The truncated moment problem on unital commutative real algebras

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The classical truncated $K-$moment problem

Let $d, n \in \mathbb{N}$.

- $\left.\begin{array}{l}\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \\ \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d\end{array}\right\} \to \underline{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.
- $J_n := \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d : \ \alpha_1 + \cdots + \alpha_d \leq n\} \subset \mathbb{N}_0^d$.

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Given $n \in \mathbb{N}$, $m = (m_\alpha)_{\alpha \in J_n}$ with $m_\alpha \in \mathbb{R}$ and a closed subset K of $\mathbb{R}^d$, does there exist a nonnegative Radon measure μ supported in K s.t.

$$m_\alpha = \underbrace{\int_K \underline{X}^\alpha \mu(d\underline{X})}_{\alpha\text{-th moment of } \mu}, \quad \forall \alpha \in J_n?$$

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If such a μ exists, we say that μ is a $K-$**representing measure for** m.

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$J_n \rightsquigarrow \mathbb{N}_0^d$ $\rightsquigarrow$ classical **full** $K-$moment problem

Need for a more general formulation...

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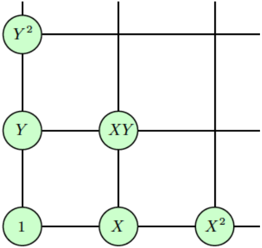
Examples of monomial diagrams for $d = 2$



classical $\to \alpha \in J_2$

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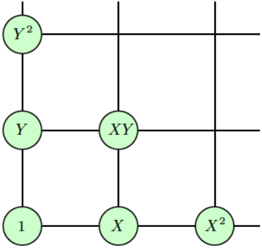

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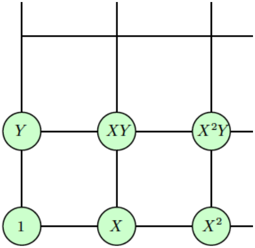

rectangular $\rightarrow \alpha \in \{0,1,2\} \times \{0,1\}$

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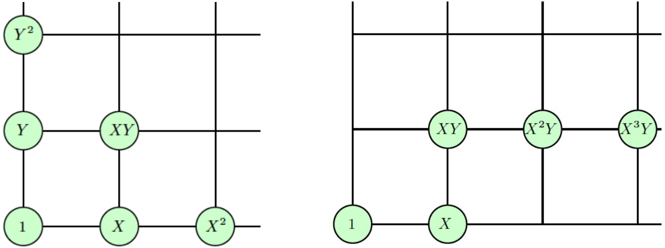

classical $\to \alpha \in J_2$

sparse $\to \alpha \in \{0,1\} \times \{0\} \cup \{1,2,3\} \times \{1\}$

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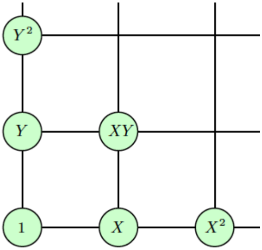

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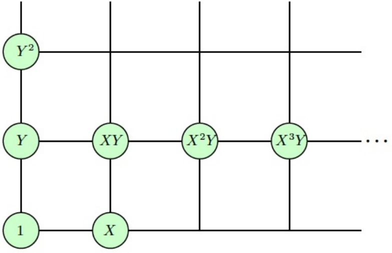

sparse hybrid $\to \alpha \in J \subsetneq \mathbb{N}_0^2$ infinite

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- need to prescribe more general sets of moments than all the ones up to a degree

$J_n \rightsquigarrow$ general $J \subsetneq \mathbb{N}_0^d$ (finite or infinite)

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$\mathbb{R}[X_1, \ldots, X_d] \rightsquigarrow$ any other unital commutative real algebra (not necessarily finitely generated)

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Given a linear subspace $B \subsetneq \mathbb{R}[\underline{X}]$, $L: B \to \mathbb{R}$ linear and $K \subseteq \mathbb{R}^d$ closed, does there exist a nonnegative Radon measure μ supported in K s.t.

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Finite dimensional setting

- $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_d]$

General setting

- $A =$ **unital commutative** $\mathbb{R}-$**algebra**

The $d-$dim. $B-$truncated $K-$MP

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- $A =$ **unital commutative** $\mathbb{R}-$**algebra**
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- For $a \in A$ the **Gelfand transform** $\hat{a} : X(A) \to \mathbb{R}$ is $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.

