Non-interpenetration conditions in the passage from nonlinear to linearized Griffith fracture

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Compensated compactness and applications to materials
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Non-interpenetration in large-strain elasticity: a (very short) recap

A body should not be allowed to interpenetrate itself during elastic deformations. Extreme compressions should lead to a blow-up of the elastic energy, therefore being energetically unfavorable.

How to enforce that, in practice? J. Ball, V. Sˇverák, I. Fonseca, W. Gangbo,...

- Positivity of the determinant of ∇y?
  Not enough to have injectivity everywhere nor global invertibility.
- Positivity of the determinant of $\nabla y +$ Ciarlet-Nečas condition?
  Injectivity almost everywhere and non-interpenetration.

$$\int_\Omega \det \nabla y(x)\, \mathrm{d}x \leq \mathcal{L}^d(y(\Omega)).$$

Griffith's functional in 2D [A. Griffith, B. Bourdin, G. Francfort, J. Marigo,...]

$$\mathcal{E}(y) = \int_\Omega W(\nabla y(x))\,\mathrm{d}x + \kappa\mathcal{H}^1(J_y),$$

- Frame-indifferent bulk energy vs surface term.
- $W\colon \mathbb{M}^{2\times 2} \to [0, +\infty)$ is a nonlinear elastic energy density,
- $\kappa > 0$ is a material constant,
- deformations $y\colon \Omega \to \mathbb{R}^2$ in $GSBV(\Omega)$,
- ∇y denotes the absolutely continuous part of their gradient,
- J_y is their jump set.

How to define $y(\Omega)$ if y is not continuous?

Definition (Measure theoretical image)

Let $y \in GSBV(\Omega; \mathbb{R}^2)$ and let $\Omega_d \subseteq \Omega$ be the set of points where y is approximately differentiable. We define y_d by

$$y_d(x) := \begin{cases} \tilde{y}(x) & \text{for } x \in \Omega_d, \\ 0 & \text{else,} \end{cases}$$

where $\tilde{y}(x)$ denotes the Lebesgue value of y at $x \in \Omega_d$. Given a measurable set $E \subseteq \Omega$, we say that $y_d(E)$ is the *measure theoretic image* of E under the map y, and we denote it by $[y(E)]$.

Non-interpenetration in (G)SBV [A. Giacomini, M. Ponsiglione]

Definition (Ciarlet-Nečas condition for *GSBV*-maps)

We say that $y \in GSBV(\Omega;\mathbb{R}^2)$ satisfies the *Ciarlet-Nečas non-interpenetration condition* if $\det \nabla y(x) > 0$ for a.e. $x \in \Omega$ and

$$\int_\Omega \det \nabla y \,\mathrm{d}x \le \mathcal{L}^2([y(\Omega)])\,. \tag{CN}$$

- Equivalent to a.e. injectivity (in the domain).
- Minimizers of Griffith under (CN) exist.
- *Its linearized counterpart is the contact condition:*

$$[u](x)\cdot \nu_u(x) \ge 0 \text{ for } \mathcal{H}^1\text{-a.e. } x \in J_u, \tag{CC}$$

$u \in GSBD^2(\Omega) := \{u \in GSBD(\Omega) : e(u) \in L^2, \mathcal{H}^1(J_u) < +\infty\}.$

A counterexample [S. Almi-E.D.-M. Friedrich '22]

We construct a sequence of deformations

- $(y_\varepsilon)_\varepsilon \subset GSBV^2(\Omega;\mathbb{R}^2)$
- satisfying **CN**,
- such that their associated rescaled displacements

$$u_\varepsilon := \frac{1}{\varepsilon}(y_\varepsilon - \mathrm{id}),$$

  have uniformly bounded linearized energies, i.e.,

$$\sup_{\varepsilon>0} \mathcal{F}(u_\varepsilon) < +\infty, \qquad \text{where } \mathcal{F}(u_\varepsilon) := \|e(u_\varepsilon)\|^2_{L^2(\Omega)} + \mathcal{H}^1(J_{u_\varepsilon}).$$

- u_ε goes in measure to a displacement u violating **CC**.

