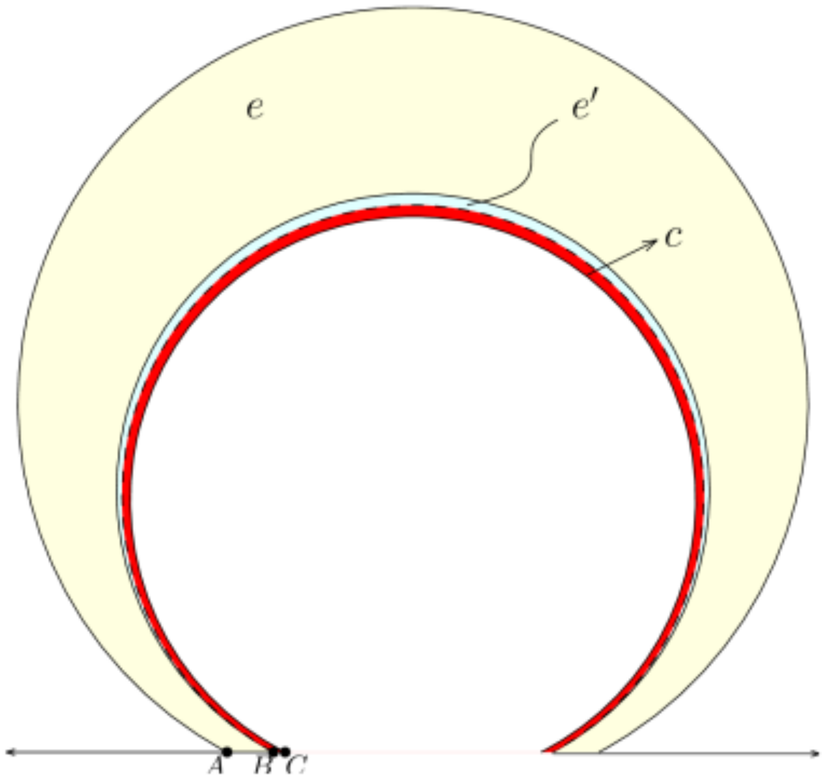
Before the deformation:



An intermediate configuration:

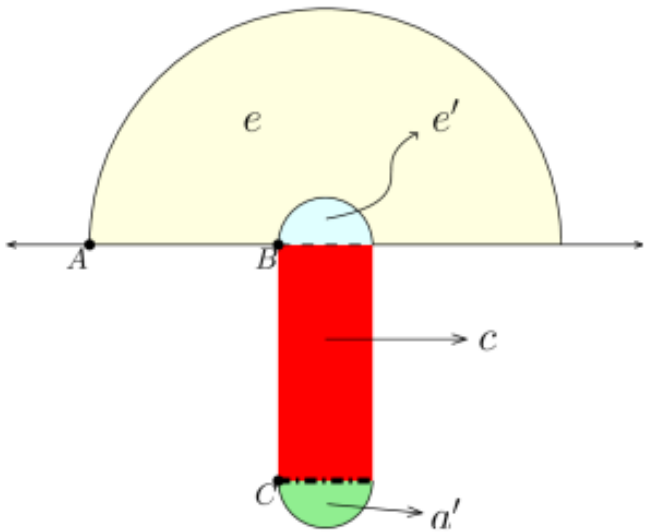


After the deformation:

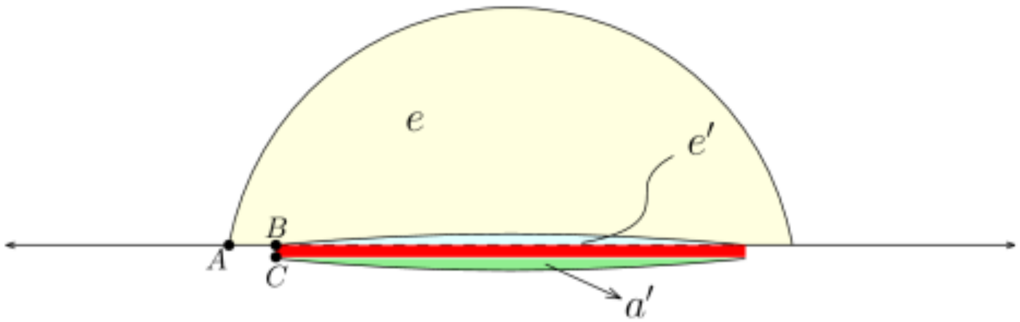


Step 4.

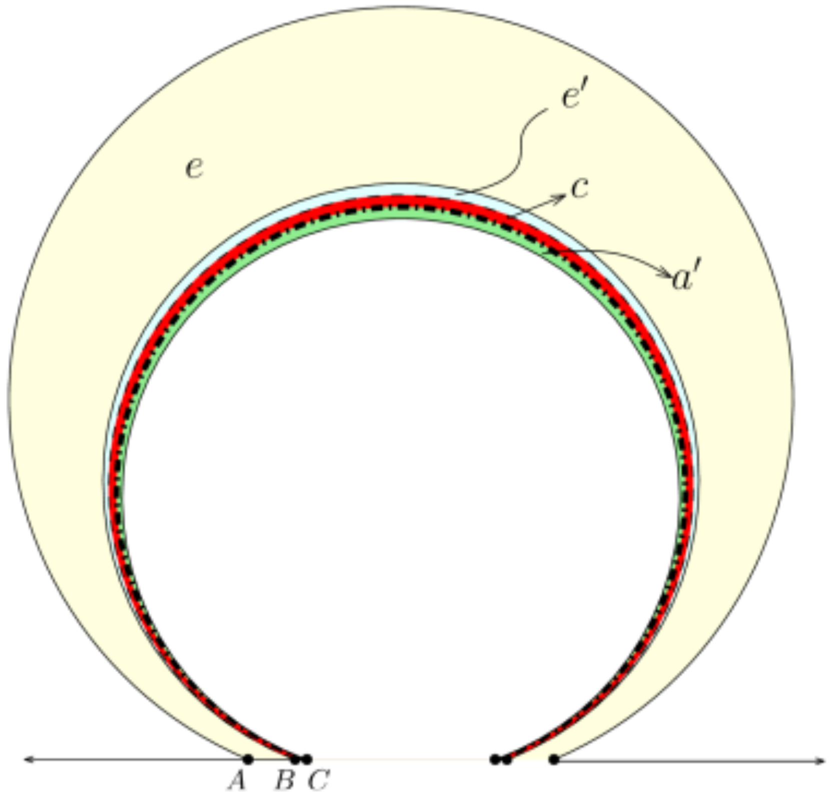
Before the deformation:



An intermediate configuration:

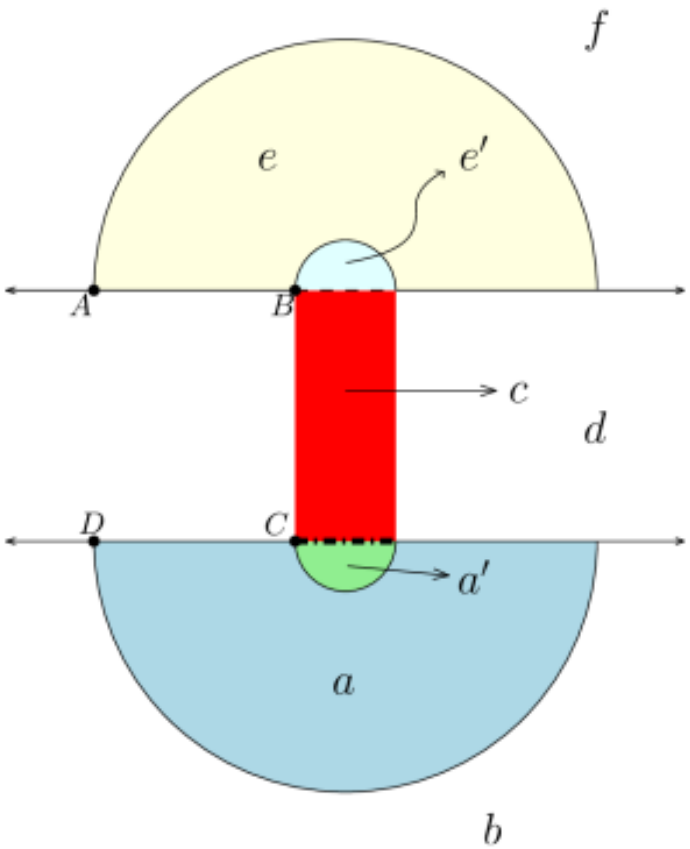


After the deformation:

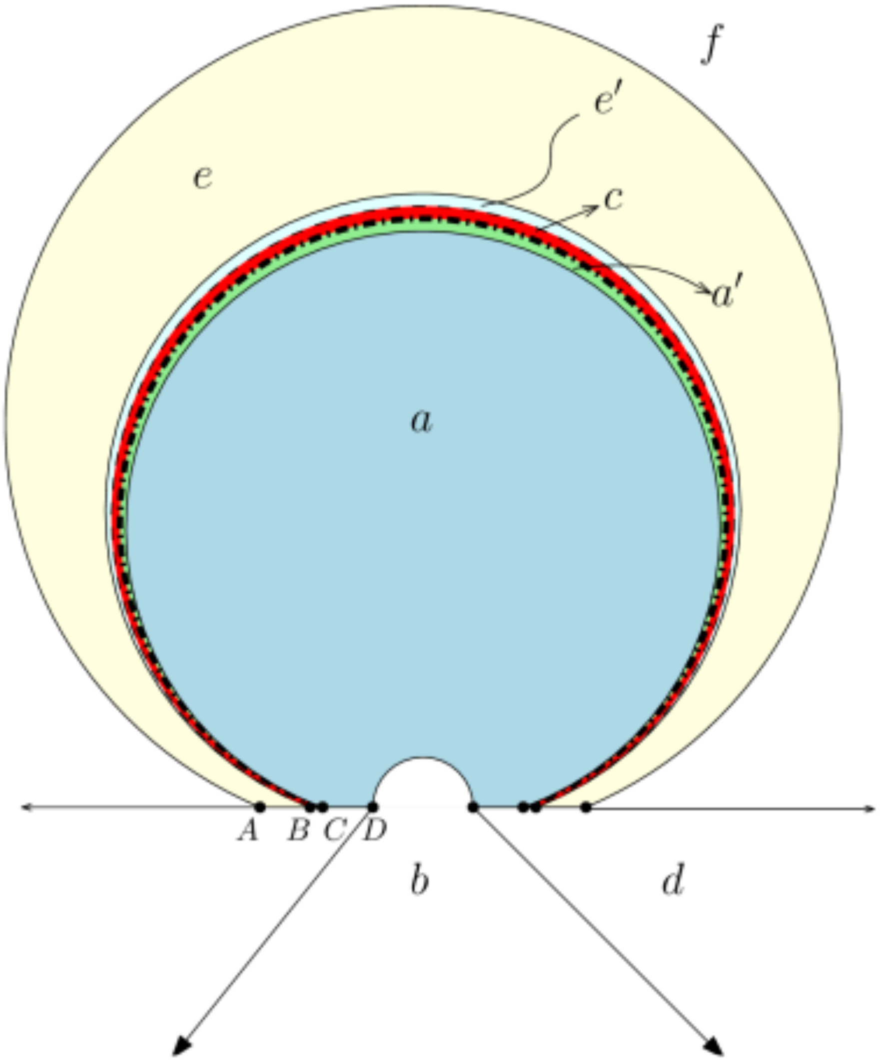


Step 5.

