

MINICOURSE: INTRODUCTION TO THE ALGEBRAIC AND SPATIAL
SPECTRAL THEORY FOR SEVERAL COMMUTING OPERATORS

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REVIEW FROM MONDAY'S LECTURE

Let \mathcal{B} be a **unital commutative Banach algebra**, and let $M_{\mathcal{B}}$ be the **maximal ideal space** of \mathcal{B} , and $\hat{\cdot}: \mathcal{B} \rightarrow C(M_{\mathcal{B}})$ be the **Gelfand transform**. For $a \in \mathcal{B}$, the joint spectrum of a in \mathcal{B} is

$$\sigma_{\mathcal{B}}(a) = \hat{a}(M_{\mathcal{B}}).$$

Moreover, there exists a homomorphism $f \mapsto f(a)$ from the algebra $H(\sigma_{\mathcal{B}}(a))$ of (germs of) functions **analytic** in a neighborhood of $\sigma_{\mathcal{B}}(a)$ into \mathcal{B} such that

$$1(a) = 1;$$

$$\xi(a) = a, \quad \text{where } \xi(z) := z \text{ for all } z;$$

$$f(\hat{a}) = f \circ \hat{a} \quad (f \in H(\sigma_{\mathcal{B}}(a)))$$

$$\sigma_{\mathcal{B}}(f(a)) = f(\sigma_{\mathcal{B}}(a)) \quad (\text{Spectral Mapping Theorem}).$$

When \mathcal{B} is no longer commutative, say $\mathcal{B} = \mathcal{L}(\mathcal{X})$ (\mathcal{X} a Banach space), one looks for a commutative subalgebra \mathcal{A} containing a .

DEFINITION 1.4. Let K be a bounded subset of \mathbb{C}^n . The **polynomially convex hull** of K is

$$\hat{K} := \{ \lambda \in \mathbb{C}^n : |p(\lambda)| \leq \|p\|_K \text{ for all } p \in \mathbb{C}[z] \}.$$

(Here $\mathbb{C}[z]$ stands for the algebra of polynomials in z .)

Clearly $K \subseteq \hat{K}$. When $K = \hat{K}$ we say that K is **polynomially convex**.

Observe that \hat{K} is always closed.

THEOREM 1.6. (Oka-Weil) Let K be **polynomially convex**, and let Ω be an open subset of \mathbb{C}^n containing K . If $f \in H(\Omega)$, the algebra of functions analytic in Ω , then f is the uniform limit on K of a sequence of polynomials.

THEOREM 1.8. Let $a \subset \mathcal{B}^n$ and assume that a generates \mathcal{B} . Then $\sigma_{\mathcal{B}}(a)$ is polynomially convex, and will be denoted by $\hat{\sigma}(a)$.

To construct an analytic functional calculus for σ_B , we first consider the case of a polydomain D :

$$f(a) := \frac{1}{(2\pi i)^n} \int_{\partial D} f(w) \prod_{i=1}^n (w_i - a_i)^{-1} d_1 \cdot \dots \cdot dw_n.$$

Next, we look at a polynomially convex set K , and use Oka-Weyl to approximate a function analytic in a neighborhood of K by polynomials.

From polynomially convex sets we jump to a special type of polynomially convex sets, the polynomial polyhedra.

THEOREM 1.12. (Oka-Cartan-Waelbroeck)

$$H(D \times \Delta)/I(g) \simeq H(\Omega(p, D, \Delta))$$

THEOREM 1.13. (Waelbroeck) Let $a \in \mathcal{B}^n$ and let Ω be an open subset of \mathbb{C}^n containing $\sigma_R(a)$. Then there exist polydomains D and Δ and a polynomial mapping $p : D \rightarrow \Delta$ such that $\sigma_R(a) \subseteq \Omega(p, D, \Delta) \subseteq \Omega$.

To get the functional calculus for σ_B one needs the following clever observation.

LEMMA 1.14. (**The Arens-Calderón trick**) Let $a \in \mathcal{B}^n$ and let $\Omega \supseteq \sigma_B(a)$. Then there exists $b = (b_1, \dots, b_k) \in \mathcal{B}^k$ such that

$$\Omega \supseteq P_a(\hat{\sigma}(a, b)),$$

where $P_a : \mathbb{C}^{n+k} \rightarrow \mathbb{C}^n$ takes (z, w) to z , and where $\hat{\sigma}(a, b) := (\sigma_B(a, b))^\wedge$.

To further appreciate the value of this result, recall that $\hat{\sigma}$ is the largest of the algebraic spectra. Actually,

$$\sigma_R \subseteq \sigma_B \subseteq \hat{\sigma}.$$

Thus, if we start with $\Omega \supset \sigma_B(a)$, a priori we don't know that $\Omega \supset \sigma_B(a)$. However, as we transition from (a) to $((a, b))$, the likelihood that a would be invertible increases.

The Arens-Calderón trick states that for some b , Ω will contain the projection of $\hat{\sigma}(a, b)$.

THEOREM 1.15. (Shilov, Arens-Calderón, Waelbroeck) There exists a continuous homomorphism $f \mapsto f(a)$ from $H(\sigma_{\mathcal{B}}(a))$ into \mathcal{B} such that

(i) $1(a) = 1$;

(ii) $z_i(a) = a_i$ ($i = 1, \dots, n$); and

(iii) $f(\hat{a}) = f \circ \hat{a}$, for all $f \in H(\sigma_{\mathcal{B}}(a))$.

Consequently,

$$\sigma_{\mathcal{B}}(f(a)) = f(\sigma_{\mathcal{B}}(a)).$$

THEOREM 1.17. (Uniqueness of the Functional Calculus) ([59, III.4.1]) Suppose that

$f \mapsto \tilde{f}(a)$ is another functional calculus satisfying (i), (ii), and

(iv) $\tilde{f}(a) = (f \circ P_a)(a, b)$, for all a, b and $f \in H(\sigma_{\mathcal{B}}(a))$. Then $\tilde{f}(a) = f(a)$ for all

$f \in H(\sigma_{\mathcal{B}}(a))$.

EXAMPLE 2.2. (E. Albrecht) There exist a Banach space, \mathcal{X} , an n -tuple $a \in \mathcal{L}(\mathcal{X})$, and maximal abelian subalgebras \mathcal{A}_1 and \mathcal{A}_2 containing a such that

$$\sigma_{\mathcal{A}_1}(a) \neq \sigma_{\mathcal{A}_2}(a).$$

Given a set X , we shall let $\mathcal{P}(X)$ denote the power set of X . Also, \mathbb{C}^ω will denote the Cartesian product of denumerably many copies of \mathbb{C} .

DEFINITION 2.3. Let \mathcal{B} be a Banach algebra. A **spectral system** for \mathcal{B} is a map

$$\tilde{\sigma} : \bigcup_{n=1}^{\infty} \mathcal{B}_{\text{com}}^n \longrightarrow \mathcal{P}(\mathbb{C}^\omega)$$

such that

- (i) $a \in \mathcal{B} \implies \tilde{\sigma}(a) \neq \emptyset$
- (ii) $a \in \mathcal{B}_{\text{com}}^n \implies \tilde{\sigma}(a) \subseteq \mathbb{C}^n \subseteq \mathbb{C}^\omega$ and
- (iii) $a \in \mathcal{B} \implies \tilde{\sigma}(a)$ is compact.

DEFINITIONS 2.4. Let \mathcal{B} be a Banach algebra. A spectral system $\tilde{\sigma}$ for \mathcal{B} possesses the **projection property** if

$$P_a[\tilde{\sigma}(a, b)] = \tilde{\sigma}(a)$$

for all $a, b \in \mathcal{B}$, where $P_a : \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}^n$, $(z, w) \mapsto z$.

$\tilde{\sigma}$ possesses the **spectral mapping property** for polynomials if $\tilde{\sigma}(p(a)) = p(\tilde{\sigma}(a))$ for every polynomial map $p : \mathbb{C}^n \rightarrow \mathbb{C}^k$ and for every n -tuple $a \in \mathcal{B}_{\text{com}}^n$.

EXAMPLES 2.5. (i) If \mathcal{B} is commutative, $\sigma_{\mathcal{B}}$, σ_R and $\hat{\sigma}$ are all spectral systems. However, only $\sigma_{\mathcal{B}}$ has the projection property.

(ii) For $a \in \mathcal{B}$ we let (a) , $(a)'$ and $(a)''$ denote the unital Banach subalgebras of \mathcal{B} generated by a_1, \dots, a_n , its **commutant** (relative to \mathcal{B}) and its **double commutant**, respectively. We define

$$\hat{\sigma}(a) := \sigma_{(a)}(a) \quad (\text{using the Arens-Calderón trick})$$

$$\sigma''(a) := \sigma_{a''}(a),$$

and

$$\sigma'(a) := \{\lambda \in \mathbb{C}^n : (a - \lambda) \circ (a)' \neq (a)'\}.$$

More generally, if \mathcal{A} is any (closed) subalgebra of \mathcal{B} containing a in its center, we let

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C}^n : (a - \lambda) \circ \mathcal{A} \neq \mathcal{A}\}.$$

Observe that $\hat{\sigma}, \sigma'$ and σ'' are spectral systems **without** the projection property (Słodkowski - Żelazko). Also,

$$\sigma' \subseteq \sigma'' \subseteq \hat{\sigma}.$$

In general, when we refer to an **algebraic spectrum** we mean $\sigma_{\mathcal{A}}$ for some \mathcal{A} .