A counterexample cont'd [S. Almi-E.D.-M. Friedrich '22]

- $\Omega = (-1, 1)^2$
- $u = (-1, 0)\chi_{\{x_1 > 0\}}$.
  - $J_u = \{0\} \times (-1, 1)$,
  - $\nu_u = e_1$,
  - $[u] = -e_1$.

$\Rightarrow [u] \cdot \nu_u = -1 < 0$ on $J_u \Rightarrow$ **No CC**.

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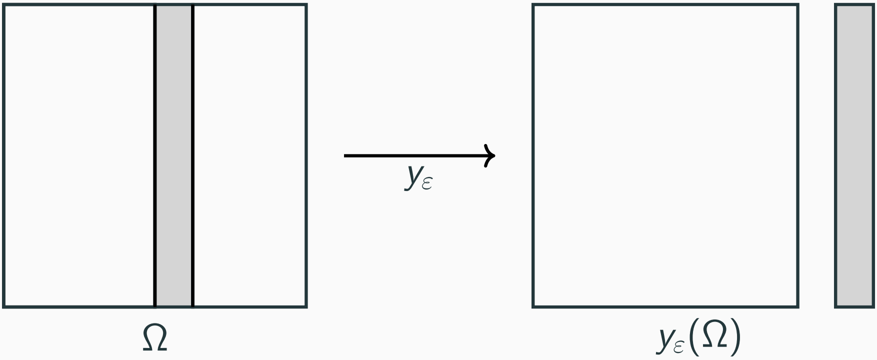
  $\Rightarrow [u] \cdot \nu_u = -1 < 0$ on $J_u \Rightarrow$ **No CC**.
- $u_\varepsilon := (-1,0)\chi_{\{x_1>0\}} + \left(\frac{2}{\varepsilon}, 0\right)\chi_{\{-2\varepsilon<x_1<0\}}\,, \quad y_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon\,.$
  - $\nabla y_\varepsilon = \mathrm{Id}$ on Ω
  - $\mathcal{H}^1(J_{y_\varepsilon}) = 4$
  - $u_\varepsilon \to u$ in measure on Ω.
  - y_ε satisfy **CN** since for ε small

    $$[y_\varepsilon(\{x_1 < -2\varepsilon\})], \quad [y_\varepsilon(\{-2\varepsilon < x_1 < 0\})], \quad [y_\varepsilon(\{x_1 > 0\})]$$

    are pairwise disjoint.

A counterexample cont'd [S. Almi-E.D.-M. Friedrich '22]



Key point: The length of the jump along the sequence has twice the size of the limiting jump.

Griffith type models for nonsimple materials

$$\mathcal{E}_\varepsilon(y) = \begin{cases} \varepsilon^{-2}\displaystyle\int_{\Omega'} W(\nabla y(x))\,dx + \varepsilon^{-2\beta}\displaystyle\int_{\Omega'} |\nabla^2 y(x)|^2\,dx + \kappa\mathcal{H}^1(J_y) \\ \qquad \text{if } J_{\nabla y} \subseteq J_y, \\ +\infty \quad \text{else in } GSBV^2_2(\Omega;\mathbb{R}^2). \end{cases}$$

- W is a continuous, frame-indifferent, one-well density with quadratic growth from $SO(2)$ from below
- $\kappa > 0, \quad \beta \in (\frac{2}{3}, 1), \quad \Omega \subseteq \Omega'$.
- $GSBV^2_2(\Omega;\mathbb{R}^2) := \{y \in GSBV^2(\Omega;\mathbb{R}^2) : \nabla y \in GSBV^2\}$.

Linearized Griffith models under non-interpenetration [A. Chambolle, S. Conti, V. Crismale, G. Francfort, M. Focardi, F. Iurlano, M. Friedrich...][M. Friedrich '20]

$$\mathcal{E}(u) := \int_{\Omega'} \frac{1}{2} Q(e(u))\,\mathrm{d}x + \kappa \mathcal{H}^1(J_u),$$

- $Q(\mathrm{F}) = D^2 W(\mathrm{Id})\mathrm{F} : \mathrm{F}$ for all $\mathrm{F} \in \mathbb{R}^{2\times 2}$.
- $u \in GSBD^2(\Omega; \mathbb{R}^2)$.