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Searching solvability criteria in this general setting....

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Standing on the shoulders of giants....

Full $d-$dim. KMP:
$A = B = \mathbb{R}[X_1, \ldots, X_d]$

→

Riesz-Haviland theorem (1923, $d = 1$; 1936, $d \geq 2$)

$L(\mathrm{Pos}_{\mathbb{R}[\underline{X}]}(K)) \subseteq [0, \infty) \Leftrightarrow \exists\ K$–representing measure for L.

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Truncated $d-$dim. KMP:
$A = \mathbb{R}[X_1, \ldots, X_d]$
$B = \mathbb{R}[X_1, \ldots, X_d]_n$

→ (K cmpt) **Truncated Riesz-Haviland thm (Tchakaloff 1957)**

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$A = B = \mathbb{R}[X_1, \ldots, X_d]$

→ **Riesz-Haviland theorem (1923, $d = 1$; 1936, $d \geq 2$)**

$L(\mathrm{Pos}_{\mathbb{R}[\underline{X}]}(K)) \subseteq [0, \infty)$ ⇔ ∃ K–representing measure for L.

Truncated d−dim. KMP:
$A = \mathbb{R}[X_1, \ldots, X_d]$
$B = \mathbb{R}[X_1, \ldots, X_d]_n$

→ (K cmpt) **Truncated Riesz-Haviland thm (Tchakaloff 1957)**

$L(\mathrm{Pos}_{\mathbb{R}[\underline{X}]_n}(K)) \subseteq [0, \infty)$ ⇔ ∃ K–representing measure for L.

→ (K non-cmpt) exact analogue of Riesz-Haviland fails!

Searching solvability criteria in this general setting....

The general $B-$truncated $K-$moment problem

Let A be unital commutative $\mathbb{R}$–algebra. Given a linear subspace $B \subseteq A$, $L : B \to \mathbb{R}$ linear and $K \subset X(A)$ closed, $\exists\ \mu$ Radon supported on K s.t. $L(b) = \int_K \hat{b}\, d\mu, \forall b \in B$?

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Truncated Riesz-Haviland thm (Curto,Fialkow 2008)

$L(\mathrm{Pos}_{\mathbb{R}[\underline{X}]_{2n+i}}(K)) \subseteq [0, \infty)$, $i \in \{0, 1\}$
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$\exists\ \overline{L}$ extension of L to $\mathbb{R}[\underline{X}]_{2n+2}$ s.t. $\overline{L}(\mathrm{Pos}_{\mathbb{R}[\underline{X}]_{2n+2}}(K)) \subseteq [0, \infty)$

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$\mathrm{Pos}_B(K) := \{b \in B : \hat{b} \geq 0 \text{ on } K\} \rightsquigarrow L$ is $K-$**positive** if $L(\mathrm{Pos}_B(K)) \subseteq [0, +\infty)$

Standing on the shoulders of giants....

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Our first goal

finding an analogue of Riesz-Haviland thm for the general $B-$truncated $K-$MP!

Our generalized Riesz-Haviland theorem: the compact case

Theorem (Curto, Ghasemi, I., Kuhlmann, 2023)

Let A be a unital commutative $\mathbb{R}$–algebra, $K \subseteq X(A)$ compact
$B \subseteq A$ linear subspace s.t. $\exists\ q \in B$ with $\hat{q} > 0$ on K, and $L : B \longrightarrow \mathbb{R}$ linear.

$$L(\mathrm{Pos}_B(K)) \subseteq [0, \infty) \Longleftrightarrow \exists\ K - \text{representing measure for } L$$

Our generalized Riesz-Haviland theorem: the compact case

Thm (*) (Curto, Ghasemi, I., Kuhlmann)

Let $(\mathcal{A}, \rho)$ be a seminormed algebra, $\mathcal{B} \subseteq \mathcal{A}$ linear subsp, S quadratic module, and $\ell : \mathcal{B} \longrightarrow \mathbb{R}$ linear.

$$\exists D > 0 : \ell(g) \leq D\|g\|_{S;\rho}, \forall g \in \mathcal{B}$$

$$\Updownarrow$$

$\exists$ $(\mathfrak{sp}_\rho(\mathcal{A}) \cap \mathcal{K}_S)-$ representing meas. for L