Before the deformation:

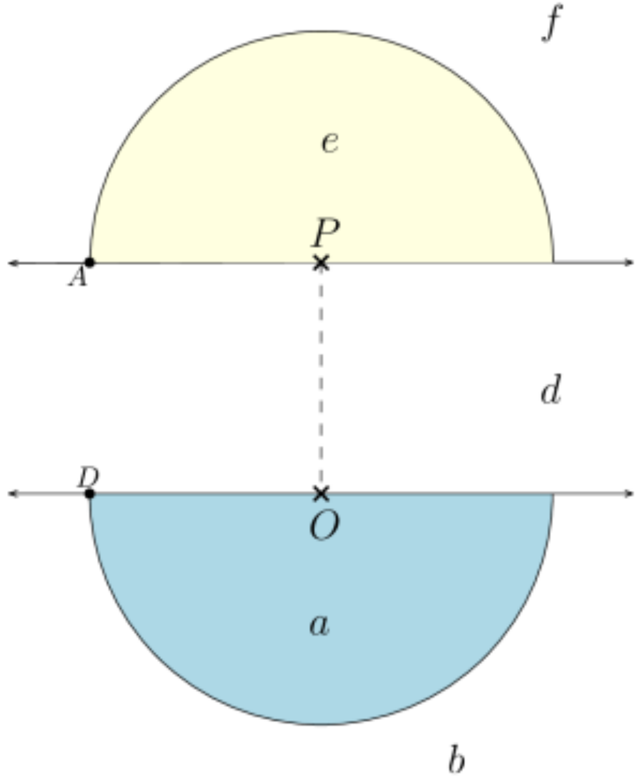


After the deformation:

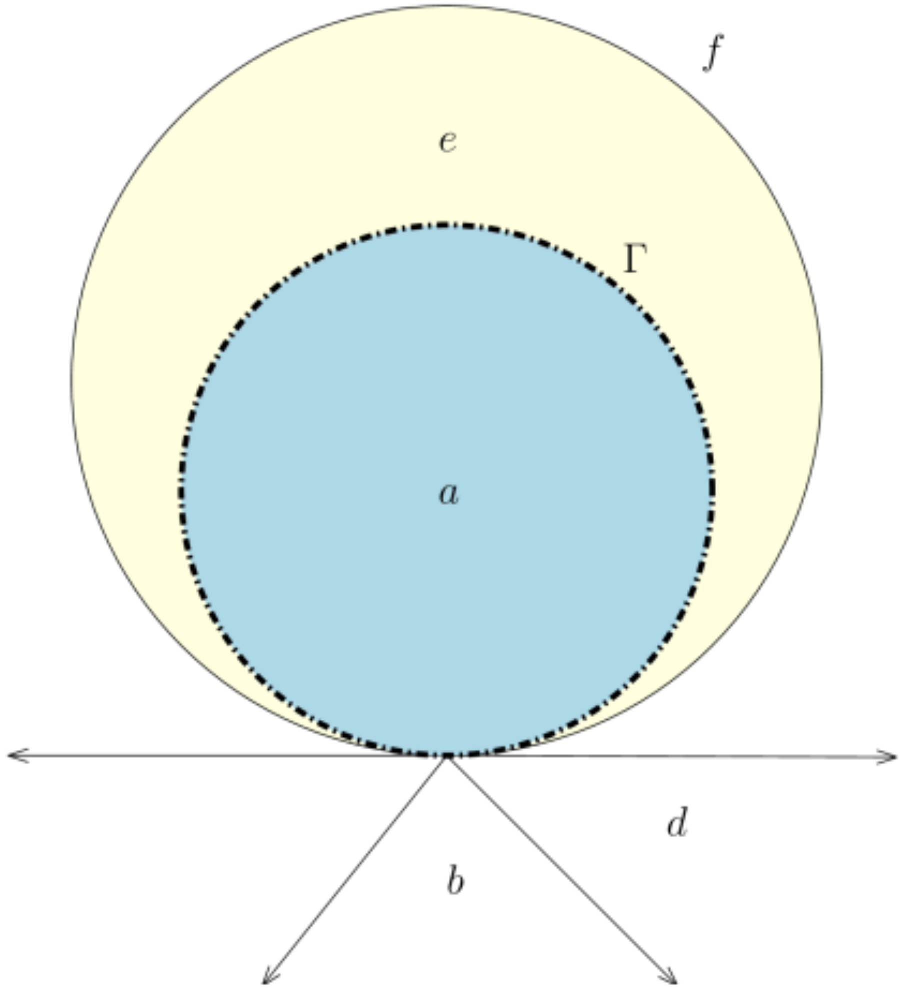


In the limit:

Before the deformation:

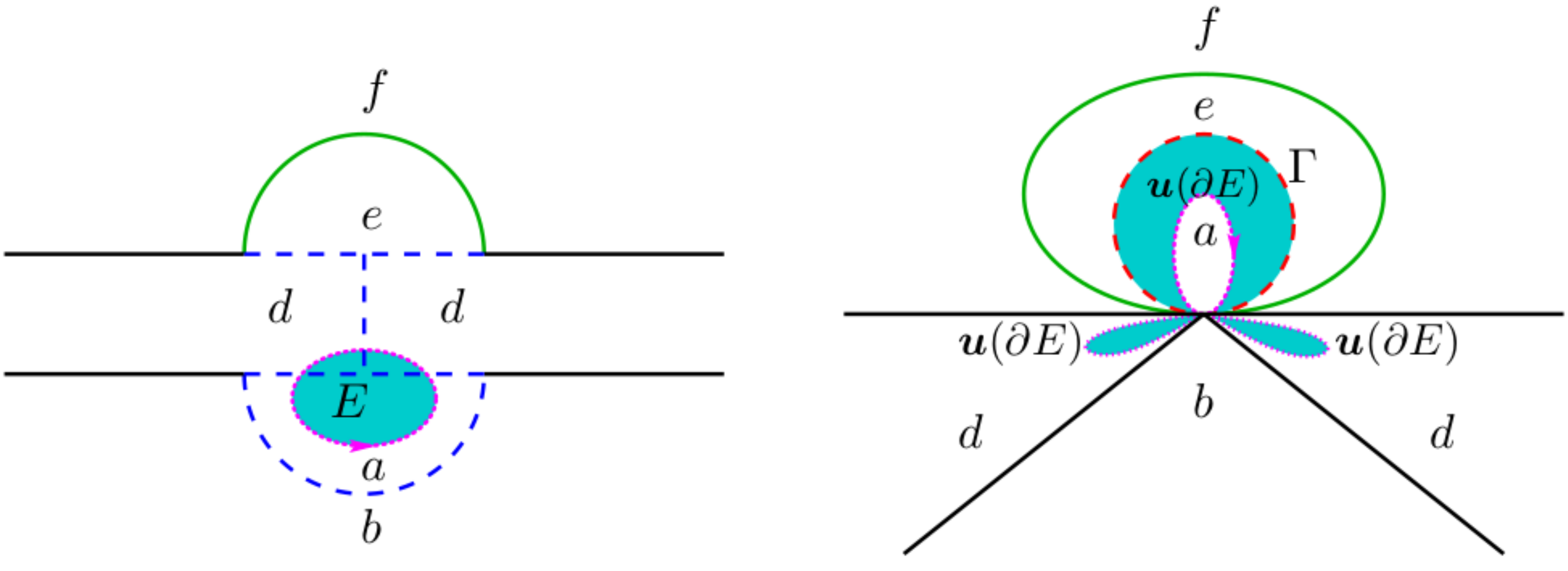


After the deformation:



$$\int_{B(0,r)} (\operatorname{div} g)(u(x)) \det Du(x) = \int_{u(\partial B)} g\cdot\nu + \int_{\Gamma} g\cdot\nu$$

$$\int_{\mathbb{R}^3} \deg(u, B(x,r), y)\, \operatorname{div} g(y)\, dy = \int_{u(\partial B)} g\cdot\nu$$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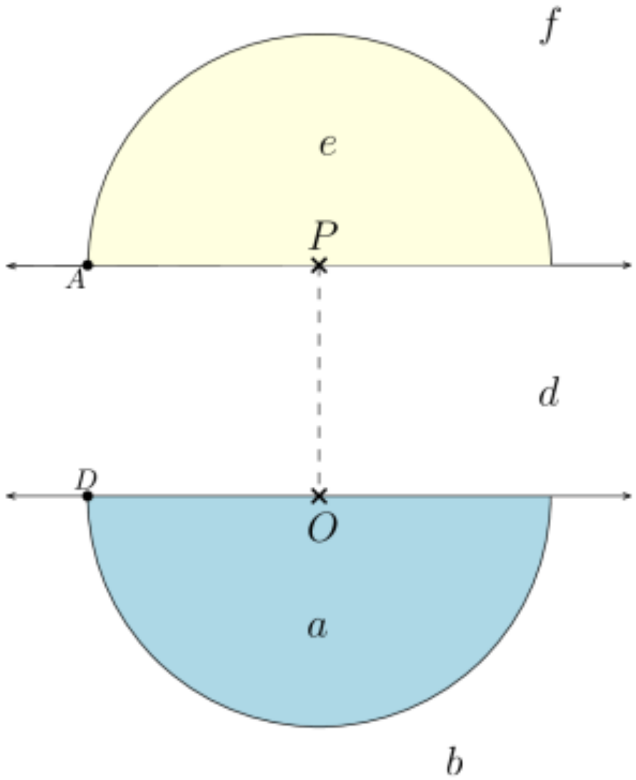
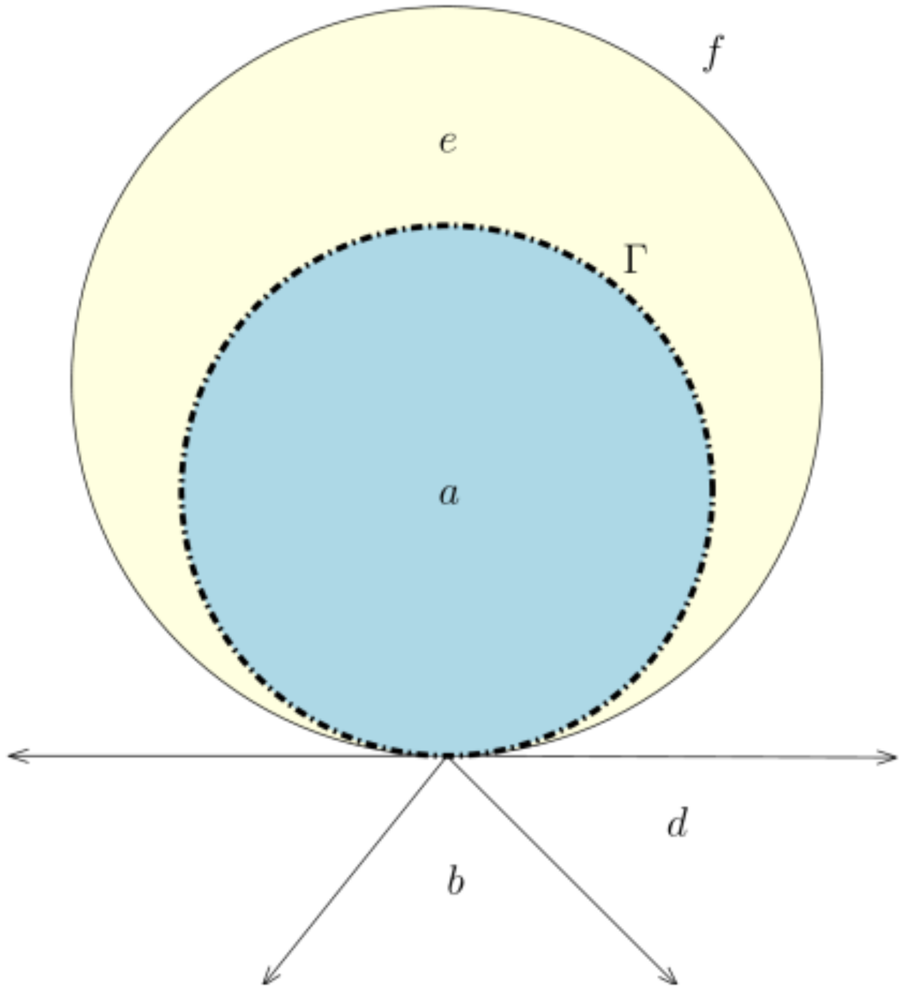
In the limit:

Before the deformation:



After the deformation:



$$J_{u^{-1}} = \Gamma \Rightarrow u^{-1} \in SBV(\tilde{\Omega}_b, \mathbb{R}^3)$$

It is not $W^{1,1}$.