Natural question:

1. Is $\mathcal{E}$ with **CC** the right linearization for $\mathcal{E}_\varepsilon$ with **CN**?

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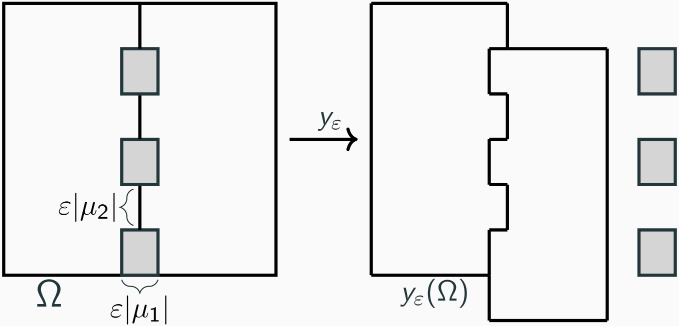
Natural question:

1. Is $\mathcal{E}$ with **CC** the right linearization for $\mathcal{E}_\varepsilon$ with **CN**? No$\Rightarrow$ Second counterexample

A second counterexample [S. Almi-E.D.-M. Friedrich'22]

- $\Omega = (-1, 1)^2$,
- $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\mu_1, \mu_2 < 0$,
- $u = (\frac{\mu_1}{2}, \mu_2)\chi_{\{x_1>0\}}$.
- $J_u = \{0\} \times (-1, 1)$ has length $\mathcal{H}^1(J_u) = 2$ and normal vector $\nu_u = e_1$. Hence, $[u] \cdot e_1 = \frac{\mu_1}{2} < 0$ on $J_u \Rightarrow$ **No CC**

A second counterexample cont'd [S. Almi-E.D.-M. Friedrich'22]



$$\lim_{\varepsilon\to 0}\mathcal{H}^1(J_{y_\varepsilon}) = 3 + 2\frac{|\mu_1|}{|\mu_2|}\,, \quad \mathcal{H}^1(J_u) = 2$$

$\Rightarrow$ Besides $\kappa\mathcal{H}^1(J_u)$, there should be an **additional anisotropic surface term** being positive whenever **CC** is violated, depending on the orientation and on the amplitude of the jump of u.

Positive results for energy-convergent sequences

- Boundary conditions on $\Omega' \setminus \overline{\Omega}$?
  For $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$ and $\varepsilon > 0$ we set

$$\mathcal{S}_{\varepsilon,h} = \{y \in GSBV_2^2(\Omega';\mathbb{R}^d)\colon \ y = \mathrm{id} + \varepsilon h \text{ on } \Omega' \setminus \overline{\Omega}\}.$$

- Which notion of convergence?

• In general, compactness for $(u_\varepsilon)_\varepsilon$ if $\sup_\varepsilon \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$.

• For bodies undergoing fracture no compactness can be expected: take, e.g., $y_\varepsilon := \mathrm{id}\chi_{\Omega' \setminus B} + \mathrm{R}\,\mathrm{id}\chi_B$, for a small ball $B \subset \Omega$ and a rotation $\mathrm{R} \in SO(2)$, $\mathrm{R} \neq \mathrm{Id}$. Then $|u_\varepsilon|, |\nabla u_\varepsilon| \to \infty$ on B as $\varepsilon \to 0$.

• This phenomenon can be avoided if the deformation is *rotated back to the identity* on the set B.