(iii) The **left spectrum** is defined by

$$\sigma_{\ell}(a) := \{\lambda \in \mathbb{C}^n : \mathcal{B} \circ (a - \lambda) \neq \mathcal{B}\} \quad (a \in \mathcal{B}).$$

Similarly, the **right spectrum** is given by

$$\sigma_r(a) := \{\lambda \in \mathbb{C}^n : (a - \lambda) \circ \mathcal{B} \neq \mathcal{B}\} \quad (a \in \mathcal{B}).$$

The **Harte spectrum** is

$$\sigma_H := \sigma_{\ell} \cup \sigma_r.$$

We easily see that

$$\sigma_{\ell}, \sigma_r \subseteq \sigma_H \subseteq \sigma';$$

moreover, σ_{ℓ} , σ_r and σ_H **all possess the projection property**.

(iv) If $\mathcal{B} = \mathcal{L}(\mathcal{X})$, where \mathcal{X} is a Banach space, the **approximate point spectrum** and **defect spectrum** are defined by

$$\sigma_{\pi}(\mathbf{a}) := \{ \lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \text{ is not jointly bounded below} \}$$

and

$$\sigma_{\delta}(\mathbf{a}) := \{ \lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \text{ is not jointly onto} \}.$$

$\mathbf{a} \subset \mathcal{L}(\mathcal{X})$ is **jointly bounded below** if there exists a positive constant $\epsilon > 0$ such that

$$\sum_{i=1}^n \|a_i x\| \geq \epsilon \|x\| \quad (\text{all } x \in \mathcal{X});$$

\mathbf{a} is **jointly onto** if

$$\sum_{i=1}^n a_i \mathcal{X} = \mathcal{X}.$$

σ_{π} and σ_{δ} are **spectral systems with the projection property**.

(v) Let $\{\sigma_{\pi,k}\}_{k=0}^{n-1}$ and $\{\sigma_{\delta,k}\}_{k=1}^n$ be the **generalized** approximate point and defect spectra introduced by Z. Slodkowski. As shown by Slodkowski, $\sigma_{\pi,k}$ and $\sigma_{\delta,k}$ are spectral systems **with the projection property**.

(vi) For $a \subset \mathcal{L}(\mathcal{X})$ let

$$\sigma_{\Pi}(a, \mathcal{X}) := \sigma(a_1, \mathcal{X}) \times \cdots \times \sigma(a_n, \mathcal{X}),$$

where $\sigma(a_i, \mathcal{X})$ denotes the ordinary spectrum of a_i as an element of $\mathcal{L}(\mathcal{X})$.

σ_{Π} is a spectral system **with the projection property**; however, σ_{Π} does not have the spectral mapping property for polynomial mappings. (**Example**. Let p be a nontrivial idempotent in $\mathcal{L}(\mathcal{X})$. Then $\sigma_{\Pi}(p, p) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, so that

$$\{\lambda_1 + \lambda_2 : (\lambda_1, \lambda_2) \in \sigma_{\Pi}(p, p)\} = \{0, 1, 2\},$$

while $\sigma(2p) = \{0, 2\}$.

(vii) (**Taylor's spectrum**) Let $\Lambda = \Lambda[e] = \Lambda_n[e]$ be the exterior algebra on n generators e_1, \dots, e_n , with identity $e_0 = 1$. Λ is the algebra of forms in e_1, \dots, e_n with complex coefficients, subject to the collapsing property

$$e_i e_j + e_j e_i = 0 \quad (1 \leq i, j \leq n);$$

in particular, $e_i^2 = 0 \quad (i = 1, \dots, n)$.

Let $E_i : \Lambda \rightarrow \Lambda$ be given by $E_i \xi = e_i \xi \quad (i = 1, \dots, n)$.

E_1, \dots, E_n are the so-called **creation operators**. Clearly,

$$E_i E_j + E_j E_i = 0 \quad (1 \leq i, j \leq n).$$

Λ can be regarded as a **Hilbert space**, if we declare

$\{e_{i_1} \cdots e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ as an **orthonormal basis**. Observe that every form

$\xi \in \Lambda$ admits a **unique decomposition**

$$\xi = \alpha + \xi' + \xi''$$

Actually, each E_i is a **partial isometry**, and

$$E_i^* E_j + E_j E_i^* = \delta_{ij} \quad (1 \leq i, j \leq n).$$

(In a C^* -algebra \mathcal{A} , an element v is a partial isometry if $v^* v v^* = v^*$; equivalently, if

$v^* v$ is an orthogonal projection p , i.e., $p^2 = p$ and $p^* = p$.)

For \mathcal{X} a vector space, we let $\Lambda(\mathcal{X}) := \mathcal{X} \otimes_{\mathbb{C}} \Lambda$. For $A \in \mathcal{L}(\mathcal{X})$, we define

$D_A : \Lambda(\mathcal{X}) \rightarrow \Lambda(\mathcal{X})$ by

$$D_A := \sum_{i=1}^n A_i \otimes E_i.$$

Then for $x \in \mathcal{X}$ and $\xi \in \Lambda$, we get

$$D_A^2(x \otimes \xi) = \sum_{i,j=1}^n A_j A_i x \otimes E_j E_i \xi = \sum_{i < j} A_i A_j x \otimes (E_i E_j + E_j E_i) \xi = 0.$$

It follows that $R(D_A) \subseteq N(D_A)$, where R and N denote range and kernel, resp.

We say that A is **nonsingular** on \mathcal{X} if $R(D_A) = N(D_A)$.

When $n = 1$, for instance, A is nonsingular if and only if A is one-to-one and onto, and therefore invertible.

The **Taylor spectrum** of A on \mathcal{X} is

$$\sigma_T(A, \mathcal{X}) := \{\lambda \in \mathbb{C}^n : R(D_{A-\lambda}) \neq N(D_{A-\lambda})\}.$$

One can also construct a cochain complex $K(A, \mathcal{X})$, called the **Koszul complex** associated to A on \mathcal{X} , as follows:

$$K(A, \mathcal{X}) : 0 \longrightarrow \Lambda^0(\mathcal{X}) \xrightarrow{D_A^0} \Lambda^1(\mathcal{X}) \xrightarrow{D_A^1} \Lambda^2(\mathcal{X}) \cdots \Lambda^{n-1}(\mathcal{X}) \xrightarrow{D_A^{n-1}} \Lambda^n(\mathcal{X}) \longrightarrow 0,$$

where $\Lambda^k(\mathcal{X})$ is the tensor product of the vector space of k forms and \mathcal{X} . Then

$$\sigma_T(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : K(A - \lambda, \mathcal{X}) \text{ is not exact}\}.$$

Illustrative Example: Consider the space $C_0^\infty(\mathbb{R}^3)$, the operators $\partial_x, \partial_y, \partial_z$ of partial differentiation, and the de Rham complex associated with them; that is,

$$0 \longrightarrow C_0^\infty(\mathbb{R}^3) \xrightarrow{\nabla} C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \xrightarrow{\text{curl}} C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \xrightarrow{\text{div}} C_0^\infty(\mathbb{R}^3) \longrightarrow 0,$$

with $\text{curl } \nabla = 0$ and $\text{div curl} = 0$.

(J.L. Taylor) $\sigma_T(A, \mathcal{X})$ is compact and nonempty, and $\sigma_T(A, \mathcal{X}) \subseteq \sigma'(A)$. Moreover, σ_T carries an analytic functional calculus (so that in particular σ_T has the projection property).

(J.L. Taylor) If \mathcal{B} is a commutative Banach algebra and $a \in \mathcal{B}$, and if we let L_a denote the n -tuple of left multiplications by the a_i 's, then $\sigma_T(L_a, \mathcal{B}) = \sigma_{\mathcal{B}}(a)$.

Thus, **every algebraic joint spectrum** $\sigma_{\mathcal{A}}(a)$ can be thought of as $\sigma_T(L_a, \mathcal{A})$.

For $n = 2$,

$$D_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & -A_2 & A_1 & 0 \end{bmatrix} \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_1 e_2 \end{matrix}$$

and

$$N(D_A) = [N(A_1) \cap N(A_2) \oplus \{(x_1, x_2) : A_2 x_1 = A_1 x_2\} \oplus \mathcal{X}$$

$$R(D_A) = 0 \oplus \{(A_1 x_0, A_2 x_0) : x_0 \in \mathcal{X}\} \oplus [R(A_1) + R(A_2)].$$

Observe that

$$\sigma_\pi(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : D_{A-\lambda}^0 \text{ is not bounded below}\}$$

and

$$\sigma_\delta(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : D_{A-\lambda}^{n-1} \text{ is not onto}\}$$

so that $\sigma_\pi, \sigma_\delta \subseteq \sigma_T$.