↙

Theorem (Curto, Ghasemi, I., Kuhlmann, 2023)

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$$L(\mathrm{Pos}_B(K)) \subseteq [0, \infty) \Longleftrightarrow \exists\ K - \text{representing measure for } L$$

Notation

- $S \subseteq A$ quadratic module, i.e. $1 \in S$, $S + S \subseteq S$ and $a^2 S \subseteq S$ for all $a \in A$
- $\|g\|_{S;\rho} := \inf_{h\in S} \rho(g + h)$
- $\mathfrak{sp}_\rho(\mathcal{A}) := \{\alpha \in X(\mathcal{A}) : \alpha \text{ is } \rho-\text{continuous}\}$ Gelfand spectrum
- $\mathcal{K}_S := \{\alpha \in X(\mathcal{A}) : \alpha(s) \geq 0, \forall\ s \in S\}$

Our generalized Riesz-Haviland theorem: the compact case

Choquet's Lemma (Choquet, 1969)

Let C convex cone in a real vector space V
$W \subseteq V$ linear subsp, $L : W \longrightarrow \mathbb{R}$ linear

$$L(W \cap C) \subseteq [0, \infty)$$
$$\Downarrow$$

$\exists\ \bar{L}$ lin. extension of L to $(W + C) \cap (W - C)$
s.t. $\bar{L}((W + C) \cap (W - C) \cap C) \subseteq [0, \infty)$.

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Applications to finite dimensional truncated $K-$MPs

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Taking in our theorem

- $A = \mathbb{R}[X_1, \ldots, X_d]$
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Truncated Riesz-Haviland theorem (Tchakaloff 1957)

Let $d, n \in \mathbb{N}$, $K \subseteq \mathbb{R}^d$ compact, and $L : \mathbb{R}[X_1, \ldots, X_d]_n \longrightarrow \mathbb{R}$ linear.

$$L(\mathrm{Pos}_{\mathbb{R}[\underline{X}]_n}(K)) \subseteq [0, \infty) \iff \exists\ K - \text{representing measure for } L$$

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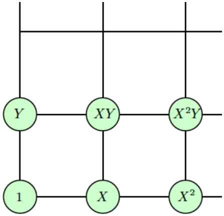

- $A = \mathbb{R}[X, Y]$
- $B = span\{X^i Y^j : 0 \leq i \leq n_1, 0 \leq j \leq n_2\}$
- $q := 1 \in B$

Rectangular truncated $K-$MP (Putinar 1990)

Let $n_1, n_2 \in \mathbb{N}$, $K \subseteq \mathbb{R}^2$ compact and
$L : span\{X^i Y^j : 0 \leq i \leq n_1, 0 \leq j \leq n_2\} \to \mathbb{R}$ linear.

$$L \text{ is } K-\text{positive} \Longleftrightarrow \exists\ K - \text{repr. meas. for } L$$

Applications to finite dimensional truncated $K-$MPs

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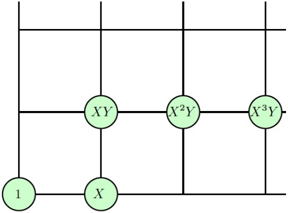

$C = \{1, X, XY, X^2Y, X^3Y\}$

A set C of monomials in $\mathbb{R}[X, Y]$ is connected if every monomial in C is the endpoint of a staircase path starting at 1.

- $A = \mathbb{R}[X, Y]$
- $B = span(C)$
- $q := 1 \in B$

Sparse truncated $K-$MP (Laurent, Mourrain 2009)

Let C be a finite and connected set of monomials in $\mathbb{R}[X, Y]$, $K \subseteq \mathbb{R}^2$ compact and $L : span(C) \to \mathbb{R}$ linear.

L is $K-$positive $\iff\ \exists\ K - $ repr. meas. for L

Applications to finite dimensional truncated $K-$MPs

Theorem (Curto, Ghasemi, I., Kuhlmann, 2023)

Let A be a unital commutative $\mathbb{R}$–algebra, $K \subseteq X(A)$ compact
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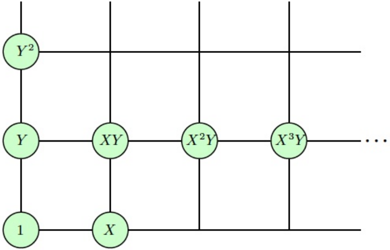

- $A = \mathbb{R}[X, Y]$
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Sparse hybrid truncated $K-$MP

Let C be a (not necessarily finite) connected set of monomials in $\mathbb{R}[X, Y]$, $K \subseteq \mathbb{R}^2$ compact and $L : span(C) \to \mathbb{R}$ linear.