Positive results for energy-convergent sequences cont'd

Definition (Asymptotic representation)

Fix $\gamma \in (\frac{2}{3}, \beta)$. We say that $(y_\varepsilon)_\varepsilon$ with $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ is *asymptotically represented* by $u \in GSBD^2_h(\Omega')$, and write $y_\varepsilon \rightsquigarrow u$, if there exist sequences of Caccioppoli partitions $(P_j^\varepsilon)_j$ of Ω' and corresponding rotations $(R_j^\varepsilon)_j \subset SO(2)$ such that, setting

$$y_\varepsilon^{\mathrm{rot}} := \sum_{j=1}^{\infty} R_j^\varepsilon \, y_\varepsilon \, \chi_{P_j^\varepsilon} \qquad \text{and} \qquad u_\varepsilon := \frac{1}{\varepsilon}(y_\varepsilon^{\mathrm{rot}} - \mathrm{id}),$$

the following conditions hold:

$$\|\mathrm{sym}(\nabla y_\varepsilon^{\mathrm{rot}}) - \mathrm{Id}\|_{L^2(\Omega')} \le C\varepsilon,$$

$$\|\nabla y_\varepsilon^{\mathrm{rot}} - \mathrm{Id}\|_{L^2(\Omega')} \le C\varepsilon^\gamma,$$

$$|\nabla y_\varepsilon^{\mathrm{rot}} - \mathrm{Id}| \le C\big(\varepsilon^\gamma + \mathrm{dist}(\nabla y_\varepsilon^{\mathrm{rot}}, SO(2))\big) \text{ a.e. on } \Omega'$$

Positive results for energy-convergent sequences cont'd

Definition (Asymptotic representation cont'd)

Additionally:

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega' \setminus E_u,$$
$$e(u_\varepsilon) \rightharpoonup e(u) \ \text{ weakly in } L^2(\Omega' \setminus E_u; \mathbb{R}^{2\times 2}_{\mathrm{sym}}),$$
$$\mathcal{H}^1(J_u) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon}) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon}),$$
$$e(u) = 0 \quad \text{on} \quad E_u, \quad \mathcal{H}^1\big((\partial^* E_u \cap \Omega') \setminus J_u\big) = \mathcal{H}^1(J_u \cap (E_u)^1) = 0,$$

where $E_u := \{x \in \Omega : |u_\varepsilon(x)| \to \infty\}$ is a set of finite perimeter (compactness result in [A. Chambolle-V. Crismale '21]).

Key point: u is not unique. It depends on partitions and rotations.

Positive results for energy-convergent sequences cont'd

We have the following compactness result for asymptotic representations.

Proposition (Compactness [M. Friedrich '20])

Let $(y_\varepsilon)_\varepsilon$ *be a sequence satisfying* $y_\varepsilon \in \mathcal{S}_{\varepsilon,h}$ *and* $\sup_\varepsilon \mathcal{E}_\varepsilon(y_\varepsilon) < +\infty$. *Then there exists a subsequence (not relabeled) and* $u \in GSBD^2_h(\Omega')$ *such that* $y_\varepsilon \rightsquigarrow u$.

Our results [S. Almi-E.D.-M. Friedrich '22]

Theorem (From CN to CC)

Let $(y_\varepsilon)_\varepsilon$ be a sequence satisfying $y_\varepsilon \in \mathcal{S}_{\varepsilon,\mathbf{h}}$ and **CN**. *Let $u \in GSBD^2_h(\Omega')$ be such that $y_\varepsilon \rightsquigarrow u$ and $\mathcal{E}_\varepsilon(y_\varepsilon) \to \mathcal{E}(u)$ as $\varepsilon \to 0$. Then, u satisfies* **CC** *on $J_u \setminus \partial^* E_u$.*

Theorem (Existence of energy-convergent sequences)

Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ be bounded Lipschitz domains. Then, for every $u \in GSBD^2_h(\Omega')$ satisfying **CC** *there exists a sequence $(y_\varepsilon)_\varepsilon$ satisfying* **CN** *and such that $y_\varepsilon \in \mathcal{S}_{\varepsilon,\mathbf{h}}$, $y_\varepsilon \rightsquigarrow u$, and*

$$\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(y_\varepsilon) = \mathcal{E}(u).$$

Going from CN to CC–proof idea

Area formula for a.e approximably differentiable maps: for every measurable set $E \subset \Omega$ the function $z \mapsto m(y, z, E \cap \Omega_d)$ is measurable and

$$\int_E |\det \nabla y(x)| \, \mathrm{d}x = \int_{\mathbb{R}^2} m(y, z, E \cap \Omega_d) \, \mathrm{d}z \, .$$