- * Interpenetration of matter
- * Orientation reversal

The divergence identities express a higher regularity of u.

Suppose that u coincides on $\partial\Omega$ with an orientation-preserving diffeomorphism $b : \tilde{\Omega} \to \mathbb{R}^3$, where $\Omega \subset\subset \tilde{\Omega}$.

Suppose, further, that when u is extended to $\tilde{\Omega}$ by $u \equiv b$ in $\tilde{\Omega} \setminus \Omega$, the extension is still one-to-one a.e., $\det Du > 0$ a.e. and satisfies the divergence identities in $\tilde{\Omega}$. Then, applying [H. & Mora-Corral, JFA '15, Thm. 3.4]

- $\operatorname{im}_G(u, \tilde{\Omega}) \overset{a.e.}{=} b(\tilde{\Omega}) =: \tilde{\Omega}_b$
- $u^{-1} \in W^{1,1}(\tilde{\Omega}_b, \mathbb{R}^3)$.

$$\int_{\tilde{\Omega}} g(u(x)) \cdot \operatorname{cof} Du(x) D\phi = \int_{\tilde{\Omega}_b} g(y) \cdot (\nabla u^{-1}(y))^T D\phi(u^{-1}(y))$$

$$\int_{\tilde{\Omega}} (\operatorname{div} g)(u(x)) \det Du(x)\, \phi(x)\, dx = \int_{\tilde{\Omega}_b} \operatorname{div} g(y)\, \phi(u^{-1}(y))\, dy$$

Open problem: to prove that the weak H^1 limit of a minimizing sequence of regular axisymmetric maps has a Sobolev inverse.

Malý '93: Suppose $W(F)$ is polyconvex on $\mathbb{R}^{3\times 3}$,

$u_j \rightharpoonup u$ in H^1

u, u_j are orientation-preserving **diffeomorphisms**

Then $\int W(Du) \le \liminf \int W(Du_j)$.

Def.

$\mathcal{A}_s = \{ u \in H^1 :$ injective a.e., axisymmetric, $\det Du > 0$ a.e., $u = b$ in $\tilde{\Omega} \setminus \Omega$, $E(u) \le E(b) \}$

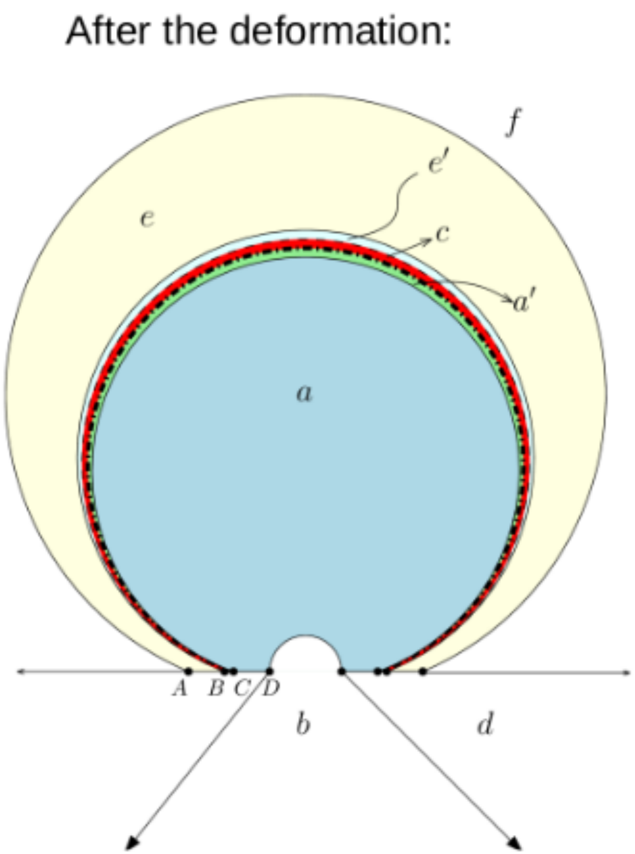
$\mathcal{A}_s^r = \{ u \in \mathcal{A} : \forall g \ \mathrm{Div}((\mathrm{adj}\, Du) g) = (\mathrm{div}\, g) \det Du \}$

$E_{rel}(u) := \inf \{ \liminf E(u_j) : u_j \rightharpoonup u$ in H^1, $u_j \in \mathcal{A}_s^r \ \forall j \}$

Thm.: Define $\mathcal{B} := \{ u \in \mathcal{A}_s : \tilde{\Omega}_b = \mathrm{im}_G(u, \tilde{\Omega})$ and $u^{-1} \in W^{1,1}(\tilde{\Omega}_b, \mathbb{R}^2) \times BV(\tilde{\Omega}_b) \}$.

Then $\int_\Omega |Du|^2 + H(\det Du)\,dx + 2\|D^s u_3^{-1}\|_{\mathcal{M}(\tilde{\Omega}_b, \mathbb{R}^3)}$

attains its infimum in $\mathcal{B}$.



$$\int_{x_3=0}^{1} \int_{x_1^2+x_2^2<\varepsilon^2} |\partial_1 u|^2 + |\partial_2 u|^2 + |\partial_3 u|^2$$

$$\geq \int_0^1 \int 2|\partial_1 u||\partial_2 u|$$

$$\geq \int_0^1 \int 2|\partial_1 u \wedge \partial_2 u|$$

$$= 2\mathcal{H}^2(\Gamma)$$

$$= 2\|D^s u_3^{-1}\|_{\mathcal{M}}.$$

Theorem 1.2. *Let* $\boldsymbol{u}$ *be the* $H^1\big(B(\mathbf{0},3),B(\mathbf{0},3)\big)$ *axisymmetric map of Conti–De Lellis defined in Section 3.1.*

Let $H:(0,+\infty)\to[0,+\infty)$ *be any function satisfying the following hypotheses:*

- H *is convex (hence continuous) and satisfies* (1.1).
-

$$\int_{B(\mathbf{0},3)} H(\det D\boldsymbol{u})\mathrm{d}\boldsymbol{x} < \infty. \tag{1.8}$$

Then there exist a sequence of axisymmetric maps $(\boldsymbol{u}_n)_n \subset H^1(B(\mathbf{0},3),B(\mathbf{0},3))$ *such that:*

i) $\boldsymbol{u}_n$ *is bi-Lipschitz for every* $n\in\mathbb{N}$,
ii) $\mathcal{E}(\boldsymbol{u}_n)=0$ *and* $\boldsymbol{u}_n$ *satisfies INV for every* $n\in\mathbb{N}$,
iii) $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ *in* $H^1(B(\mathbf{0},3),B(\mathbf{0},3))$
iv) $\operatorname{im}_{\mathrm{T}}(\boldsymbol{u},L)=\partial B((0,0,\frac{1}{2}),\frac{1}{2})$, $\operatorname{Det} D\boldsymbol{u}=\frac{\pi}{6}(\delta_{(0,0,1)}-\delta_{(0,0,0)})$, $\mathcal{E}(\boldsymbol{u})>0$, $\boldsymbol{u}$ *does not satisfy condition INV,*
v) $\lim_{n\to\infty}\int_{B(\mathbf{0},3)}|D\boldsymbol{u}_n|^2\mathrm{d}\boldsymbol{x}=\int_{B(\mathbf{0},3)}|D\boldsymbol{u}|^2\mathrm{d}\boldsymbol{x}+2\pi$,
vi) $\lim_{n\to\infty}\int_{B(\mathbf{0},3)}H(\det D\boldsymbol{u}_n)=\int_{B(\mathbf{0},3)}H(\det D\boldsymbol{u})$,
vii) $\boldsymbol{u}$ *is one-to-one a.e.,* $\boldsymbol{u}^{-1}\in SBV(B(\mathbf{0},3),B(\mathbf{0},3))$ *and*

$$\|D^s\boldsymbol{u}^{-1}\|_{\mathcal{M}(B(\mathbf{0},3),\mathbb{R}^3))}=\|D^s u_3^{-1}\|_{\mathcal{M}(B(\mathbf{0},3))}=\pi.$$

In particular, the last three items show that

$$E_{rel}(\boldsymbol{u})=E(\boldsymbol{u})+2\|D^s(\boldsymbol{u}^{-1})_3\|_{\mathcal{M}(B(\mathbf{0},3),\mathbb{R}^3)}=F(\boldsymbol{u}). \tag{1.9}$$

Quantification of failure of divergence identities

Divergence identities:

$$\int_\Omega (\det D\mathbf{u}(\mathbf{x})\,\phi(\mathbf{x})\,\operatorname{div}\mathbf{g}(\mathbf{u}(\mathbf{x})) + [\operatorname{adj} D\mathbf{u}(\mathbf{x})\,\mathbf{g}(\mathbf{u}(\mathbf{x}))]\cdot D\phi(\mathbf{x}))\,\mathrm{d}\mathbf{x} = 0$$

for $\mathbf{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\phi \in C_c^\infty(\Omega)$.

Taking functions of separate variables: $\mathbf{f} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$,

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x})\,\mathbf{g}(\mathbf{y}),$$

the divergence identites read as

$$\int_\Omega [\operatorname{cof} \nabla\mathbf{u}(\mathbf{x}) \cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det \nabla\mathbf{u}(\mathbf{x})\,\operatorname{div}_{\mathbf{y}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x} = 0.$$

Definition of surface energy:

$$\mathcal{E}(\mathbf{u}) :=$$

$$\sup_{\|\mathbf{f}\|_\infty \le 1} \int_\Omega [\operatorname{cof} \nabla\mathbf{u}(\mathbf{x}) \cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det \nabla\mathbf{u}(\mathbf{x})\,\operatorname{div}_{\mathbf{y}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x}.$$

Geometric interpretation of the surface energy

Theorem. (Henao, M.-C. 11)
$\mathcal{E}(\mathbf{u}) = \mathcal{H}^{n-1}(\Gamma_V(\mathbf{u})) + 2\mathcal{H}^{n-1}(\Gamma_I(\mathbf{u}))$.