L is $K-$positive $\iff \exists\ K-$repr. meas. for L

Applications to infinite dimensional truncated $K-$MPs

applications in statistical mechanics $\rightsquigarrow$ **truncated MP for random measures**

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Can we embed such a K in a character space of some algebra?

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- $\mathcal{M}(X) :=$ space of all signed Radon measures supported in X
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- $\forall \nu \in \mathcal{M}(X), f_n \in C_c(X^n),\ \ f_n \nu^{\otimes n} := \int_{X^n} f_n(x_1, \dots, x_n)\nu^{\otimes n}(dx_1, \dots, dx_n).$

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- $\mathscr{P} :=$ polynomials in the variable ν in $\mathcal{M}(X)$ and coefficients in $C_c(X)$, i.e.

$$a \in \mathscr{P} \rightsquigarrow a(\nu) := \sum_{j=0}^{N} f_j \nu^{\otimes j}, \;\; N \in \mathbb{N}_0, \;\; f_0 \in \mathbb{R}, \;\; f_j \in C_c(X^j)$$

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$(\mathcal{M}(X), \tau)$ is topologically embedded in $\left(X(\mathscr{P}), \omega_{X(\mathscr{P})}\right)$

Applications to infinite dimensional truncated $K-$MPs

Taking

- $A := \mathscr{P}$
- $K \subset \mathcal{M}(X) \subset X(\mathscr{P})$
- $B := \mathscr{P}_N :=$ polynomials in $\mathscr{P}$ of degree N
- $q := 1 \in \mathscr{P}_N$

Theorem (Curto, Ghasemi, I., Kuhlmann, 2023)

Let A be a unital commutative $\mathbb{R}$–algebra, $K \subseteq X(A)$ compact, $B \subseteq A$ linear subspace s.t. $\exists\ q \in B$ with $\hat{q} > 0$ on K, and $L : B \longrightarrow \mathbb{R}$ linear.

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$$L(\mathrm{Pos}_{\mathscr{P}_N}(K)) \subseteq [0, +\infty) \Longleftrightarrow \exists\ K - \text{representing measure for } L.$$

$\rightarrow$ generalizes some results in **Kuna, Lebowitz, Speer 2011** for compact subsets of

$$\mathcal{N}(X) := \left\{ \sum_{i\in I} \delta_{x_i} : x_i \in X,\ I \subseteq \mathbb{N} \text{ with either } |I| < \infty \text{ or } I = \mathbb{N} \right\} \subset \mathcal{M}(X)$$

Our generalized Riesz-Haviland theorem: the non-compact case

Theorem (Curto, Ghasemi, I., Kuhlmann, 2023)

Let A be a unital commutative $\mathbb{R}$–algebra, $K \subseteq X(A)$ non-compact and closed, $B \subseteq A$ linear subspace s.t.

1. $\exists\, q \in A \setminus B$ s.t. $\hat{q} \geq 1$ on K
2. $1 \in B_q := \text{Span}(B \cup \{q\})$
3. B_q generates A
4. $\forall\, b \in B$, $\sup_{\alpha \in K} \left| \frac{\hat{b}(\alpha)}{\hat{q}(\alpha)} \right| < \infty$

and let $L : B \longrightarrow \mathbb{R}$ be linear.

$L(\text{Pos}_B(K)) \subseteq [0, +\infty)$
$+$
$\exists\, \overline{L}$ extension of L to B_q
s.t. $\overline{L}(\text{Pos}_{B_q}(K)) \subseteq [0, \infty)$

$\iff$

$\exists\, K$–representing measure for L
i.e. $L(b) = \int \hat{b}\, d\nu, \quad \forall\, b \in B$

Proof's idea

- From $\bar{L}$ construct a $\widetilde{K}-$positive linear functional $\tilde{L}$ on a subspace of an algebra $\mathcal{B} \subset \mathcal{C}_b(K)$, where $\widetilde{K}$ is the Hausdorff compactification of K
- use our result in the compact case to show $\exists$ a $\widetilde{K}-$representing measure for L
- show that the representing measure is actually supported in K

Applications to finite dimensional truncated $K-$MPs

To apply our result to $A = \mathbb{R}[\underline{X}]$, $B = \mathbb{R}[\underline{X}]_{2n+i}$ with $i \in \{0, 1\}$, $K \subseteq \mathbb{R}^d$ non-cmpt, we need to find q s.t.