First remark: combining **CN** and the area formula

$$\int_E \det \nabla y_\varepsilon \, \mathrm{d}x = \mathcal{L}^2([y_\varepsilon(E)]) \quad \text{for all } E \subset \Omega \text{ measurable.}$$

Strategy: by contradiction, suppose there exists a rectifiable set $J^{\mathrm{int}} \subset J_u$ with $\mathcal{H}^1(J^{\mathrm{int}}) > 0$ such that $[u](x) \cdot \nu_u(x) < 0$ for all $x \in J^{\mathrm{int}}$. By blow-up around points in J^{int}, we construct $E_\varepsilon \subseteq \Omega$ such that

$$\int_{E_\varepsilon} \det(\nabla y_\varepsilon) \, \mathrm{d}x > \mathcal{L}^2([y_\varepsilon(E_\varepsilon)]). \quad \square$$

Existence of energy-convergent sequences–proof idea

Lemma (Stronger contact condition)

Given $h \in W^{r,\infty}(\Omega;\mathbb{R}^2)$ *for* $r \in$, *let* $u \in GSBD^2_h(\Omega')$ *satisfy* **CC**. *Then, there exist sequences* $(\tau_n)_n$ *in* $(0,+\infty)$ *and* $(u_n)_n$ *in* $GSBD^2_h(\Omega')$ *such that*

$$u_n \to u \text{ in measure on } \Omega',$$

$$\lim_{n\to\infty} \|e(u_n) - e(u)\|_{L^2(\Omega')} = 0,$$

$$\lim_{n\to\infty} \mathcal{H}^1(J_{u_n}) = \mathcal{H}^1(J_u),$$

$$\lim_{n\to\infty} \mathcal{H}^1\big(\{x \in J_{u_n} \colon [u_n](x)\cdot\nu_{u_n}(x) \le 2\tau_n\}\big) = 0.$$

Existence of energy-convergent sequences–proof idea

First step: prove that you can approximate maps in $GSBD^2_h$ satisfying the stronger **CC** up to small sets with maps in SBV^2 satisfying the same. [Adaptation of [A.Chambolle-V.Crismale '19], [G.Cortesani-R. Toader'99], [M. Friedrich'20]]

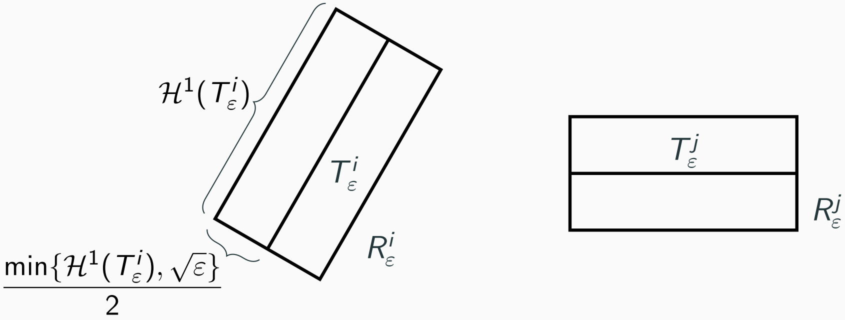
Second step: Approximate u by $v_\varepsilon \in W^{2,\infty}(\Omega' \setminus J_{v_\varepsilon}; \mathbb{R}^2)$ and

$$J^{\mathrm{bad}}_{v_\varepsilon} := \{x \in J_{v_\varepsilon} : \; [v_\varepsilon](x) \cdot \nu_{v_\varepsilon}(x) \leq \tau_\varepsilon\}$$

consists of a finite number of segments $(T^i_\varepsilon)_{i=1}^{n_\varepsilon}$.