Theorem: (Henao, M.-C. 12)
Let $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfy INV, $\det D\mathbf{u} > 0$, $\mathcal{E}(\mathbf{u}) < \infty$. Then

$$\mathcal{E}(\mathbf{u}) = \sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}).$$

and

$$\Gamma_I(\mathbf{u}) \overset{\mathcal{H}^{n-1}}{=} \varnothing, \qquad \Gamma_V(\mathbf{u}) \overset{\mathcal{H}^{n-1}}{=} \bigcup_{\mathbf{a} \in C(\mathbf{u})} \partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}).$$

Cavitation is the only process of creation of surface.

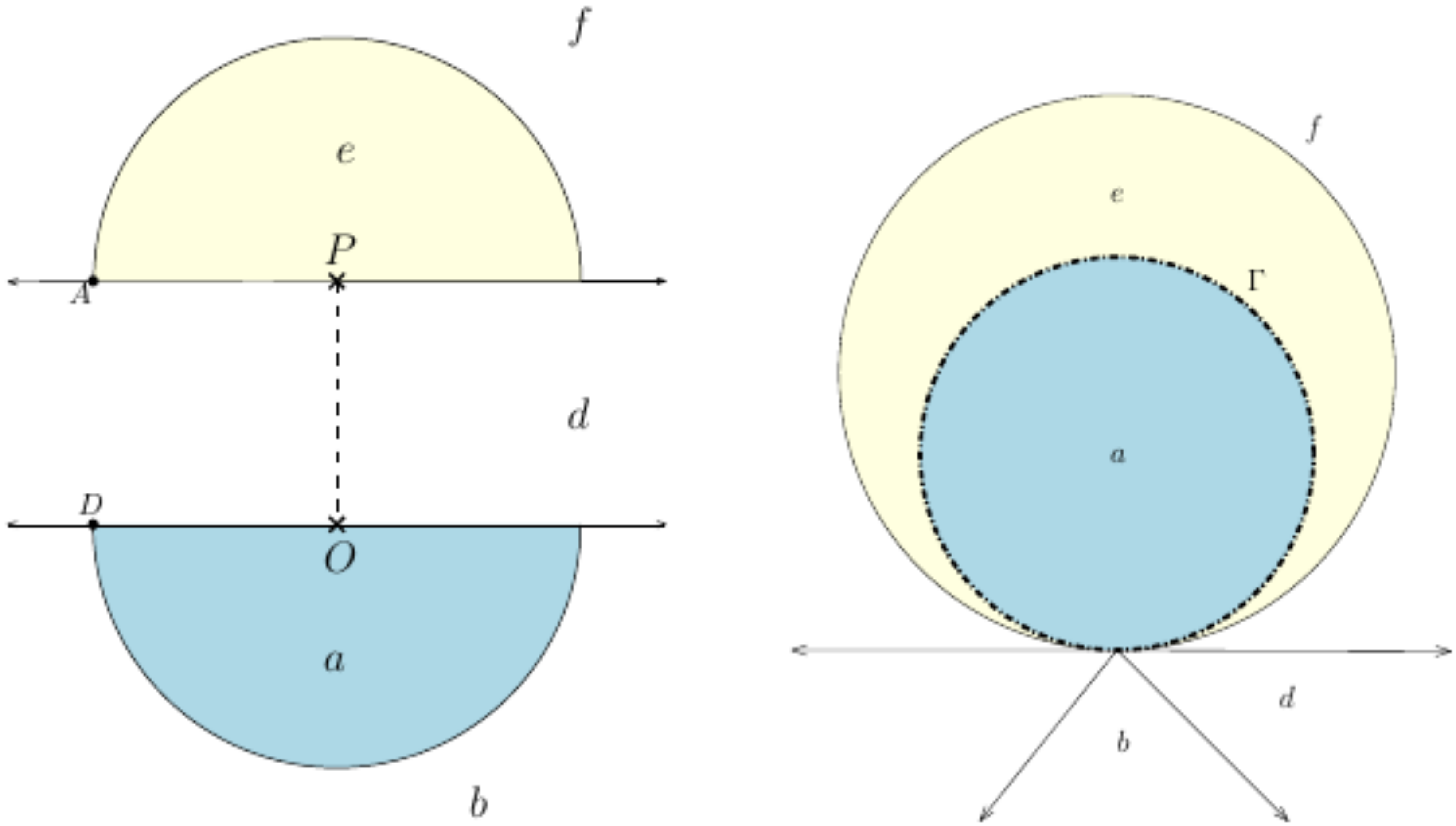
Theorem. (Barchiesi, Henao, M.C., Rodiac 23) If $\mathbf{u} \in \bar{\mathcal{A}}_s^r$ and $\mathcal{E}(\mathbf{u}) < \infty$ then

- $\mathbf{u}^{-1} \in SBV$
- $\operatorname{Det} D\mathbf{u} = (\det D\mathbf{u})\mathcal{L}^3 + \sum_i v_i \, \delta_{\boldsymbol{\xi}_i}$
- For $\mathcal{H}^2$-a.e. $\mathbf{y} \in J_{\mathbf{u}^{-1}}$ both $(\mathbf{u}^{-1})^{\pm}(\mathbf{y})$ are among the $\{\boldsymbol{\xi}_i\}_{i\in\mathbb{N}}$
- $\Gamma_V(\mathbf{y}) \overset{\mathcal{H}^2}{=} \varnothing$ and $\Gamma_I(\mathbf{y}) \overset{\mathcal{H}^2}{=} J_{\mathbf{u}^{-1}}$
- $\Delta_{\mathbf{x}} := \lim_{r\to 0} \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) - \chi_{\mathbf{u}(B(\mathbf{x},r))}$. Non-zero iff $\mathbf{x} = \boldsymbol{\xi}_i$
- $\sum_i \Delta_i = 0$

We illustrate each conclusion in Conti & De Lellis example.

Reference configuration

Deformed configuration

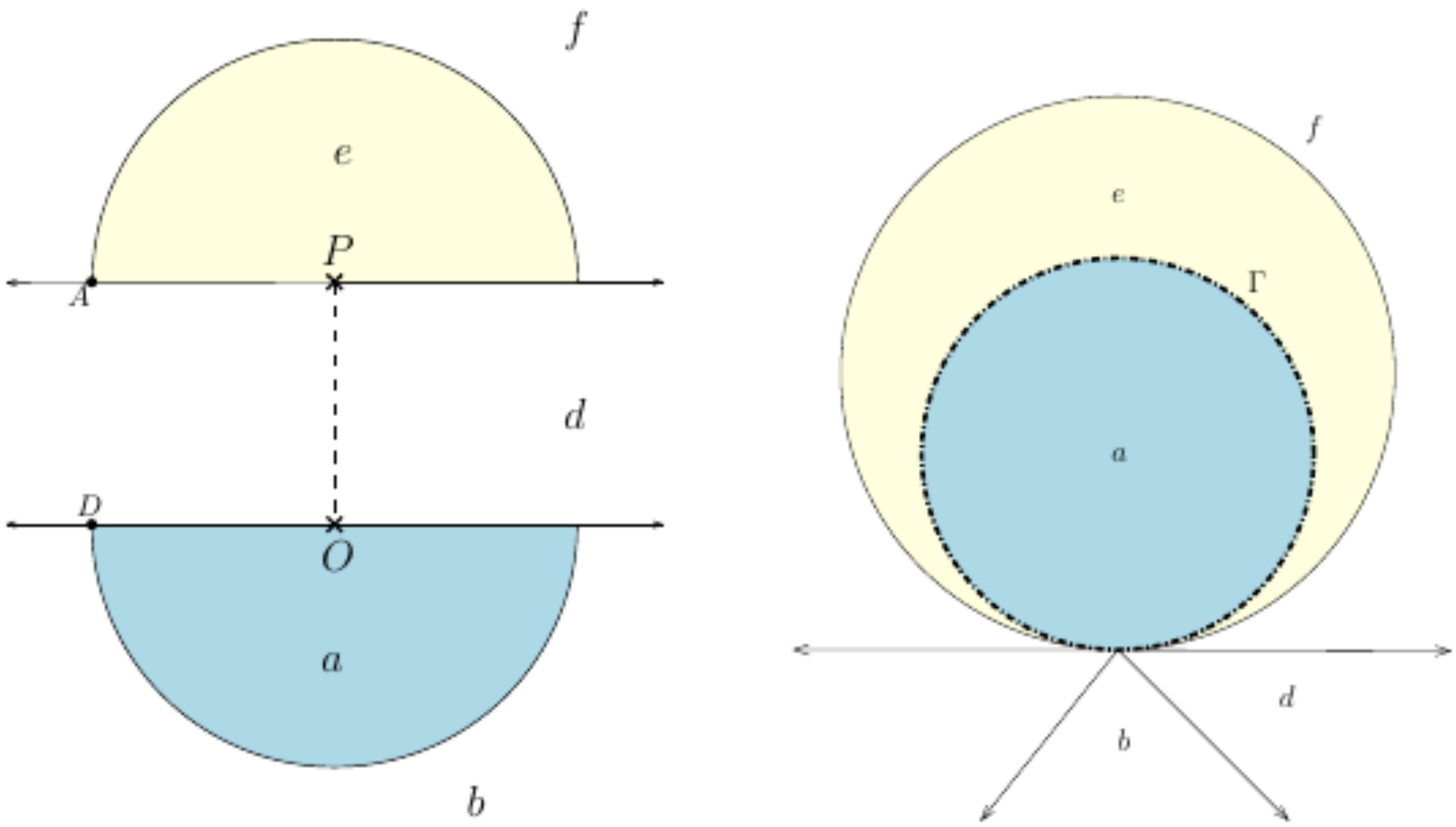


- $\mathbf{u}^{-1} \in SBV$

The only singularities of $\mathbf{u}^{-1}$ are jumps.