(Spectral Permanence; RC 1982) When \mathcal{B} is a C^* -algebra, say represented as $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$, and $a \in \mathcal{B}$, then

$$\sigma_T(L_a, \mathcal{B}) = \sigma_T(L_a, \mathcal{B}(\mathcal{H})) = \sigma_T(a, \mathcal{H}) \quad (RC, [37]).$$

In particular, if $a \in \mathcal{B} \subseteq \mathcal{C}$, then $\sigma_T(L_a, \mathcal{B}) = \sigma_T(L_a, \mathcal{C})$.

This fact, known as [spectral permanence for the Taylor spectrum](#), is a consequence of the following result:

L_a is nonsingular on \mathcal{B} if and only if $D_a + D_{a^*}^t$ is invertible (as a matrix over \mathcal{B}),

where $a^* = (a_1^*, \dots, a_n^*)$ and t denotes the transpose of a matrix.

As a consequence, we also see that $D_a + D_{a^*}^t$ determines the nonsingularity of a on \mathcal{H} . It is because of this that we shall usually refer to nonsingularity as invertibility when a is an n -tuple of C^* -algebra elements.

In the special case of a normal k -tuple $N \equiv (N_1, \dots, N_n)$, we know that $C^*(N)$ is abelian, and

$$\sigma_{C^*(N)}(N) = \sigma_T(N);$$

that is, the Taylor spectrum agrees with the algebraic joint spectrum.

It is also the case that for normal k -tuples on Hilbert space,

$$\sigma_\ell(N) = \sigma_r(N) = \sigma_H(N) = \sigma_{\pi,i}(N) = \sigma_{\delta,j}(N) = \sigma_T(N) \quad (0 \leq i \leq k-1, 1 \leq j \leq k).$$

More can be said:

$\sigma_{C^*(N)}(N)$ is intimately related to $\sigma_{C^*(N)}(N, N^*)$.

For, $C^*(N)$ is generated, as a **Banach algebra**, by N and N^* . Therefore, $\sigma_{C^*(N)}(N, N^*)$ is polynomially convex. Moreover, the map

$$\Phi : \sigma_{C^*(N)}(N) \longrightarrow \sigma_{C^*(N)}(N, N^*) \quad (\lambda \mapsto (\lambda, \bar{\lambda}))$$

is a homeomorphism. (Recall that if $\varphi \in M_{C^*(N)}$, then $\varphi(N^*) = \overline{\varphi(N)}$.)

Now, any compact nonempty set can be regarded as the spectrum of a normal operator. Thus, although $\sigma_{C^*(N)}(N)$ is just an arbitrary compact nonempty set, $\sigma_{C^*(N)}(N, N^*)$ is always polynomially convex.

We have, therefore, an operator theoretic proof of the following fact:

(RC) Given a compact nonempty set $K \subset \mathbb{C}^n$, the set $\{(\lambda, \bar{\lambda}) : \lambda \in K\}$ is polynomially convex in \mathbb{C}^{2n} .

When \mathcal{X} is a finite dimensional space and $A \in \mathcal{L}(\mathcal{X})$, then

$$\begin{aligned}\sigma_p(A, \mathcal{X}) &= \sigma_\ell(A, \mathcal{X}) = \sigma_\pi(A, \mathcal{X}) = \sigma_r(A, \mathcal{X}) = \sigma_\delta(A, \mathcal{X}) \\ &= \sigma_T(A, \mathcal{X}) = \sigma'(A) = \sigma''(A) = \hat{\sigma}(A)\end{aligned}$$

The reason for this lies in the fact that A_1, \dots, A_n can be **simultaneously triangularized**

as

$$A_i = \begin{bmatrix} \lambda_i^{(1)} & * & \cdots & * \\ 0 & \lambda_i^{(2)} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i^{(k)} \end{bmatrix} \quad (i = 1, \dots, n),$$

and it follows that all of the above spectra equal $\{\lambda^{(1)}, \dots, \lambda^{(k)}\}$.

To conclude this preliminary discussion of σ_T , we list another important property:

(Taylor) Let \mathcal{X} be a Banach space, let $A \in \mathcal{L}(\mathcal{X})$, and let \mathcal{Y} be a subspace of \mathcal{X} invariant under A (i.e., $A_i\mathcal{Y} \subseteq \mathcal{Y}$, $(i = 1, \dots, n)$). Then union of any two of the sets $\sigma_T(A, \mathcal{X})$, $\sigma_T(A, \mathcal{Y})$ and $\sigma_T(A, \mathcal{X}/\mathcal{Y})$ contains the third.

(RC) Put differently, if $A \in \mathcal{L}(\mathcal{X})$, $B \in \mathcal{L}(\mathcal{Y})$, and if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{Y})$$

then

$$\sigma_T \left[\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \mathcal{X} \oplus \mathcal{Y} \right] \subseteq \sigma_T(A, \mathcal{X}) \cup \sigma_T(B, \mathcal{Y}).$$

To end this section, we shall present a table summarizing the various notions of joint spectra and their properties.

Spectral Systems on \mathcal{X}

(\mathcal{B} unital, commutative Banach algebra; \mathcal{X} Banach space)

	$\sigma_{\mathcal{B}}$	$\sigma_{\pi}, \sigma_{\delta}$	σ_{ℓ}, σ_r	σ_H	σ_T	σ'	σ''	σ_R	$\hat{\sigma}$	σ_{Π}
Proj. Prop.	✓	✓	✓	✓	✓					✓
SMP for Polyn. Maps	✓	✓	✓	✓	✓					
Funct. Calculus	✓				✓					
SMP for Anal. Fctns.	✓				✓					

SECTION 3. THE SPECTRAL MAPPING THEOREM

We will study the **functional representation** of spectral systems with the projection property. W. Żelazko defined a **subspectrum** on a Banach algebra \mathcal{B} as a spectral system $\tilde{\sigma}$ on \mathcal{B} possessing the spectral mapping property for polynomials and such that $\tilde{\sigma}(a) \subseteq \prod_{i=1}^n \sigma_{\mathcal{B}}(a_i)$ for all $a \in \mathcal{B}$. Żelazko then showed that subspectra admit a functional representation in terms of the Gelfand transform, as follows:

If \mathcal{A} is a **maximal abelian subalgebra** of \mathcal{B} , then there exists a compact nonempty subset $M_{\tilde{\sigma}}(\mathcal{A})$ of $M_{\mathcal{A}}$ such that

$$\tilde{\sigma}(a) = \hat{a}(M_{\tilde{\sigma}}(\mathcal{A}))$$

for all $a \in \mathcal{A}$. As a consequence, $\tilde{\sigma}(a) \subseteq \hat{a}(M_a) = \sigma_{\mathcal{A}}(a)$. (Żelazko proved this inclusion first, and then used it in the proof of the functional representation).

Our approach to this circle of ideas will be the following:

- (i) We start with a spectral system $\tilde{\sigma}$ on a Banach space \mathcal{X} possessing the projection property and such that $\tilde{\sigma} \subseteq \sigma_{\mathcal{B}}$ for **some** commutative Banach algebra \mathcal{B} (see Definition 3.1 below).
- (ii) We find the **functional representation** for $\tilde{\sigma}$.
- (iii) We prove the **uniqueness** of $M_{\tilde{\sigma}}(\mathcal{B})$.

Our hypotheses are **weaker** than those used by Żelazko, but we will still recover the same results.

DEFINITION 3.1. Let \mathcal{X} be a Banach space and let $\tilde{\sigma}$ be a spectral system on \mathcal{X} possessing the projection property. Let \mathcal{B} be a commutative Banach algebra acting on \mathcal{X} , i.e., there exists a map $\Phi : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{X})$ (alternatively, assume that \mathcal{X} is a Banach \mathcal{B} -module). Assume that

$$\tilde{\sigma}(\Phi(a)) \subseteq \sigma_{\mathcal{B}}(a) \quad (\text{for all } a \in \mathcal{B}).$$

Then

$$M_{\tilde{\sigma}} \equiv M_{\tilde{\sigma}}(\mathcal{B}, \mathcal{X}, \Phi) := \bigcap_{a \in \mathcal{B}} \hat{a}^{-1}[\tilde{\sigma}(\Phi(a))] \subseteq M_{\mathcal{B}}.$$

LEMMA 3.2. $M_{\tilde{\sigma}}$ is a compact and nonempty subset of $M_{\mathcal{B}}$.

The proof uses the finite intersection property.

THEOREM 3.3. Let $\mathcal{X}, \mathcal{B}, \Phi$ and $\tilde{\sigma}$ be as before. If $a \in \mathcal{B}$ then

$$\tilde{\sigma}(\Phi(a)) = \hat{a}(M_{\tilde{\sigma}})$$

Moreover, $M_{\tilde{\sigma}}$ is the only compact nonempty subset of $M_{\mathcal{B}}$ having this property.