Existence of energy-convergent sequences–proof idea

Third step: Cover these segments by pairwise disjoint rectangles R^i_ε, $i = 1, \ldots, n_\varepsilon$, of length $\mathcal{H}^1(T^i_\varepsilon)$ and height $\min\{\mathcal{H}^1(T^i_\varepsilon), \sqrt{\varepsilon}\}$ such that T^i_ε separates R^i_ε into two rectangles of length $\mathcal{H}^1(T^i_\varepsilon)$ and height $\min\{\mathcal{H}^1(T^i_\varepsilon), \sqrt{\varepsilon}\}/2$.



Existence of energy-convergent sequences–proof idea

Fourth step: define

$$w_\varepsilon := v_\varepsilon \chi_{\Omega' \setminus \bigcup_{i=1}^{n_\varepsilon} R_\varepsilon^i} + \sum_{i=1}^{n_\varepsilon} s_\varepsilon^i \chi_{R_\varepsilon^i}$$

for suitable constants $(s_\varepsilon^i)_i \subset \mathbb{R}^2$ for which the functions $y_\varepsilon := \mathrm{id} + \varepsilon w_\varepsilon$ are such that the sets

$$\big[y_\varepsilon\big(\Omega' \setminus \bigcup_{i=1}^{n_\varepsilon} R_\varepsilon^i\big)\big], \quad [y_\varepsilon(R_\varepsilon^i)],\ i = 1, \dots, n_\varepsilon, \quad \text{are pairwise disjoint.}$$

Fifth step: show that (y_ε) satisfy **CN**. $\square$

Summarizing

- In general **CN** and **CC** are not related via linearization, and the variational linearization of Griffith under **CN** is not the linearized Griffith.
- For energy-convergent sequences, instead, the passage **CN** to **CC** holds true and we also have the converse approximation result.

Summarizing

- In general **CN** and **CC** are not related via linearization, and the variational linearization of Griffith under **CN** is not the linearized Griffith.
- For energy-convergent sequences, instead, the passage **CN** to **CC** holds true and we also have the converse approximation result.

Preprint *https://arxiv.org/abs/2204.10622* available on *asc.tuwien.ac.at/~edavoli/*

Thank you for your attention!

Technical condition on Ω and Ω'

Geometrical assumption on the Dirichlet part of the boundary $\partial_D\Omega := \Omega' \cap \partial\Omega$:
$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$ with

$$\partial_D\Omega, \partial_N\Omega \text{ relatively open}, \quad \mathcal{H}^{d-1}(N) = 0,$$
$$\partial_D\Omega \cap \partial_N\Omega = \emptyset, \quad \partial(\partial_D\Omega) = \partial(\partial_N\Omega),$$

and there exist $\bar{\delta} > 0$ small and $x_0 \in \mathbb{R}^d$ such that for all $\delta \in (0, \bar{\delta})$ there holds

$$O_{\delta,x_0}(\partial_D\Omega) \subset \Omega,$$

where $O_{\delta,x_0}(x) := x_0 + (1-\delta)(x - x_0)$.

Definition of *GBD*

Basic notation for the slicing technique. For $\xi \in \mathbb{S}^1$, we let

$$\Pi^\xi := \{w \in \mathbb{R}^2 \colon w \cdot \xi = 0\},$$

and for any $w \in \mathbb{R}^2$, $B \subset \mathbb{R}^2$, and $u : B \to \mathbb{R}^2$ we let

$$B^\xi_w := \{t \in \mathbb{R} \colon w + t\xi \in B\}, \qquad \hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi.$$

Let also

$$J^1 \hat{u}^\xi_y := \{t \in J\hat{u}^\xi_y(t) : [(\hat{u}^\xi_y)^+(t) - (\hat{u}^\xi_y)^-(t)] \geq 1\}.$$

$GBD(\Omega)$ is the space of $\mathcal{L}^2$-measurable functions such that there exists a bounded Radon measure λ such that

$$\int_{\Pi^\xi} (|D\hat{u}^\xi_y|(B^\xi_y \setminus J^1 \hat{u}^\xi_y) + \mathcal{H}^0(B^\xi_y \cap J^1 \hat{u}^\xi_y)) \mathrm{d}\mathcal{H}^1(y) \leq \lambda(B).$$