- Det $D\mathbf{u} = (\det D\mathbf{u})\mathcal{L}^3 + \sum_i v_i\,\delta_{\boldsymbol{\xi}_i}$

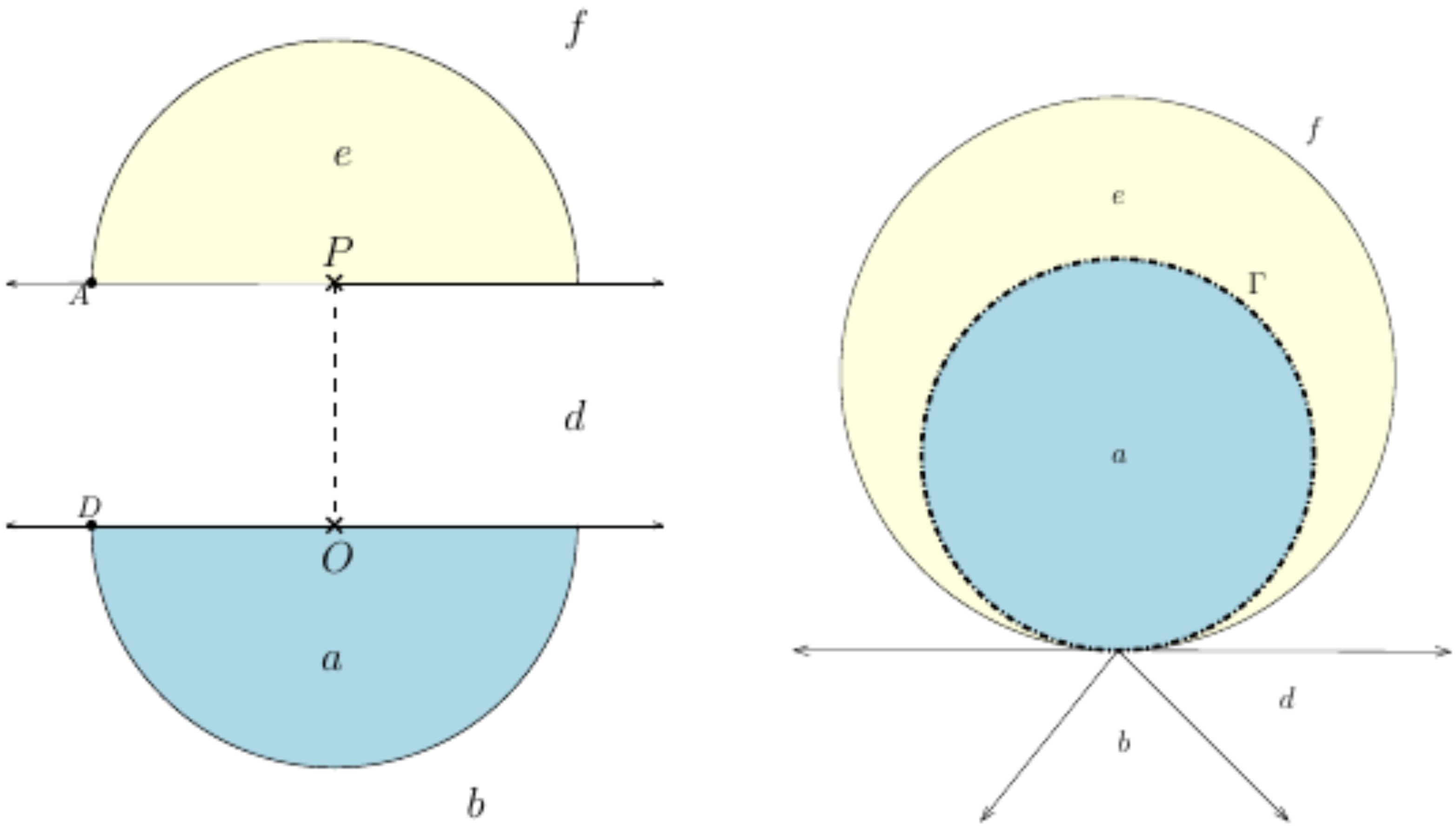
$|v_i|$ = volume of cavity. If $v_i < 0$, orientation reversal



In the example, Det $D\mathbf{u} = \det D\mathbf{u} + \frac{\pi}{6}\delta_P - \frac{\pi}{6}\delta_Q$

- For $\mathcal{H}^2$-a.e. $\mathbf{y} \in J_{\mathbf{u}^{-1}}$ both $(\mathbf{u}^{-1})^{\pm}(\mathbf{y})$ are among the $\{\boldsymbol{\xi}_i\}_{i\in\mathbb{N}}$

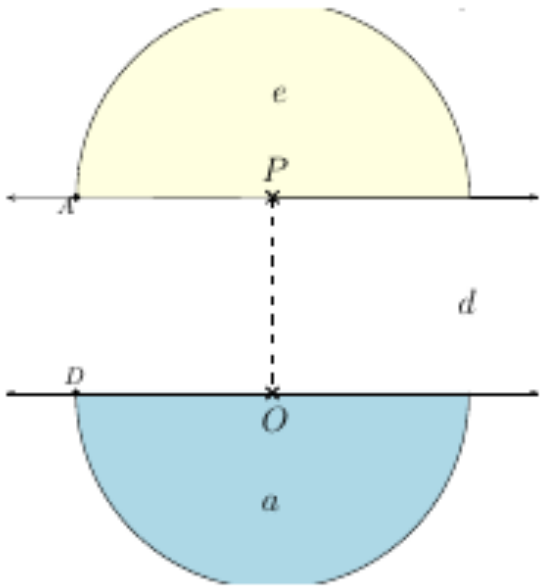
Jumps of the inverse are only produced by cavities.

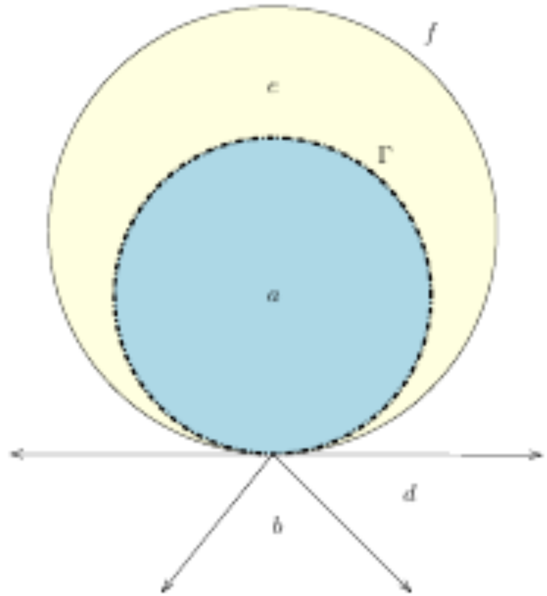


In the example,

- For $\mathbf{y} \in \Gamma$, $(\mathbf{u}^{-1})^{+}(\mathbf{y}) = P$ and $(\mathbf{u}^{-1})^{-}(\mathbf{y}) = O$

- $\Gamma_V(\mathbf{y}) \overset{\mathcal{H}^2}{=} \varnothing$ and $\Gamma_I(\mathbf{y}) \overset{\mathcal{H}^2}{=} J_{\mathbf{u}^{-1}}$
- $\Gamma_i^{\pm} := \{\mathbf{y} \in J_{\mathbf{u}^{-1}} : (\mathbf{u}^{-1})^{\pm} = \boldsymbol{\xi}_i\}$ and $\Gamma_{ij} := \Gamma_i^- \cap \Gamma_j^+$. Then $\|D^s\mathbf{u}^{-1}\|_{\mathcal{M}} = \sum_{ij} |\xi_i - \xi_j| \, \mathcal{H}^2(\Gamma_{ij})$





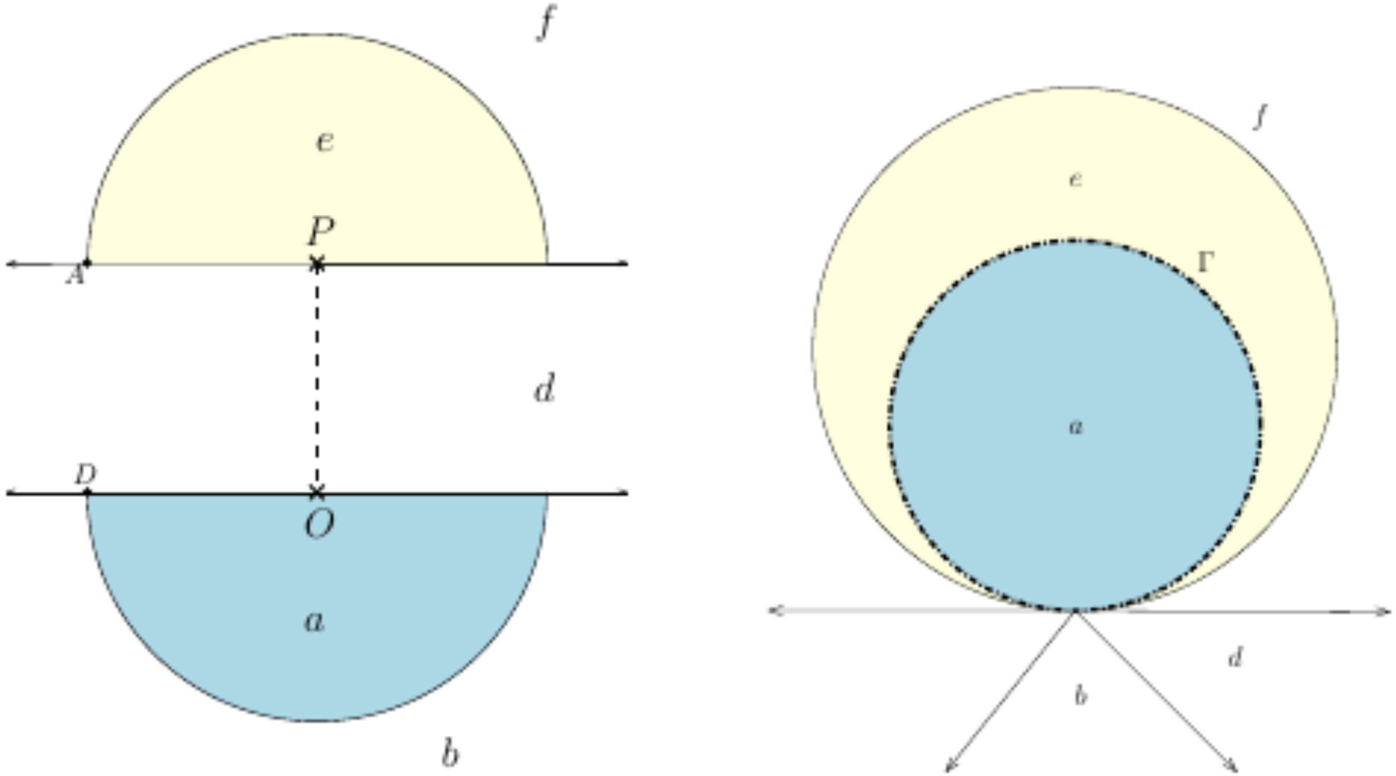
In the example,

- $\Gamma_V(\mathbf{u}) = \varnothing$ and $\Gamma_I(\mathbf{u}) = \Gamma$
- $\Gamma_P^+ = \Gamma, \;\; \Gamma_P^- = \varnothing, \;\; \Gamma_O^+ = \varnothing, \;\; \Gamma_O^- = \Gamma, \;\; \Gamma_{PO} = \varnothing, \;\; \Gamma_{OP} = \Gamma,$ $\|D^s\mathbf{u}^{-1}\|_{\mathcal{M}} = |P - O| \, \mathcal{H}^2(\Gamma)$

- $\Delta_{\mathbf{x}} := \lim_{r\to 0} \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) - \chi_{\mathbf{u}(B(\mathbf{x},r))}$. Non-zero iff $\mathbf{x} = \boldsymbol{\xi}_i$
- $\sum_i \Delta_i = 0$

$\Delta_{\mathbf{x}}$ indicates if $\mathbf{x}$ is cavity point, volume enclosed and orientation.

Cancellation: cavity formed at one point is filled from another.



In the example,

$$\Delta_P = \chi_a, \qquad \Delta_O = -\chi_a, \qquad \Delta_P + \Delta_O = 0.$$

Construction similar to bubbling of harmonic maps (Giaquinta, Modica, Souček 98, Lin 99, Lin, Rivière 02).