REMARKS 3.4. (i) The importance of the functional representation of $\tilde{\sigma}$ is that one recovers the formula “ $\sigma(a) = \text{range } \hat{a}$ ”; in the noncommutative case, however, one must restrict \hat{a} to a certain **subset** of a maximal ideal space. The uniqueness of $M_{\tilde{\sigma}}$ is of significance in some applications (see Corollary 3.11 below). (ii) For the special case of σ_T , J.L. Taylor found a corresponding version of Theorem 3.3. We basically follow his outline for our proof below.

Recall:

THEOREM 3.3. Let x, \mathcal{B}, Φ and $\tilde{\sigma}$ be as before. If $a \in \mathcal{B}$ then

$$\tilde{\sigma}(\Phi(a)) = \hat{a}(M_{\tilde{\sigma}})$$

Moreover, $M_{\tilde{\sigma}}$ is the only compact nonempty subset of $M_{\mathcal{B}}$ having this property.

Proof of Theorem 3.3.

$$\hat{a}(M_{\tilde{\sigma}}) \subseteq \hat{a}\left(\hat{a}^{-1}[\tilde{\sigma}(\Phi(a))]\right) \subseteq \tilde{\sigma}(\Phi(a)).$$

Conversely, let $\lambda \in \tilde{\sigma}(\Phi(a))$. We claim that there exists $\varphi \in M_{\tilde{\sigma}}$ such that $\varphi(a) = \lambda$.

Consider the family $\{M_b \cap \hat{a}^{-1}(\lambda)\}_{b \in \mathcal{B}}$. One shows that

$$\bigcap_{b \in \mathcal{B}} (M_b \cap \hat{a}^{-1}(\lambda)) \neq \emptyset$$

provided $M_b \cap \hat{a}^{-1}(\lambda) \neq \emptyset$ for all $b \in \mathcal{B}$.

To prove the latter fact, consider $(a, b) \in \mathcal{B}$. Then

$$P_a[\tilde{\sigma}(\Phi(a), \Phi(b))] = \tilde{\sigma}(\Phi(a)),$$

so that for some $\mu \in \tilde{\sigma}(\Phi(b))$,

$$(\lambda, \mu) \in \tilde{\sigma}(\Phi(a), \Phi(b)) \subseteq \sigma_{\mathcal{B}}(a, b),$$

and therefore there exists $\varphi \in M_{\mathcal{B}}$ such that $\varphi(a) = \lambda$, and $\varphi(b) = \mu$.

In particular, $\varphi \in M_b \cap \hat{a}^{-1}(\lambda)$.

We have thus shown that

$$\tilde{\sigma}(\Phi(a)) \subseteq \hat{a}(M_{\tilde{\sigma}}).$$

To prove that $M_{\tilde{\sigma}}$ is unique, one uses the fact that $\{\hat{a} : a \in \mathcal{B}\}$ separates points in $M_{\mathcal{B}}$.

COROLLARY 3.5. (Spectral Mapping Theorem for Polynomial Mappings) Let $\mathcal{X}, \mathcal{B}, \Phi$ and $\tilde{\sigma}$ be as before. Let $a \in \mathcal{B}$ and let $p : \mathbb{C}^n \rightarrow \mathbb{C}^k$ be a polynomial mapping. Then

$$\tilde{\sigma}[\Phi(p(a))] = p[\tilde{\sigma}(\Phi(a))]$$

Proof.

$$\tilde{\sigma}[\Phi(p(a))] = [p(a)]^{\wedge}(M_{\tilde{\sigma}}) = (p \circ \hat{a})(M_{\tilde{\sigma}}) = p[\hat{a}(M_{\tilde{\sigma}})] = p[\tilde{\sigma}(\Phi(a))].$$

REMARK 3.6. What the corollary really shows is that whenever the formula $f(a) = f \circ \hat{a}$ holds then the spectral mapping theorem also holds. For instance, that is the case for $f \in H(\sigma_B(a))$.

The following special case of Corollary 3.5 encompasses the basic ideas of this section.

COROLLARY 3.7. Let $\tilde{\sigma}$ be a spectral system on a Banach space \mathcal{X} . Assume that $\tilde{\sigma}$ possesses the projection property and that $\tilde{\sigma}(a) \subseteq \hat{\sigma}(a)$ for all $a \in \mathcal{L}(\mathcal{X})$. Then $\tilde{\sigma}$ possesses the spectral mapping property for polynomial mappings.

APPLICATION 3.8. (RC) Let $\tilde{\sigma}$ be a spectral system on \mathcal{X} possessing the projection property. Assume that $\tilde{\sigma}(A) = \sigma(A)$ for every $A \in \mathcal{L}(\mathcal{X})$. Then

$$\tilde{\sigma} \circ \tilde{\sigma} = \sigma_T \circ \sigma_T;$$

in other words,

$$\tilde{\sigma}(A) \circ \tilde{\sigma}(B) = \sigma_T(A) \circ \sigma_T(B)$$

for every $A, B \in \mathcal{L}(\mathcal{X})_{\text{com}}^{(n)}$. Moreover, $\tilde{\sigma}(A)$ and $\sigma_T(A)$ have identical rationally convex hulls, namely $\sigma_R(A)$, for all $A \in \mathcal{L}(\mathcal{X})$.

APPLICATION 3.9. (RC) For $A \equiv (A_1, \dots, A_n), B \equiv (B_1, \dots, B_n) \subset \mathcal{L}(\mathcal{X})$, let

$$\mathcal{E}(T) := \sum_{i=1}^n A_i T B_i \quad (T \in \mathcal{L}(\mathcal{X}))$$

be the [elementary operator](#) associated with the n -tuples A and B . Then

$$\sigma(\mathcal{E}) = \sigma_T(A) \circ \sigma_T(B).$$

To conclude this section we shall establish a functorial property of $M_{\bar{\sigma}}$.

SECTION 4. THE PROJECTION PROPERTY FOR $\sigma_{\mathcal{T}}$

We shall devote this section to a proof of the projection property for the Taylor spectrum. We shall present a simplification of F.-H. Vasilescu's proof, making more evident the use of [nilpotent operators of index 2](#) and pointing out the relevance of [Voiculescu's Theorem](#) in this context.

Vasilescu's proof, which basically follows the outline of J. Bunce's proof of the projection property for the left spectrum, makes implicit use of Voiculescu's Theorem. Our proof shows explicitly the role played by that theorem.

We shall begin with a series of lemmas.

LEMMA 4.1. Let T be an operator on a Hilbert space \mathcal{H} , and assume that $T^2 = 0$.

The following statements are equivalent:

- (i) $R(T) = N(T)$.
- (ii) $R_T := T + T^*$ is invertible.
- (iii) $\square_T := T^*T + TT^*$ is invertible.

\square is called the [Laplacian](#) of T . When (i) holds, we say that T is [exact](#).

Proof. (i) \Rightarrow (ii) Let $y \in \mathcal{H}$. Since $R(T) = N(T)$, we can write $y = Tx + T^*z$, where $x \perp N(T)$ and $z \perp N(T^*)$. Then $y = (T + T^*)(x + z)$, since $x \in R(T^*)$ and $z \in R(T)$. Thus $T + T^*$ is onto, i.e., $T + T^*$ is invertible.

(ii) \Rightarrow (iii): $R_T^2 = (T + T^*)^2 = T^*T + TT^* = \square_T$.

(iii) \Rightarrow (i): Let $x \in N(T)$. Then $u_T x = T^*Tx + TT^*x = TT^*x$, so that $x = a_T^{-1}T^*x$.

Since \square_T commutes with T^* , we have $x = T^*a_T^{-1}x \in R(T)$. Therefore $N(T) \subseteq R(T)$.

The other inclusion follows from the condition " $T^2 = 0$."

COROLLARY 4.2. Let $A \in \mathcal{L}(\mathcal{H})$. Then $\sigma_T(A, \mathcal{H})$ is a closed subset of \mathbf{C}^n .

Proof. The map $A \mapsto R_{D_A}$ is real analytic in the norm topology. Since the set of invertible operators in $\mathcal{L}(\mathcal{H})$ is open, it follows that

$$\{\lambda \in \mathbf{C}^n : R(D_{A-\lambda}) = N(D_{A-\lambda})\}$$

is an open subset of \mathbf{C}^n .

The following result contains a new formula for the orthogonal projection onto the range of T , in the case when T is exact.

LEMMA 4.3. Let $T \in \mathcal{L}(\mathcal{H})$ with $T^2 = 0$. Then

(i) $\overline{R(T)} = N(T) \Leftrightarrow N(\square_T) = 0$.

(ii) R_T invertible $\Rightarrow P_{R(T)} = TR_T^{-1}$. (Here $P_{R(T)}$ is the orthogonal projection of \mathcal{H} onto $R(T)$.)

LEMMA 4.4. Let $T \in \mathcal{L}(\mathcal{X})$ with $T^2 = 0$. If there exists $S \in \mathcal{L}(\mathcal{X})$ such that $TS + ST = I$, then $N(T) = R(T)$.

LEMMA 4.5. (J.L. Taylor) Let $A \in \mathcal{L}(\mathcal{X})$. Then $\sigma_T(A, \mathcal{X}) \subseteq \sigma'(A)$.