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Lemma

Let $\mathcal{P} \subseteq \mathbb{R}[X_1, \ldots, X_d]_k$. Then $\exists\, p$ with $deg(p) = \begin{cases} k+1 & \text{when } k \text{ is odd} \\ k+2 & \text{when } k \text{ is even} \end{cases}$ s.t. $p \geq 1$ on $\mathbb{R}^d$ and $\sup_{\underline{y} \in \mathbb{R}^d} \left| \frac{f(\underline{y})}{p(\underline{y})} \right| < \infty$, $\forall f \in \mathcal{P}$.

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$\mathcal{P} = B \implies$ (1) and (4) hold for $q := p \in \mathbb{R}[\underline{X}]_{2n+2}$!

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4. $\forall\, b \in B$, $\sup_{\alpha \in K} \left| \frac{\hat{b}(\alpha)}{\hat{q}(\alpha)} \right| < \infty$

$1 \in \mathbb{R}[\underline{X}]_{2n+i}$ and $\mathbb{R}[\underline{X}]_{2n+i}$ generates $\mathbb{R}[\underline{X}] \Longrightarrow$ (2), (3) hold for all q!

Lemma

Let $\mathcal{P} \subseteq \mathbb{R}[X_1, \ldots, X_d]_k$. Then $\exists\, p$ with $deg(p) = \begin{cases} k+1 & \text{when } k \text{ is odd} \\ k+2 & \text{when } k \text{ is even} \end{cases}$ s.t. $p \geq 1$ on $\mathbb{R}^d$ and $\sup_{\underline{y} \in \mathbb{R}^d} \left| \frac{f(\underline{y})}{p(\underline{y})} \right| < \infty$, $\forall f \in \mathcal{P}$.

$\mathcal{P} = B \Longrightarrow$ (1) and (4) hold for $q := p \in \mathbb{R}[\underline{X}]_{2n+2}$!

Improved version of truncated Riesz-Haviland thm (Curto, Fialkow, 2008)

Let $K \subseteq \mathbb{R}^d$ be non-compact, $n \in \mathbb{N}$, and $L : \mathbb{R}[\underline{X}]_{2n+i} \to \mathbb{R}$ with $i \in \{0,1\}$.

$L(\text{Pos}_{\mathbb{R}[\underline{X}]_{2n+i}}(K)) \subseteq [0, +\infty)$
$+$
$\exists\, \overline{L}$ extension of L to B_p
s.t. $\overline{L}(\text{Pos}_{B_p}(K)) \subseteq [0, \infty)$

$\Longleftrightarrow$ $\exists\, K$–representing measure for L.

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BUT if we balance unboundedness and compactness...

Theorem (Curto, Ghasemi, I. Kuhlmann, 2023)

Let
$A := \mathbb{R}[\underline{X}, \underline{Y}]$ with $\underline{X} \equiv (X_1, \dots, X_d)$ and $\underline{Y} \equiv (Y_1, \dots, Y_s)$,
$K := K_1 \times K_2 \subseteq \mathbb{R}^d \times \mathbb{R}^s$, with K_1 compact in $\mathbb{R}^d$ and K_2 non-compact in $\mathbb{R}^s$
$B := \mathbb{R}[\underline{X}][\underline{Y}]_{2n-1}$
$L : B \to \mathbb{R}$ linear
$p \in \mathbb{R}[\underline{X}][\underline{Y}]_{2n}$

$$\begin{array}{c} L(\mathrm{Pos}_{\mathbb{R}[\underline{X}][\underline{Y}]_{2n-1}}(K)) \subseteq [0, +\infty) \\ + \\ \exists\, \overline{L} \text{ extension of } L \text{ to } B_p \\ \text{s.t. } \overline{L}(\mathrm{Pos}_{B_p}(K)) \subseteq [0, \infty) \end{array} \iff \exists\ K\text{–representing measure for } L.$$