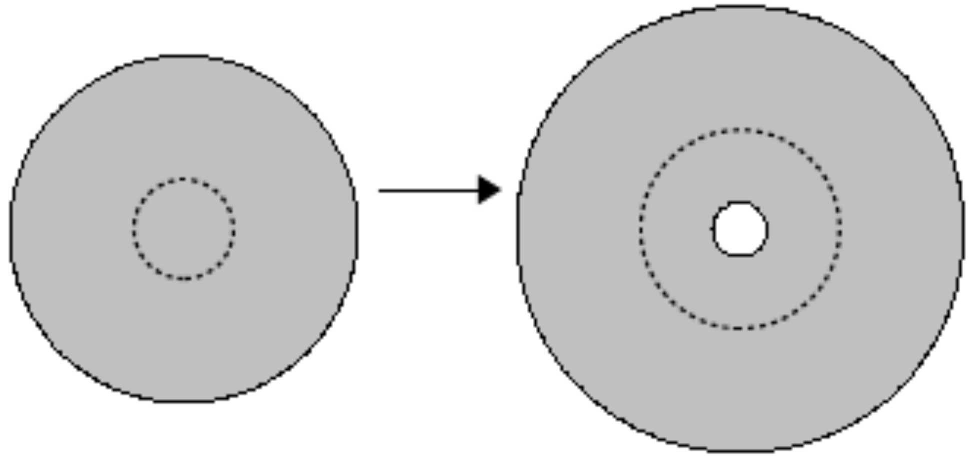
Dipole phenomenon (Brezis, Coron 86).

Example by Conti & De Lellis is generic: point singularities, interpenetration of matter, orientation-reversal, cavities are filled, bubbling, dipoles.

How to define cavities using degree

$\text{im}_T(\mathbf{u}, B) := \{\mathbf{y} \in \mathbb{R}^n : \deg(\mathbf{u}, B, \mathbf{y}) \neq 0\}.$

(Šverák 88, Müller, Spector 95)



For $\mathbf{x}_0 \in \Omega$,

$$\text{im}_T(\mathbf{u}, \mathbf{x}_0) := \bigcap_{r>0} \text{im}_T(\mathbf{u}, B(\mathbf{x}_0, r)).$$

A *cavity point* is an $\mathbf{x}_0$ such that $\mathcal{L}^n(\text{im}_T(\mathbf{u}, \mathbf{x}_0)) > 0$.

We call $C(\mathbf{u})$ the set of all cavity points.

Condition INV

(Müller, Spector 95)

Let $p > n - 1$ and $\mathbf{x}_0 \in \Omega$. By co-area formula,

$$\mathbf{u}|_{\partial B(\mathbf{x}_0, r)} \in W^{1,p}(\partial B(\mathbf{x}_0, r), \mathbb{R}^n) \qquad \text{for a.e. } r > 0.$$

Hence $\mathbf{u}|_{\partial B(\mathbf{x}_0, r)}$ is continuous and has a degree

$$\deg(\mathbf{u}, B(\mathbf{x}_0, r), \mathbf{y}) \qquad \text{for } \mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{x}_0, r)).$$

We say that $\mathbf{u}$ satisfies INV if

$$\deg(\mathbf{u}, B(\mathbf{x}_0, r), \mathbf{u}(\mathbf{x})) \neq 0 \quad \text{a.e. } \mathbf{x} \in B(\mathbf{x}_0, r),$$
$$\deg(\mathbf{u}, B(\mathbf{x}_0, r), \mathbf{u}(\mathbf{x})) = 0 \quad \text{a.e. } \mathbf{x} \in \Omega \setminus B(\mathbf{x}_0, r).$$

Roughly, *almost every sphere is impenetrable.*

Recent related results:

Doležalová, Hencl, Malý 21: $n = 3$. If $\mathbf{u}_j$ homeomorphisms with bounded energy

$$\int_\Omega |D\mathbf{u}_j|^2 + \frac{1}{(\det D\mathbf{u}_j)^2}\,\mathrm{d}\mathbf{x},$$

then $\mathbf{u}_j \rightharpoonup \mathbf{u}$ with $\mathbf{u}$ satisfying INV.

Doležalová, Hencl, Molchanova 22: If $\mathbf{u}_j$ homeomorphisms satisfying Lusin's N, with $\det D\mathbf{u}_j > 0$ and bounded energy

$$\int_\Omega |D\mathbf{u}_j|^{n-1} + A\left(|\mathrm{cof}\, D\mathbf{u}_j|\right) + \varphi\left(\det D\mathbf{u}_j\right)\mathrm{d}\mathbf{x},$$

with

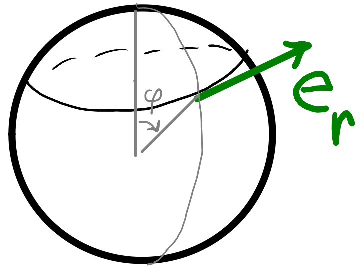
$$\lim_{t\to 0^+} \varphi(t) = 0, \qquad \lim_{t\to\infty} \frac{A(t)}{t} = \infty$$

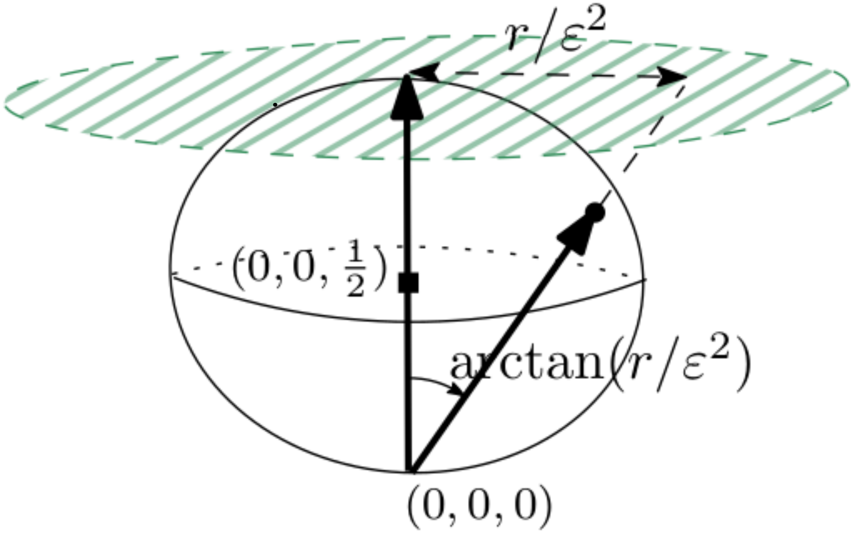
then $\mathbf{u}_j \rightharpoonup \mathbf{u}$ with $\mathbf{u}$ satisfying INV and Lusin's N.

$$C_\varepsilon := \{x_1^2+x_2^2<\varepsilon,\ 0<x_3<1\}$$

$$\underset{\sim}{x}(r,\theta,x_3) = r\,\underset{\sim}{e_r} + x_3\,\underset{\sim}{e_3},$$

$$0<r<\varepsilon,\quad 0\le\theta\le 2\pi,\quad 0\le x_3\le 1$$



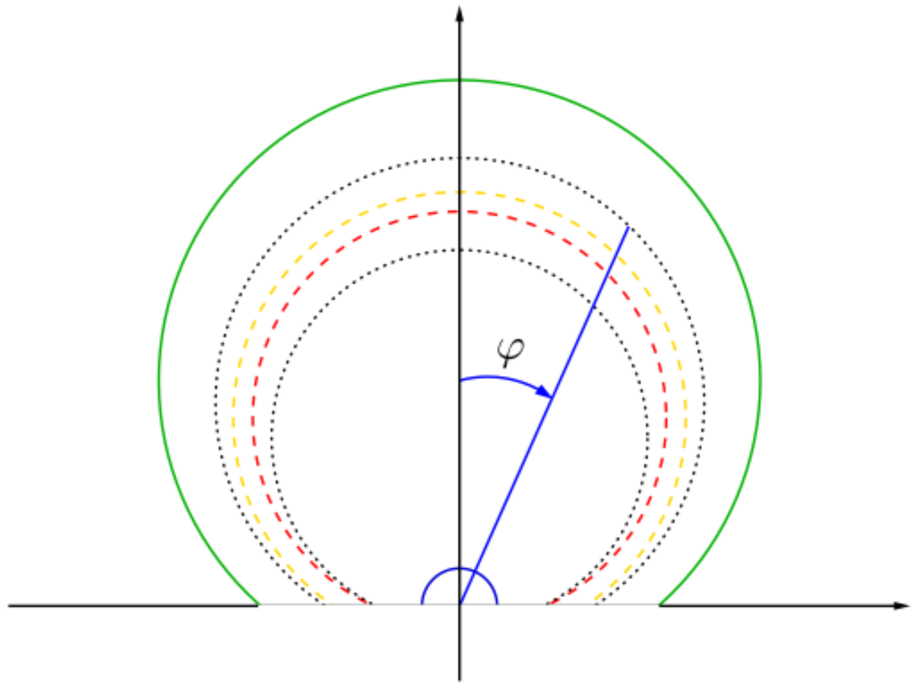


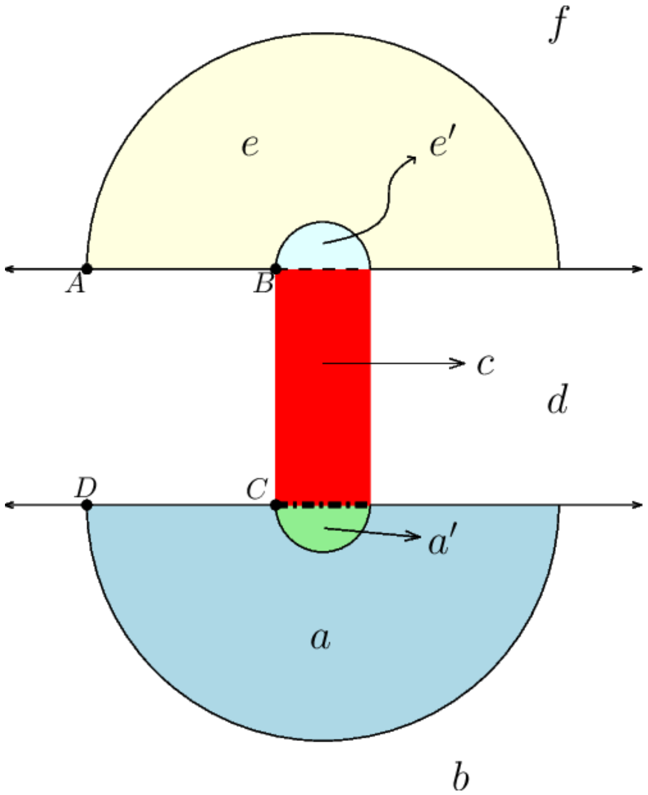
$$u_\varphi^\varepsilon = f_\varepsilon(r) := \arctan\left(\frac{r}{\varepsilon^2}\right) + \alpha_\varepsilon \frac{r}{\varepsilon},$$

$$\alpha_\varepsilon = \arctan(\varepsilon)$$

$f_\varepsilon(0)=0$, $f_\varepsilon(\varepsilon)=\frac{\pi}{2}$, and

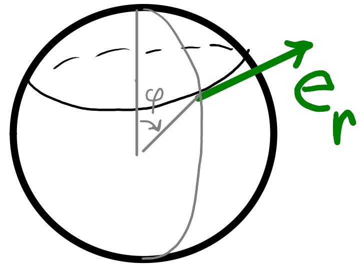
$f_\varepsilon'(r)>0$ for all r





$$C_\varepsilon := \{\, x_1^2 + x_2^2 < \varepsilon \,,\ 0 < x_3 < 1 \,\}$$

$$\underset{\sim}{x}(r,\theta,x_3) = r\,\underset{\sim}{e_r} + x_3\,\underset{\sim}{e_3} \,, \quad \begin{array}{l} 0 \le r < \varepsilon \\ 0 \le \theta \le 2\pi \\ 0 < x_3 < 1 \end{array}$$



$$\underset{\sim}{u_\varepsilon} = u_\rho^\varepsilon \sin u_\varphi^\varepsilon \,\underset{\sim}{e_r} + u_\rho^\varepsilon \cos u_\varphi^\varepsilon \,\underset{\sim}{e_3}$$