Proof. Assume A is invertible relative to $(A)'$, i.e., there exists $B \in (A)'$ such that $A \circ B = I$. Let

$$S := \sum_{i=1}^n B_i E_i^*.$$

Then S satisfies the equation

$$\begin{aligned} D_A S + S D_A &= \sum_{i,j} (A_i B_j E_i E_j^* + B_j A_i E_j^* E_i) \\ &= \sum_{i,j} A_i B_j (E_i E_j^* + E_j^* E_i) \\ &= \sum_{i=1}^n A_i B_i = I. \end{aligned}$$

By the previous lemma, $N(D_A) = R(D_A)$, so that A is Taylor invertible.

In the next lemma, we establish that for $(A, B) \in \mathcal{L}(\mathcal{H})$, the invertibility of A implies the invertibility of (A, B) ; as a result, $\sigma_T((A, B), \mathcal{H}) \subseteq \sigma_T(A, \mathcal{H}) \times \sigma_T(B, \mathcal{H})$.

LEMMA 4.6. Let $A, B \in \mathcal{L}(\mathcal{H})$ and assume that $N(D_A) = R(D_A)$ and that $(A, B) \in \mathcal{L}(\mathcal{H})$. Then $N(D_{(A,B)}) = R(D_{(A,B)})$.

Proof. Observe that $D_{(A,B)} = D_A + \tilde{D}_B$, where $\tilde{D}_B := \sum_{j=1}^m B_j E_{n+j}$.

We shall show that if $\eta \in \Lambda_{n+m}(\mathcal{H})$ and $(D_A + \tilde{D}_B)\eta = 0$ then $\eta = (D_A + \tilde{D}_B)\xi$, for some $\xi \in \Lambda_{n+m}(\mathcal{H})$. (We shall actually give a formula for ξ in terms of η). It suffices to assume that η is homogeneous of degree p ($0 \leq p \leq n+m$), i.e., η is a linear combination of forms of degree p in e_1, \dots, e_{n+m} ; this is because $D_A + \tilde{D}_B$ maps homogeneous p -forms to homogeneous $(p+1)$ -forms. Now write $\eta = \eta_0 + \dots + \eta_p$, where η_i has degree i in e_{n+1}, \dots, e_{n+m} . We get:

$$\left\{ \begin{array}{l} D_A \eta_0 = 0 \\ D_A \eta_1 + \tilde{D}_B \eta_0 = 0 \\ \dots \quad \cdot \quad \dots \\ D_A \eta_p + \tilde{D}_B \eta_{p-1} = 0 \\ \tilde{D}_B \eta_p = 0. \end{array} \right.$$

Standard manipulations with these equations and the well-known properties of kernel and range for operators on Hilbert space lead to the following formula for the components of ξ :

$$(*) \xi_{p-1} = \sum_{k=0}^{p-1} (-1)^k R_A^{-1} (\tilde{D}_B R_A^{-1})^k \eta_{p-1-k}.$$

Then

$$(D_A + \tilde{D}_B) (\xi_0 + \cdots + \xi_{p-1}) = \eta,$$

as desired.

To prove the remaining half of the projection property (i.e., that

$\sigma_T(A, \mathcal{H}) \subseteq P_A \sigma_T((A, B), \mathcal{H})$, one must show that whenever $R(D_A) \neq N(D_A)$ then

$R(D_{(A, B-\mu)}) \neq N(D_{(A, B-\mu)})$ for at least one $\mu \in \mathbb{C}^m$.

Suppose for simplicity that \mathcal{H} is finite dimensional, $A = (A_1)$ and $B = (B_1)$. If A is not invertible then $N(A) \neq 0$; since B maps $N(A)$ into $N(A)$, we can look at $B|_{N(A)}$ and find an eigenvalue $\mu \in \mathbb{C}$. Then $(0, \mu)$ is a joint eigenvalue for (A, B) , which shows that $(0, \mu) \in \sigma_T((A, B), \mathcal{H})$.

The general case is of course much more complicated than this simple example, but the idea of the proof is basically the same. One finds, however, that there are two distinctive cases whenever A is not invertible on \mathcal{H} :

either $\overline{R(D_A)} \neq N(D_A)$ or $\overline{R(D_A)} = N(D_A)$.

In the former case, one can imitate the above construction with $N(A)$ replaced by $N(D_A) \ominus R(D_A)$. In the latter case, however, there is not enough room between $N(D_A)$ and $R(D_A)$!

It is to handle this case that one needs Voiculescu's Theorem.

First, we need a definition.

DEFINITION 4.7. Let $A, \tilde{A} \in \mathcal{L}(\mathcal{H})$. We say that A is approximately unitarily equivalent to \tilde{A} (in symbols, $A \simeq_a \tilde{A}$) if there exists a sequence $\{U_k\}_{k=1}^\infty$ of unitary operators on \mathcal{H} such that $U_k A U_k^* := (U_k A_1 U_k^*, \dots, U_k A_n U_k^*) \longrightarrow \tilde{A}$, as $k \longrightarrow \infty$ (naturally, this implies that $\tilde{A} \in \mathcal{L}(\mathcal{H})$).

\simeq_a is an equivalence relation on commuting n -tuples of operators on \mathcal{H} , which preserves all spectral properties, as the reader can easily verify. In particular, $\sigma_T(A) = \sigma_T(\tilde{A})$ whenever $A \simeq_a \tilde{A}$.

The next lemma, which slightly generalizes the Gap Theorem proved by RC and L.

Fialkow, states that if A is not invertible on \mathcal{H} , one can always assume

$\overline{R(D_A)} \neq N(D_A)$, up to approximate unitary equivalence. (For $n = 1$, this fact was first proved by C. Apostol).

LEMMA 4.8. Let $(A, B) \in \mathcal{L}(\mathcal{H})$. If $R(D_A) \neq \overline{R(D_A)} = N(D_A)$ then there exists $(\tilde{A}, \tilde{B}) \simeq_a (A, B)$ such that $\overline{R(D_{\tilde{A}})} \neq N(D_{\tilde{A}})$.

Proof. Recall that

$$N(\square_A) = N(D_A) \cap N(D_A^*) = \{0\},$$

so that \square_A cannot have closed range (being a self-adjoint operator). Therefore, there exists a projection $Q \in \mathcal{L}(\Lambda_n(\mathcal{H}))$ such that Q is **not compact** but $\square_A Q$ is a **compact operator**.

Let \mathcal{A} be the C^* -algebra generated by $A_1, \dots, A_n, B_1, \dots, B_m, I$ and $\{Q_{ij}\}_{i,j=1}^{2^n}$, and let $\rho : \pi(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{K})$ be a faithful $*$ -representation of $\pi(\mathcal{A})$ on a separable Hilbert space \mathcal{K} , where π is the **Calkin map**. By Voiculescu's Theorem,

$$\text{id}_{\mathcal{A}} \simeq_a \text{id}_{\mathcal{A}} \oplus (\rho \circ \pi)$$

so that $(A, B) \simeq_a (A', B') \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$. Let

$$Q' = 0 \oplus (\rho \circ \pi)(Q)$$

Then $\square_{A'} Q' = 0$ and $Q' \neq 0$, so that $\overline{R(D_{A'})} \neq N(D_{A'})$. Since $\mathcal{H} \oplus \mathcal{K}$ is isometrically isomorphic to \mathcal{H} , we obtain $(\tilde{A}, \tilde{B}) \subset \mathcal{L}(\mathcal{H})$, $(\tilde{A}, \tilde{B}) \simeq_a (A, B)$ and $N(D_{\tilde{A}}) \neq \overline{R(D_{\tilde{A}})}$.

THEOREM 4.9 (J.L. Taylor). Let $(A, B) \subset \mathcal{L}(\mathcal{H})$. Then

$$P_A(\sigma_T((A, B), \mathcal{H})) = \sigma_T(A, \mathcal{H}).$$

PROOF OF THEOREM 4.9 (Conclusion). Since $R(Q(\tilde{B} - \mu)Q) \neq R(Q)$, Lemma 4.12

(iii) says that $R(C) \neq R(P)$, where $P = P_{N(D_{A-\lambda})}$ and $C = PD_{A-\lambda} + P(\tilde{B} - \mu)P'$. By

Lemma 4.11, this then implies that $N(D_{A-\lambda}) \neq R(D_{A-\lambda}) + (\tilde{B} - \mu)N(D_{A-\lambda})$, and by

Lemma 4.10, we must therefore have $N(D_{A-\lambda, B-\mu}) \neq R_{A-\lambda, B-\mu}$, so that

$(\lambda, \mu) \in \sigma_T((A, B), \mathcal{H})$. This completes the proof.

SECTION 5. THE ANALYTIC FUNCTIONAL CALCULUS FOR σ_T

J.L. Taylor developed the analytic functional calculus for σ_T . He based the construction on a Cauchy-Weil integral formula in homology. Taylor later simplified the construction, but it was F.-H. Vasilescu who found a much simpler version (for the case of n -tuples of elements in a C^* -algebra). We shall present here an outline of Vasilescu's construction of the functional calculus.