Applications to infinite dimensional truncated $K-$MP:

applications in statistical mechanics $\rightsquigarrow$ truncated $\mathcal{N}(X)-$MP

$$\mathcal{N}(X) := \left\{ \sum_{i \in I} \delta_{x_i} : x_i \in X, \quad \text{with either } |I| < \infty \text{ or } I = \mathbb{N} \right\} \text{ is non-compact in } \mathcal{M}(X)$$

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<u>CASE 1</u>: X compact

Applying our result in the non-compact case for:

- $A := \mathscr{P}$
- $K := \mathcal{N}(X) \subseteq \mathcal{M}(X) \subseteq X(\mathscr{P})$
- $B := \mathscr{P}_2 :=$ polynomials in $\mathscr{P}$ of degree 2
- $q := 1 + \mathbb{1}_X^{\otimes 3}\eta^{\otimes 3} \Rightarrow$ **PROBLEM**: $B_q = span\{B \cup \{q\}\}$ does not generate $\mathscr{P}$!

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Applying our result in the non-compact case for:

- $A := \langle \mathcal{R} \rangle$ where

$$\mathcal{R} := \left\{ f_0 + f_1\eta + f_2\eta^{\otimes 2} + f_3 \mathbb{1}_X^{\otimes 3}\eta^{\otimes 3} : f_0, f_3 \in \mathbb{R}, f_1 \in \mathcal{C}_c(X), f_2 \in \mathcal{C}_c(X^2) \right\}$$

- $K := \mathcal{N}(X) \subseteq \mathcal{M}(X) \subseteq X(\mathscr{P}) \subseteq X(\langle \mathcal{R} \rangle)$
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$\Downarrow$

Theorem (Kuna, Lebowitz, Speer, 2011)

Let X be compact and $L : \mathscr{P}^{(2)} \to \mathbb{R}$ be linear and $\mathcal{N}(X)-$positive.

$$\Big(\exists\ \mathcal{N}(X)-\text{repr. measure for } L \Big) \iff \left(\begin{array}{c} \exists\ R > 0 \text{ s.t. } \forall\ q_{f_0,f_1,f_2,f_3} \in \mathrm{Pos}_{\mathcal{R}}(\mathcal{N}(X)), \\ L(f_0 + f_1\eta + f_2\eta^{\otimes 2}) + f_3 R \geq 0. \end{array} \right)$$

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Applying our result in the non-compact case for:

- $A := \langle \mathcal{R}_\Gamma \rangle$ where $0 < \Gamma \in \mathcal{C}_0(X)$ and
  $$\mathcal{R}_\Gamma := \left\{ f_0 + f_1\eta + f_2\eta^{\otimes 2} + f_3\Gamma^{\otimes 3}\eta^{\otimes 3} : \ f_0, f_3 \in \mathbb{R}, f_1 \in \mathcal{C}_c(X), f_2 \in \mathcal{C}_c(X^2) \right\}$$
- $K = \mathcal{N}(X) \subseteq \mathcal{M}(X) \subseteq X(\mathscr{P}) \subseteq X(\mathscr{P}) \times \mathbb{R} \cong X(\langle \mathcal{R}_\Gamma \rangle)$
- $B := \mathscr{P}_2 :=$ polynomials in $\mathscr{P}$ of degree $2 \subseteq \langle \mathcal{R}_\Gamma \rangle$
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Final remarks and open problems

Further remarks on our results

- they open the way towards a more systematic approach to truncated MP in infinite dimensional settings, for which very few results are known
- they produce new insights also in the finite dimensional case
- they do not yield very concrete solutions but are certainly a first step toward more concrete ones

Open questions

- Can we identify classes of supports or of algebras for which the assumptions of our generalized Riesz-Haviland theorem can be simplified?
- When the starting algebra is a topological one, can we make our criteria more concrete?

Thank you for your attention

For more details see:

R. Curto, M. Ghasemi, M. Infusino, S. Kuhlmann, *The truncated moment problem on unital commutative real algebras*, to appear in Journal of Operator Theory, 2023

R. Curto, M. Infusino, *The realizability problem as a special case of truncated infinite-dimensional moment problem*, https://arxiv.org/abs/2305.10343