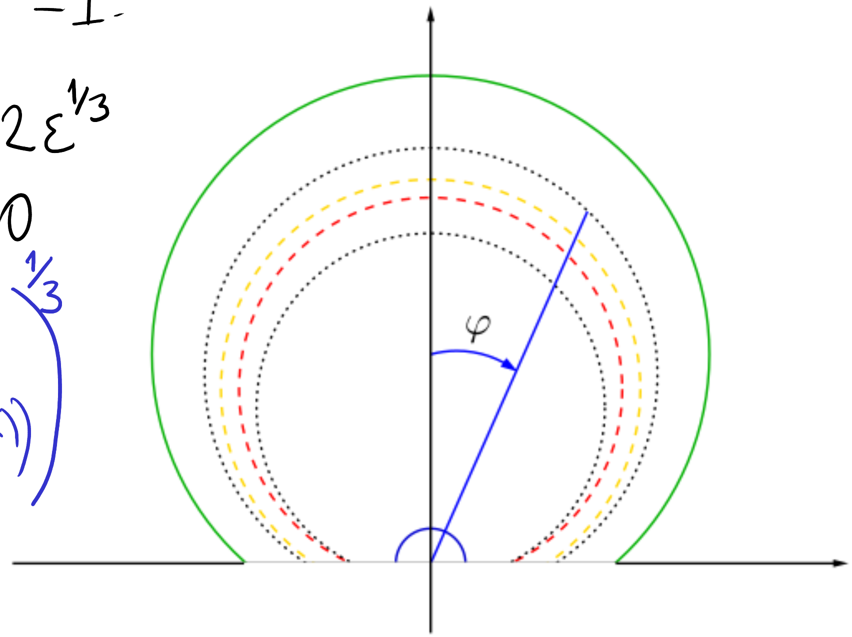
$$u_\varphi^\varepsilon = f_\varepsilon(r) := \arctan\left(\frac{r}{\varepsilon^2}\right) + \alpha_\varepsilon \frac{r}{\varepsilon} \,,$$

$$\alpha_\varepsilon = \arctan(\varepsilon)$$

$$\det D\underset{\sim}{u_\varepsilon} = \frac{1}{3}\,\frac{\sin f_\varepsilon'(r)\, f_3'(r)}{r}\,\partial_{x_3}\left(\left(u_\rho^\varepsilon\right)^3\right) \equiv 1.$$

with "initial condition" $u_\rho^\varepsilon = \cos(f_\varepsilon(r)) + 2\varepsilon^{1/3}$ at $x_3 = 0$

$$\Rightarrow u_\rho = \left(\left(\cos f_\varepsilon(r) + 2\varepsilon^{1/3}\right)^3 + x_3 \cdot \frac{3r}{\partial_r(-\cos f_\varepsilon(r))} \right)^{1/3}$$



arXiv: 2111.07112. Proposition 3.1.

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\left(|Du_\varepsilon|^2 + H(\det Du_\varepsilon)\right)dx = 2\pi$$

Proof.: $\displaystyle\int_{C_\varepsilon} H(\det Du_\varepsilon)\,dx = H(1)\cdot \mathrm{vol}(C_\varepsilon) \xrightarrow{\varepsilon\to 0} 0$.

$$\int_{x_3=0}^{1}\int_{r=0}^{\varepsilon}\Big(\underbrace{r|\partial_r u_\rho^\varepsilon|^2 + r|u_\rho\partial_r u_\varphi^\varepsilon|^2}_{I} + \underbrace{\frac{1}{r}|u_\rho^\varepsilon \sin u_\varphi^\varepsilon|^2}_{II} + \underbrace{r|\partial_{x_3} u_\rho^\varepsilon|^2}_{III}\Big)$$

Idea:

$$u_\rho^\varepsilon \simeq \cos f_\varepsilon(r)$$

$$\partial_r u_\rho^\varepsilon \simeq \partial_r(\cos f_\varepsilon(r))$$

$$u_\rho \partial_r u_\varphi \simeq (\cos f_\varepsilon(r))\cdot f_\varepsilon'(r)$$

$$\int_0^\varepsilon r f_\varepsilon'(r)^2\,dr = \int_0^\varepsilon\left(\frac{\varepsilon^{-2}}{1+r^2/\varepsilon^4} + \frac{\alpha_\varepsilon}{\varepsilon}\right)^2 r\,dr = \frac{1}{2} + O(\varepsilon^2|\ln\varepsilon|)$$

$$\int_0^\varepsilon \frac{1}{r}\cos^2 f_\varepsilon(r)\sin^2 f_\varepsilon(r)\,dr = \int_0^\varepsilon \frac{\varepsilon^4 r\,dr}{(\varepsilon^4+r^2)^2} = \left.\frac{\varepsilon^4}{2(\varepsilon^4+r^2)}\right|_{r=\varepsilon}^{0} \xrightarrow{\varepsilon\to 0} \frac{1}{2}$$

arXiv: 2111.07112, Proposition 3.1.

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} \left(|Du_\varepsilon|^2 + H(\det Du_\varepsilon) \right) dx = 2\pi$$

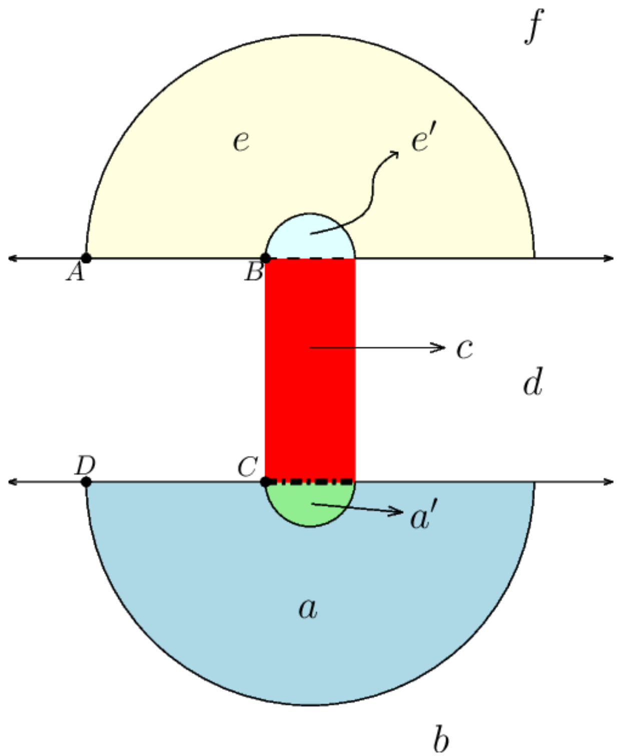
Proof:

$$\int_{C_\varepsilon} H(\det Du_\varepsilon)\, dx = H(1) \cdot \mathrm{vol}(C_\varepsilon) \xrightarrow{\varepsilon \to 0} 0 .$$

$$\int_{x_3=0}^{1} \int_{r=0}^{\varepsilon} \Big(\underbrace{r|\partial_r u_\rho^\varepsilon|^2 + r|u_\rho \partial_r u_\varphi^\varepsilon|^2}_{I} + \underbrace{\frac{1}{r}|u_\rho^\varepsilon \sin u_\varphi^\varepsilon|^2}_{II} + \underbrace{r|\partial_{x_3} u_\rho^\varepsilon|^2}_{III} \Big)$$

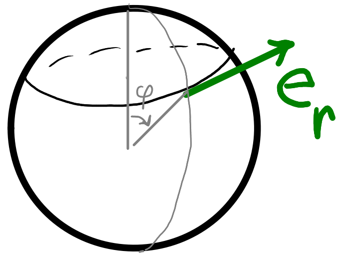
$$III: \quad \partial_{x_3} u_\rho = \frac{1}{3} u_\rho^{-2} \partial_{x_3}(u_\rho^3)$$

$$= \frac{\dfrac{r}{\partial_r(-\cos f_\varepsilon(r))}}{(u_\rho^\varepsilon)^2} \leq \frac{\dfrac{2\varepsilon^{-2}}{(\varepsilon^4 + r^2)^{3/2}}}{(2\varepsilon^{1/3})^2}$$



$a_\varepsilon^1 := \{x_1^2+x_2^2+x_3^2<\varepsilon^2,\ x_3<0\}$

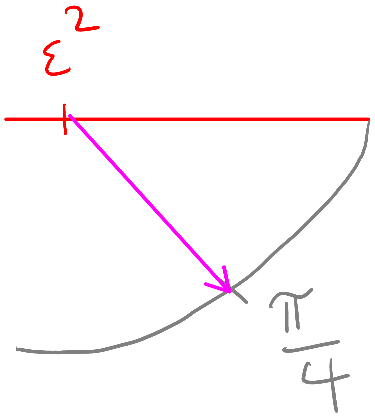
Call $g_\varepsilon:[0,\frac{\pi}{2}]\to[0,\varepsilon]$ to the inverse of the function $f_\varepsilon(r)$.



Parametrize a_ε^1 by

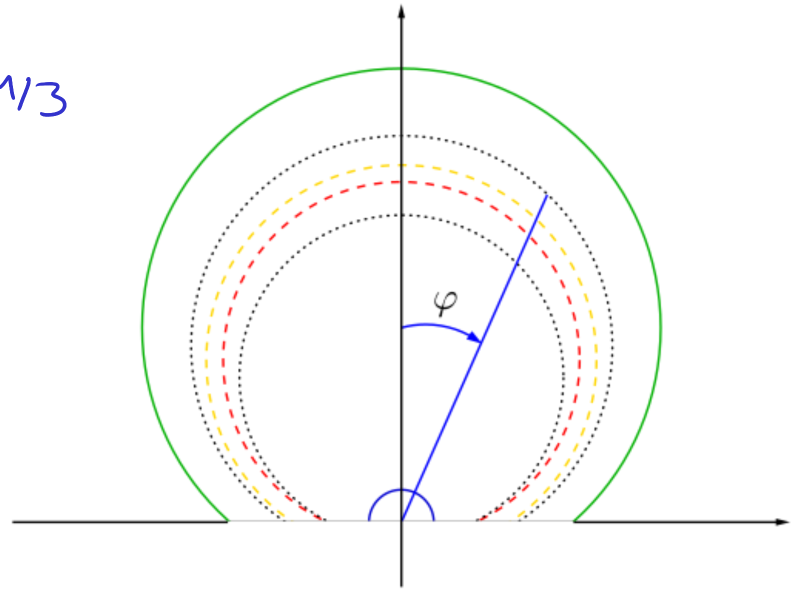
$$\underset{\sim}{x}(s,\theta,\varphi) = (1-s)\,g(\varphi)\,\underset{\sim}{e_r} + s\left(\varepsilon\sin\varphi\,\underset{\sim}{e_r} - \varepsilon\cos\varphi\,\underset{\sim}{e_3}\right)$$