The main problem is to make sense out of $f(a)$, where a is a commuting n -tuple and f is analytic in a neighborhood of $\sigma_T(a)$.

Of course, if f is analytic in a neighborhood of $\sigma_{\mathcal{B}}(a)$, where \mathcal{B} is some commutative Banach algebra containing a , $f(a)$ should agree with Waelbroeck's and Arens and Calderón's " $f(a)$." In particular, if f is analytic in a polydisc D containing $\sigma_{\mathcal{T}}(a)$, then $f(a)$ must be given by the formula

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D} f(w) \prod_{i=1}^n \frac{1}{w_i - a_i} dw_i,$$

where ∂D denotes the distinguished boundary of D .

We will discuss Vasilescu's construction in the case of Hilbert spaces. By spectral permanence, there is no loss of generality in doing so. We first need to recall some facts about the Martinelli kernel.

DEFINITION 5.1. Let Ω be a bounded and open subset of \mathbb{C}^n and assume that the boundary of Ω , $\partial\Omega$, is C^1 . Let $f \in C^1(\Omega)$. We define

$$\begin{aligned}\partial f &:= \frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f}{\partial z_n} dz_n \\ \bar{\partial} f &:= \frac{\partial f}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial f}{\partial \bar{z}_n} d\bar{z}_n\end{aligned}$$

and

$$df := \partial f + \bar{\partial} f$$

More generally, if ξ is a $C^\infty(\Omega)$ -valued (k, ℓ) -form in $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$, say

$$\xi = f dz_{i_1} \wedge \cdots \wedge dz_{i_k} \wedge d\bar{z}_j \wedge \cdots \wedge d\bar{z}_\ell,$$

we let

$$\partial\xi := \partial f \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_\ell}$$

and similarly for $\bar{\partial}$ and d . It follows that each of ∂ , $\bar{\partial}$ and d gives rise to operators on $C^\infty(\Omega) \otimes \Lambda[dz, d\bar{z}]$ such that $\partial^2 = \bar{\partial}^2 = d^2 = 0$.

The Koszul complex associated with d is called the de Rham complex over Ω ; the one associated with $\bar{\partial}$ is the $\bar{\partial}$ -complex. For us, the $\bar{\partial}$ -complex will be the restriction of the $\bar{\partial}$ -complex to $C^\infty(\Omega)\Lambda[d\bar{z}]$, that is $\bar{\partial}$ acting on $(0, \ell)$ forms. (This complex is sometimes called the Dolbeault complex.) Also, $H(\Omega) = \{f \in \Lambda_0(C^\infty(\Omega)) : \bar{\partial}f = 0\}$.

DEFINITION 5.2. For $z \neq 0, z \in \mathbb{C}^n$, let

$$M(z) := (n-1)! \sum_{j=1}^n (-1)^{j-1} \frac{\bar{z}_j}{\|z\|^{2n}} \bigwedge_{\substack{1 \leq k \leq n \\ k \neq j}} d\bar{z}_k,$$

where $\|z\|^2 := \sum_{i=1}^n |z_i|^2$. M is the [Martinelli kernel](#).

THEOREM 5.3. (Bochner-Martinelli formula) ([74, Theorem 1.14]). Let $f \in C^1(\bar{\Omega})$.

Then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega} f(w) M(w-z) \wedge dw \\ - \frac{1}{(2\pi i)^n} \int_{\Omega} \bar{\partial}f(w) \wedge M(w-z) \wedge dw \quad (z \in \Omega)$$

(generalization of Cauchy's formula).

REMARKS 5.5. (i) M is an $(n-1)$ -differential form in $d\bar{z}_1, \dots, d\bar{z}_n$ such that $\bar{\partial}M = 0$.

(ii) If $n = 1$, $M(z) = \frac{1}{z}$, the **Cauchy kernel**.

(iii) If $n \geq 2$, M is **not analytic** (in the sense that its C^∞ -coefficients are not analytic functions) and, therefore, M **does not "create" analytic functions** (as the Cauchy kernel does), but **it nevertheless reproduces analytic functions**.

We now give the definition of the Martinelli kernel, which generalizes the resolvent $(z - A)^{-1}$, used when $n = 1$.

DEFINITION 5.9. Let $\beta \in A(\Omega, \mathcal{L}(\mathcal{H}))$ and let $z \notin \sigma_T^\Omega(\beta, \mathcal{H})$. We let

$$M(\beta)(z) := R_{\beta(z)}^{-1} (\bar{\partial}_z R_{\beta(z)}^{-1})^{n-1} E|_{\mathcal{H} \otimes \Lambda^0}.$$

$M(\beta)$ is the **Martinelli kernel associated with β** .

THEOREM 5.18 (Taylor). Let $A \in \mathcal{L}(\mathcal{H})$ and let $\Omega \supseteq \sigma_T(A, \mathcal{H})$. Let Ω' be such that $\sigma_T(A, \mathcal{H}) \subseteq \Omega' \subseteq \bar{\Omega} \subseteq \Omega$ and Ω' has C^1 -boundary. Then the map

$$H(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$$

$$f \mapsto f(A) := v_\beta(f)$$

(where $\beta(z) := z - A$) is a unital continuous homomorphism such that:

(i) $z_i(A) = A_i \quad (i = 1, \dots, n)$

(ii) $f(A) \in (A)''$

(iii) If $f = 0$ on a neighborhood of $\sigma_T(A, \mathcal{H})$ then $f(A) = 0$.

(iv) For every relatively compact open subset Φ of Ω containing $\sigma_T(A, \mathcal{H})$ there exists a constant $C_\Phi > 0$ such that

$$\|f(A)\| \leq C_\Phi \sup\{|f(z)| : z \in \Phi\},$$

for all $f \in H(\Omega)$.

We shall now state the uniqueness of the analytic functional calculus for σ_T .

THEOREM 5.20. (M. Putinar) Let $A \subset \mathcal{L}(\mathcal{H})$ and let $\Omega \supseteq \sigma_T(A, \mathcal{H})$. Then there exists a **unique continuous unital $H(\Omega)$ -functional calculus extending the polynomial calculus and satisfying the spectral mapping theorem.**

COROLLARY 5.21. (W. Zame) For \mathcal{B} a commutative unital Banach algebra with identity, the Shilov-Arens-Calderón-Waelbroeck functional calculus is unique (subject to the conditions (i), (ii) and (iii) of Theorem 1.15).

Proof. $\sigma_{\mathcal{B}}(a) = \sigma_T(L_a, \mathcal{B})$, for all $a \in \mathcal{B}$.

REMARK 5.22. The importance of Putinar's uniqueness result lies in the fact that no matter which construction we use for $f(A)$ we are bound to obtain the same element of $\mathcal{L}(\mathcal{H})$. For instance, if $f \in H(\sigma_R(A))$ (recall that $\sigma_R(A)$ is the rational spectrum of A), then $f(A)$ has a priori three meanings: one in the sense of Arens-Calderón-Waelbroeck, one in the sense of Taylor, and one in the sense of Vasilescu. As it turns out, they all agree and the value $f(A)$ can be computed in whichever way.

We shall now give some applications of the results in this section.

APPLICATION 5.24. (The Shilov Idempotent Theorem for σ_T) ([105, Theorem 4.9])

Let $A \in \mathcal{L}(\mathcal{H})$ and assume that

$$\sigma_T(A, \mathcal{H}) = K_1 \dot{\cup} K_2,$$

where K_1 and K_2 are two nonempty disjoint compact subsets of $\sigma_T(A, \mathcal{H})$. Then there exist subspaces \mathcal{M}_1 and \mathcal{M}_2 of \mathcal{H} such that

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\},$$

$$\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{H},$$

$$A\mathcal{M}_i \subseteq \mathcal{M}_i \quad (i = 1, 2)$$

and

$$\sigma_T(A|_{\mathcal{M}_i}, \mathcal{M}_i) = K_i \quad (i = 1, 2).$$

APPLICATION 5.27. (Fialkow) The **Harte spectrum** σ_H **does not carry an analytic functional calculus**. Consider the following pair of operators on $\mathcal{H} = \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$, where $\ell^2(\mathbb{N}_0)$ denotes the Hilbert space of square summable sequences:

$$A_1 = \begin{bmatrix} U_+^* & & & & \\ & U_+^* & & & \\ & & U_+^* & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & & & & \\ U_+^* & 0 & & & \\ & I & 0 & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}$$

where U_+ is the unilateral shift on $\ell^2(\mathbb{N}_0)$. Suppose $P \in \mathcal{L}(\mathcal{H})$, $P^2 = P$ and $PA_i = A_iP$ ($i = 1, 2$). A matricial computation shows that P must be 0 or I . Therefore $\sigma_T(A, \mathcal{H})$ is **connected** (by Application 5.24); indeed, $\sigma_T(A, \mathcal{H}) = \overline{\mathbb{D}}^2$. However,

$$\sigma_H(A, \mathcal{H}) = \{(0, 0)\} \cup (\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T}),$$

so that $(0, 0)$ is isolated in $\sigma_H(A, \mathcal{H})$. Therefore, $(0, 0)$ **cannot be excised by means of an analytic functional calculus**.

Let \mathcal{A} be a unital C^* -algebra, let L_a, R_b denote the *left* and *right* multiplication operators induced by $a, b \in \mathcal{A}$ (i.e., $L_a(x) := ax$, $R_b(x) := xb$, $x \in \mathcal{A}$), and set $M_{a,b} := L_a R_b$.

An ideal \mathcal{I} in \mathcal{A} is said to be *prime* if, whenever $\mathcal{I}_1, \mathcal{I}_2$ are ideals in \mathcal{A} such that $\mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}$, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}$ or $\mathcal{I}_2 \subseteq \mathcal{I}$; \mathcal{A} is said to be *prime* if (0) is a prime ideal.

THEOREM

(M. Mathieu) Let \mathcal{A} be a unital C^* -algebra. The following statements are equivalent.

- (i) \mathcal{A} is prime.
- (ii) $\|M_{a,b}\| = \|a\| \|b\|$ for all $a, b \in \mathcal{A}$.
- (iii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$.

It turns out that there is a more foundational result at the heart of Mathieu's theorem.

THEOREM

(RC, C. Hernández G.) Let \mathcal{A} be a unital C^* -algebra. The following statements are equivalent.

(i) \mathcal{A} is prime.

(ii) $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b) \quad (a, b \in \mathcal{A})$.

COROLLARY

Let \mathcal{A} be a unital C^* -algebra, and let $a, b \in \mathcal{A}$. The following statements are equivalent.

(i) $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b) \quad (a, b \in \mathcal{A})$.

(ii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$.

SECTION 8. M_z ON REINHARDT DOMAINS

DEFINITION 8.1. Let Ω be a domain in \mathbb{C}^n . Ω is said to be Reinhardt if it is invariant under the action of the n -torus \mathbb{T}^n , i.e., if whenever $z \in \Omega$ and $e^{i\theta} \in \mathbb{T}^n$, it follows that $e^{i\theta z} := (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in \Omega$. Reinhardt domains are important in several complex variables because of their intrinsic relation to power series. For instance, pseudoconvex Reinhardt domains containing the origin are the natural domains of convergence of power series. Moreover, if Ω is a complete Reinhardt domain containing the origin, then $\text{int. } \hat{\Omega} = M_\Omega = \tilde{\Omega}$, where $\tilde{\Omega}$ is the pseudoconvex hull of Ω . In what follows, we shall assume that $0 \in \Omega$.

OBSERVATION 8.2. For $\alpha \in \mathbb{Z}_+^n$ let $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$; let us also denote by z^α the function whose value at z is z^α . If $c_\alpha = \|z^\alpha\|_A^2(\Omega)$ then $\left\{ \frac{z^\alpha}{c_\alpha} \right\}_{\alpha \in \mathbb{Z}_+^n}$ is an orthonormal basis for $A^2(\Omega)$. Moreover,

$$M_{z_i} \left[\frac{z^\alpha}{c_\alpha} \right] = \frac{c_{\alpha+\epsilon_j}}{c_\alpha} \left[\frac{z^{\alpha+\epsilon_j}}{c_{\alpha+\epsilon_j}} \right] \quad (\text{all } \alpha \in \mathbb{Z}_+^n)$$

where $\epsilon_j := (0, \dots, 1 \dots, 0)$, so that M_{z_i} is a weighted shift with weights

$$w_i(\alpha) = \frac{c_{\alpha+\epsilon_j}}{c_\alpha}, \alpha \in \mathbb{Z}_+^n, \quad i = 1, \dots, n.$$

EXAMPLE 8.3. Let $0 < p, q < \infty$ and let

$$\Omega_{p,q} := \left\{ z \in \mathbb{C}^2 : |z_1|^p + |z_2|^q < 1 \right\}.$$

$\Omega_{p,q}$ is **pseudoconvex**, because

$$\log |\Omega_{p,q}| := \{(\log |z_1|, \log |z_2|) : (z_1, z_2) \in \Omega_{p,q} \text{ and } z_1 \neq 0, z_2 \neq 0\}$$

is a convex set; when $p, q \geq 2$, $\Omega_{p,q}$ is Levi pseudoconvex, and $\Omega_{p,q}$ is strongly pseudoconvex if and only if $p = q = 2$. For $\Omega_{p,q}$, the coefficients c_α can be calculated explicitly:

$$c_\alpha = \frac{2\pi^2}{p(\alpha_2 + 1)} \cdot B\left(\frac{2\alpha_1 + 2}{p}, \frac{2\alpha_2 + 2}{q} + 1\right),$$

where B is the Beta function.

For **subnormal** 2-variable weighted shifts, RC-K. Yan gave in 1995 a complete description of the spectral picture, by exploiting the **groupoid machinery** in Muhly-Renault and RC-Muhly, and the presence of the **Berger measure**, which was used to analyze the **asymptotic behavior of sequences of weights**.

NOTATION

μ : compactly supported finite positive Borel measure on \mathbb{C}^n ($n \geq 1$)

$P^2(\mu)$: norm closure in $L^2(\mu)$ of $\mathbb{C}[z_1, \dots, z_n]$

$M_{\mathbf{z}} \equiv M_{\mathbf{z}}^{(\mu)} := (M_{z_1}^{(\mu)}, \dots, M_{z_n}^{(\mu)})$: multiplication operators acting on $P^2(\mu)$

$M_{\mathbf{z}}$ on $P^2(\mu)$ is the universal model for cyclic subnormal n-tuples

DEFINITION

(i) $E \subseteq \mathbb{C}^n$ is **Reinhardt** if for every $\mathbf{z} \in E$ and every $\theta \in \mathbb{R}^n$,

$$e^{i\theta} \mathbf{z} := (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in E.$$

(ii) μ is **Reinhardt** if $\mu(e^{i\theta} E) = \mu(E)$ for every Borel subset $E \subseteq \mathbb{C}^n$ and every $\theta \in \mathbb{R}^n$

(supp μ is always a Reinhardt set).

THEOREM

(RC-K. Yan, 1995) Let μ be a Reinhardt measure on \mathbb{C}^2 , and let $C := \log |\widehat{K}|$. Assume that $\partial\widehat{K} \cap \{z_1 z_2 = 0\}$ contains no 1-dimensional open disks. Then

(i) $\partial\widehat{K} \supseteq \sigma_\ell(M_{\mathbf{z}}, P^2(\mu)) = \sigma_{\ell_e}(M_{\mathbf{z}}, P^2(\mu)) \supseteq (\exp(\partial^0 C \times \mathbb{T}^2))^-$

(i') If, in addition, μ is well-behaved, then $\sigma_{\ell_e}(M_{\mathbf{z}}, P^2(\mu)) = (\exp(\partial^0 C \times \mathbb{T}^2))^-$.

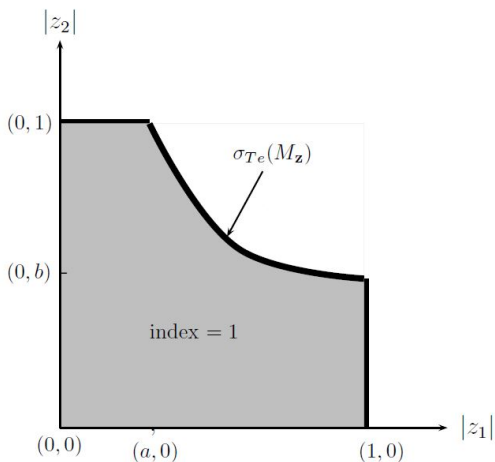
(ii) $\sigma_T(M_{\mathbf{z}}, P^2(\mu)) = \sigma_r(M_{\mathbf{z}}, P^2(\mu)) = \widehat{K}$

(iii) $\sigma_{T_e}(M_{\mathbf{z}}, P^2(\mu)) = \sigma_{r_e}(M_{\mathbf{z}}, P^2(\mu)) = \partial\widehat{K}$

(iv) $\text{int}.\widehat{K} \subseteq \sigma_p(M_{\mathbf{z}}, P^2(\mu)) = \text{b.p.e.}(\mu) \subseteq \widehat{K}$

(v) $\text{index}(M_{\mathbf{z}} - \lambda) = \begin{cases} 1 & \text{if } \lambda \in \text{int}.\widehat{K} \\ 0 & \text{if } \lambda \notin \text{int}.\widehat{K} \end{cases}$

(vi) $\ker D_{M_{\mathbf{z}} - \lambda}^1 = \text{ran} D_{M_{\mathbf{z}} - \lambda}^0$ for all $\lambda \in \text{int}.\widehat{K}$.



Spectral picture of a typical subnormal 2-variable weighted shift

We shall conclude this section with an example of a domain that has turned out to be quite an interesting one.