$$0\le s\le 1,\quad 0\le\theta\le 2\pi,\quad 0\le\varphi<\pi/2$$



$$u_\varphi^\varepsilon(s,\varphi):=\varphi$$

$$u_\rho^\varepsilon(s,\varphi) := \left((\cos\varphi+2\varepsilon^\gamma)^3 - 3\int_{\sigma=0}^{s} h_\varepsilon(\sigma,\varphi)\,d\sigma\right)^{1/3}$$



$$h_\varepsilon(\sigma,\varphi)=\varepsilon\left((1-s)\frac{g_\varepsilon(\varphi)}{\sin\varphi}+s\varepsilon\right)\left((1-s)\,g_\varepsilon'(\varphi)\cos\varphi + s\left(\varepsilon-g_\varepsilon(\varphi)\sin\varphi\right)\right)$$

$$\nabla s = \frac{s \sin\varphi\, \boldsymbol{e}_r - \big((1-s)\varepsilon^{-1} g'(\varphi) + s\cos\varphi\big)\boldsymbol{e}_3}{(1-s)g'(\varphi)\cos\varphi + s(\varepsilon - g(\varphi)\sin\varphi)},$$

$$\nabla\theta = \frac{1}{(1-s)g(\varphi) + s\varepsilon\sin\varphi}\boldsymbol{e}_\theta,$$

$$\nabla\varphi = \frac{\cos\varphi\,\boldsymbol{e}_r + (\sin\varphi - \varepsilon^{-1} g(\varphi))\boldsymbol{e}_3}{(1-s)g'(\varphi)\cos\varphi + s(\varepsilon - g(\varphi)\sin\varphi)}.$$

$$D\boldsymbol{u}_\varepsilon = \partial_s \boldsymbol{u}_\varepsilon \otimes \nabla s + \partial_\theta \boldsymbol{u}_\varepsilon \otimes \nabla\theta + \partial_\varphi \boldsymbol{u}_\varepsilon \otimes \nabla\varphi, \qquad |D\boldsymbol{u}_\varepsilon|^2 = \operatorname{tr} D\boldsymbol{u}_\varepsilon^T D\boldsymbol{u}_\varepsilon,$$

$$|D\boldsymbol{u}_\varepsilon|^2 \leq 2|\nabla s|^2 |\partial_s u_\rho^\varepsilon|^2 + |\nabla\theta|^2 |u_\rho^\varepsilon|^2 \sin^2\varphi + |\nabla\varphi|^2 (|u_\rho^\varepsilon|^2 + 2|\partial_\varphi u_\rho^\varepsilon|^2).$$

$$\int_{a'_\varepsilon} |D\boldsymbol{u}_\varepsilon|^2 \mathrm{d}\boldsymbol{x} = 2\pi \int_{s=0}^{1} \int_{\varphi=0}^{\frac{\pi}{2}} |D\boldsymbol{u}_\varepsilon|^2 h_\varepsilon(s,\varphi) \sin\varphi \mathrm{d}\varphi \mathrm{d}s.$$

Lemma A.2. *For every $\varphi \in [0, \frac{\pi}{2})$ and $s \in [0,1]$,*

$$g_\varepsilon(\varphi) < \varepsilon, \qquad \varepsilon - g_\varepsilon(\varphi)\sin\varphi > 0 \quad \textit{and} \quad h_\varepsilon(s,\varphi) > 0.$$

Lemma A.5. *For all* $\varphi \in (0, \frac{\pi}{2}]$, $s \in [0,1]$, *and* $\varepsilon \leq \frac{1}{\pi}$,

$$(1-s)\frac{g_\varepsilon(\varphi)}{\sin\varphi} + s\varepsilon \leq \sqrt{2}\varepsilon, \qquad 0 < (1-s)g'_\varepsilon(\varphi)\cos\varphi + s(\varepsilon - g_\varepsilon(\varphi)\sin\varphi) \leq \frac{3\pi}{2}\cos\varphi,$$

$$|h_\varepsilon(s,\varphi)| \leq \frac{3\pi\sqrt{2}}{2}\,\varepsilon^2\cos\varphi \quad \textit{and} \quad |\partial_\varphi h_\varepsilon(s,\varphi)| = O(\varepsilon).$$

Lemma 3.7. *For all* $\varphi \in [0, \frac{\pi}{2}]$, *all* $s \in [0,1]$, *and all positive* ε *such that* $\varepsilon^{2-2\gamma} < \frac{7}{9\pi\sqrt{2}}$,

$$\frac{1}{4}(\cos\varphi + 2\varepsilon^\gamma) \leq u^\varepsilon_\rho(s,\varphi) \leq \cos\varphi + 2\varepsilon^\gamma \leq 2, \quad |\partial_s u^\varepsilon_\rho| \leq C\varepsilon^{2-2\gamma}\cos\varphi, \quad |\partial_\varphi u^\varepsilon_\rho| = O(1).$$

Proof:

$$3\partial_\varphi u^\varepsilon_\rho = (u^\varepsilon_\rho)^{-2}\partial_\varphi\Big((u^\varepsilon_\rho)^3\Big) = -3\underbrace{\left(\frac{\cos\varphi + 2\varepsilon^\gamma}{u^\varepsilon_\rho}\right)^2}_{\leq 4^2}\sin\varphi - 3(u^\varepsilon_\rho)^{-2}\underbrace{\int_{\sigma=0}^{s}\partial_\varphi h(\sigma,\varphi)\mathrm{d}\sigma}_{=O(\varepsilon)}.$$

Lemma 3.8.

$$\int_{a'_\varepsilon}|\nabla\varphi|^2\,\mathrm{d}\boldsymbol{x} = O(\varepsilon^2|\ln\varepsilon|), \qquad \int_{a'_\varepsilon}(\varepsilon\cos\varphi)^2|\nabla s|^2\,\mathrm{d}\boldsymbol{x} = O(\varepsilon|\ln\varepsilon|),$$

$$\textit{and} \quad \int_{a'_\varepsilon}|\nabla\theta|^2\sin^2\varphi\,\mathrm{d}\boldsymbol{x} = O(\varepsilon|\ln\varepsilon|^2).$$

Lower semicontinuity

Let V be an open set such that

$$\mathrm{im}_{\mathsf{T}}(\boldsymbol{u}, L) \subset V.$$

$$\|D^s \boldsymbol{u}^{-1}\|_{\mathcal{M}(\tilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^{3\times 3})} = \|D^s \boldsymbol{u}^{-1}\|_{\mathcal{M}(V,\mathbb{R}^{3\times 3})} = \|D^s (\boldsymbol{u}^{-1})_3\|_{\mathcal{M}(V,\mathbb{R}^{3\times 3})}$$

$$\leq \|D(\boldsymbol{u}^{-1})_3\|_{\mathcal{M}(V,\mathbb{R}^{3\times 3})} \leq \liminf_{k\to\infty} \|D(\boldsymbol{u}_k^{-1})_3\|_{\mathcal{M}(V,\mathbb{R}^{3\times 3})}.$$

$$\int_V |\nabla(\boldsymbol{u}_k^{-1})_3| \mathrm{d}\boldsymbol{y} = \int_{\boldsymbol{u}_k^{-1}(V)\cap C_{\delta_1}} |\operatorname{adj} \nabla \boldsymbol{u}_k\, \boldsymbol{e}_3| + \int_{\boldsymbol{u}_k^{-1}(V)\setminus C_{\delta_1}} |\operatorname{adj} \nabla \boldsymbol{u}_k\, \boldsymbol{e}_3| \mathrm{d}\boldsymbol{x},$$

$$\liminf_{k\to\infty} \|D(\boldsymbol{u}_k^{-1})_3\|_{\mathcal{M}(V,\mathbb{R}^{3\times 3})} \leq \varepsilon + \liminf_{k\to\infty} \left[\frac{1}{2} \int_{C_{\delta_1}\cap\tilde{\Omega}} |D\boldsymbol{u}_k|^2 \mathrm{d}\boldsymbol{x} + \|D^s \boldsymbol{u}_k^{-1}\|_{\mathcal{M}(\tilde{\Omega}_{\boldsymbol{b}},\mathbb{R}^{3\times 3})} \right].$$

New Results:

- If $H(J) \geq c\, J^{-2}$ and the minimizer of $\int |Du|^2 + H(\det Du)\, dx + 2\|D^s u^{-1}\|_{\mathcal{M}}$ in $\mathcal{B} = \{ \dots\ u^{-1}_1 \in W^{1,1},\ u^{-1}_2 \in W^{1,1},\ u^{-1}_3 \in BV \}$ satisfied $u_3^{-1} \in SBV$, then $u^{-1} \in W^{1,1}$.
- $\liminf_{j \to \infty} \int |Du_j|^2 + H(\det Du_j) \geq \int |Du|^2 + H(\det Du)\, dx + 2\|D^s u^{-1}\|$ in the general 3D case (also without axisymmetry)
- Cantor

Local invertibility

Divergence identities and $\det D\mathbf{u} > 0$ also imply local invertibility:

Theorem. For a.e. $\mathbf{x} \in \Omega$ there exists $r_{\mathbf{x}} > 0$ such that $\mathbf{u}$ is injective a.e. in $B(\mathbf{x}, r_{\mathbf{x}})$.

(Fonseca, Gangbo 95, Barchiesi, Henao, C.M.-C. 17)

Global invertibility

If $\mathbf{u} \in W^{1,p}$ satisfies

$$\det D\mathbf{u} \in L^1, \text{ divergence identities, } \det D\mathbf{u} > 0$$

and some of the following:

- $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$ with $\mathbf{u}_0 : \Omega \to \mathbb{R}^n$ injective. (Ball 81, Šverák 88, Henao, Mora, Oliva 21)
- $\int_\Omega \det D\mathbf{u} \leq \mathcal{L}^n(\mathbf{u}(\Omega))$. (Ciarlet, Nečas 87, Qi 88)
- $\mathbf{u}|_{\partial\Omega}$ is limit of injective maps $\partial\Omega \to \mathbb{R}^n$. (Krömer 20)

then $\mathbf{u}$ is globally invertible.

LAND OF STANDARD CALCULUS OF VARIATIONS MACHINERY

- $p > 2$
- CONTROL OF MINORS

⋮

- INV + SLOW DETERM. OR BOUNDED DISTORTION

(SHARP →)

IMPROVEMENT BY DOLEZALOVA HENCL MALY

CONTI DE LELLIS COUNTEREXAMPLE

UNKNOWN LAND

?

Relaxation

We are looking for

A larger space $\mathcal{C}$ including $\mathcal{A}_s^r$ and compact

An effective energy F on $\mathcal{C}$, extension of E

so that

$$\min_{\mathcal{C}} F = \inf_{\mathcal{A}_s^r} E$$

The use of the relaxed energy allows us to transform a problem of lack of compactness into a problem of regularity.

Open problems

- What about the general case? (Fully open)
- Is our candidate the correct one? Recovery sequence...
- What about allowing cavitations?
- If the inverse is stable in Sobolev, is INV preserved?
- Is it possible that the inverse is fully BV and not SBV? (i.e., Cantorian part, when $\mathcal{E}(u) = \infty$)