EXAMPLE 8.5. Let $0 < \delta, \epsilon < 1$ and let

$$\Omega_{\delta, \epsilon} := \{z \in \mathbb{C}^2 : (|z_1| < \delta \text{ and } |z_2| < 1) \text{ or } (|z_1| < 1 \text{ and } |z_2| < \epsilon)\}$$

$\Omega_{\delta, \epsilon}$ is an L-shaped domain, and

$$\hat{\Omega}_{\delta, \epsilon} = \{z \in \mathbb{C}^2 : |z_1| \leq |z_2| \leq 1 \text{ and } |z_1|^{\log \epsilon} |z_2|^{\log \delta} \leq \delta^{\log \epsilon}\}$$

M_Z acting on $A^2(\Omega_{\delta, \epsilon})$ is a bivariate weighted shift whose weights are associated with the coefficients

$$c_{\alpha} = \pi \left[\frac{\delta_1^{2\alpha_1+2} + \epsilon^{2\alpha_2+2} - \delta_1^{2\alpha_1+2} 2\alpha_2 + 2}{(\alpha_1 + 1)(\alpha_2 + 1)} \right]^{1/2}.$$

Using the theory of groupoids, RC and P. Muhly studied the ideal structure of the C^* -algebra generated by multivariable weighted shifts. As an application, they obtained the following result.

THEOREM 8.6. (RC - P. Muhly) Let $\Omega_{\delta, \epsilon}$ be as above. Then $C^*(M_Z)$ is type I if and only if $\frac{\log \delta}{\log \epsilon}$ is a rational number.

REMARK 8.7. $\Omega_{\delta, \epsilon}$ is the first known example of a domain for which the Bergman C^* -algebra is not type I. Previous works indicated that $C^*(M_Z)$ might always be type I.

CONCLUDING REMARKS AND SOME OPEN PROBLEMS

Throughout these notes we have tried and emphasized the role of σ_T in the study of commuting n -tuples of Hilbert spaces operators. Although Taylor invertibility of an n -tuple $A \subset \mathcal{L}(H)$ is determined by a single operator $(R_A, \square_A \text{ or } \hat{A})$, it is still difficult to calculate σ_T for specific examples. In [39] a first step was taken in making precise the connection between σ_T and σ_H , the latter usually easier to calculate. In case $n = 2$, that connection is completely described by Theorem 6.12 above, but in the general case it remains an open problem.

PROBLEM 9.1. How does σ_H sit inside σ_T ? Or, in the language of Section 3, how does M_{σ_H} sit inside M_{σ_T} , where M_σ is calculated with respect to a commutative Banach algebra containing the n -tuple A (for instance, (A))? σ_T is certainly a preferred joint spectrum, since on one hand it is big enough to carry an analytic functional calculus,

PROBLEM 9.2. Let $\tilde{\sigma}$ be a spectral system on \mathcal{H} . Assume that $\tilde{\sigma}$ satisfies the projection property and $\tilde{\sigma} \subseteq \hat{\sigma}$ (i.e., $\hat{\sigma}(A) \subseteq \tilde{\sigma}(A)$ for all $A \in \mathcal{L}(\mathcal{H})$). Must $\tilde{\sigma} \subseteq \sigma_T$? In other words, is σ_T maximal among spectral systems contained in $\hat{\sigma}$ with the projection property?

(Problem 9.2 is basically due to W. Żelazko.)

PROBLEM 9.3. Let $\tilde{\sigma}$ be a spectral system on \mathcal{H} . Assume that $\tilde{\sigma}$ carries an analytic functional calculus, i.e., for $A \in \mathcal{L}(\mathcal{H})$ there exists a continuous homomorphism $\Phi_A : H(\tilde{\sigma}(A)) \rightarrow \mathcal{L}(\mathcal{H})$ such that $\Phi_A(1) = I$, $\Phi_A(z_i) = A_i$ ($i = 1, \dots, n$) and $\Phi_A(f) \in (A)'$ for every $f \in H(\tilde{\sigma}(A))$. Must $\sigma_T \subseteq \tilde{\sigma}$?

Let $\Phi_n(\mathcal{H}) := \left\{ A \in \mathcal{L}(\mathcal{H})(n) : 0 \notin \sigma_T(A, \mathcal{H}) \right\}$, the collection of invertible n -tuples on \mathcal{H} .

It is of course well-known that $\Psi_1(\mathcal{H})$ is contractible (Kuiper's Theorem), but even the path-connectedness of $\Phi_n(\mathcal{H})(n \geq 2)$ remains an open question. (For partial results, see [35].)

Extension theory is relevant to Problem 9.5. Let us mention briefly an important step in the proof of Theorem 9 in [36]. that is, the fact that on $H^2(\mathbb{T}^2)$, T_z and T_z^* can be joined by a path of Fredholm pairs. (Here $T_z = (U_+ \otimes I, I \otimes U_+)$.) First, one considers M_z on $H^2(S^3)$, the Hardy space for the sphere, and writes down the separate polar decompositions $M_{z_i} = V_i P_i$ ($i = 1, 2$). One then joins M_z to $V := (V_1, V_2)$ and M_z^* to V^* by showing that $[V_i, P_i] \in K(H^2(S^3))$ and then using the line segment from VP to V (the more general theory of almost commuting Fredholm n-tuples ([36],[82]) is needed. Now, V is unitarily equivalent to T_z , so that to join T_z to T_z^* we need to complete the following diagram

Here is where the BDF-theory ([18]) enters: M_z and M_z^* give rise to equivalent extensions of s^3 . Indeed, $\text{ind} : \text{Ext}(S^3) \rightarrow \mathbb{Z}$ is a group isomorphism, and $\text{ind}(M_z) = \text{ind}(T_z) = 1 = \text{ind}(T_z^*) = \text{ind}(M_z^*)$. Therefore, there exists a unitary operator U such that $U_z U^* - M_z^*$ is compact on $H^2(S^3)$. Now join U to I to complete

Concerning M_z on $A^2(\Omega)$ (or more generally, multiplication operators on the Bergman space), a number of results have been obtained for various kinds of domains Ω . There is still, however, much to learn about M_z on general Ω 's

PROBLEM 9.6. Find a complete description of the Koszul complex $K(M_z, A^2(\Omega))$.

The above-mentioned problems represent a small sample of open questions in multiparameter spectral theory. We have stated them here because of their relevance to the topics discussed in these lecture notes. Since we were committed to an elementary level of discussion, we have not had occasion to touch on many important developments that have taken place in the last few years. We shall briefly mention some of them:

(i) Building on s. Frunza's work ([58]), J. Eschmeier has developed a local analytic spectral theory ([53]), later refined by M. Putinar and F. -H. Vasilescu ([91]).

- (ii) A theory of decomposable n -tuples exists, due to the work of many authors, most notably E. Albrecht, F. -H. Vasilescu (see [1] and] [114]; see also [40], where a summary of the main results is given).
- (iii) M. Putinar has found a sheaf theoretic model for commuting n -tuples, which seems to provide a better generalization of decomposability, and is relevant to the study of Bergman n -tuples ([54],[85], [88]).
- (iv) R. Carey and J.D. Pincus have discovered an index theory for n -tuples which extends their work on trace class commutators and makes contact with geometric measure theory ([20],[21]); alternate descriptions have been independently proposed by M. Putinar and by R. Levi ([75], [89]).
- (v) The (Taylor) spectral picture of left and right multiplications has been described in detail in [42], building on previous work of various authors ([22], [36], [55], [56], [61]).
- (vi) M. Cowen and R.G. Douglas have extended to several variables their work on the

(ix) Non-analytic functional calculi have been considered by various authors ([4], [5], [6]; see also [80, F. -H. Vasilescu's problem, p.601])

Finally, we would like to say a word about noncommuting n -tuples. Their study is of a different nature, and it involves techniques different from those explained here. For the interested reader, we mention the works in [3], [63], [64], [65] and [107]

by Popescu, Helton, McCullough, McCarthy, Agler,

Mention Igor Klep's recent paper on similarity orbits.

Let \mathcal{A} be a unital C^* -algebra, let L_a, R_b denote the *left* and *right* multiplication operators induced by $a, b \in \mathcal{A}$ (i.e., $L_a(x) := ax$, $R_b(x) := xb$, $x \in \mathcal{A}$), and set $M_{a,b} := L_a R_b$.

An ideal \mathcal{I} in \mathcal{A} is said to be *prime* if, whenever $\mathcal{I}_1, \mathcal{I}_2$ are ideals in \mathcal{A} such that $\mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}$, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}$ or $\mathcal{I}_2 \subseteq \mathcal{I}$; \mathcal{A} is said to be *prime* if (0) is a prime ideal.

THEOREM

(M. Mathieu) Let \mathcal{A} be a unital C^* -algebra. The following statements are equivalent.

- (i) \mathcal{A} is prime.
- (ii) $\|M_{a,b}\| = \|a\| \|b\|$ for all $a, b \in \mathcal{A}$.
- (iii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$.

It turns out that there is a more foundational result at the heart of Mathieu's theorem.

THEOREM

(RC, C. Hernández G.) Let \mathcal{A} be a unital C^* -algebra. The following statements are equivalent.

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