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Anisotropy helps in seismology

Inverse Problems and Nonlinearity
Banff

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Based on joint work with
Maarten de Hoop, Matti Lassas, Anthony Várilly-Alvarado

Overview

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Theorem (de Hoop–I–Lassas–Várilly-Alvarado, 2023)

Generically an anisotropic stiffness tensor is uniquely determined by any of the following:

- *slowness polynomial,*
- *slowness surface,*
- *any small part of of the slowness surface for a single polarization.*

But orthorhombic stiffness tensors are not unique!

Outline

Elastic wave equation

Quantities:

- Displacement $u(t,x) \in \mathbb{R}^n$.
- Density $\rho(x) \in \mathbb{R}$.
- Stiffness tensor $c_{ijkl}(x) \in \mathbb{R}^{n^4}$.

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Properties:

- $\rho > 0$.
- $c_{ijkl} = c_{klij} = c_{jikl}$.
- $\sum_{i,j,k,l} c_{ijkl} A_{ij} A_{kl} > 0$ whenever $A = A^T \neq 0$.

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Equation of motion:
$$\rho(x)\partial_t^2 u_i(t,x) - \sum_{j,k,l} \partial_j [c_{ijkl}(x)\partial_k u_l(x)] = 0.$$

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If $u = Ae^{i\omega(t-p\cdot x)}$, then the EWE becomes

$$\rho\omega^2[-I + \Gamma(p)]A = 0,$$

where

$$\Gamma_{il}(p) = \sum_{j,k} \rho^{-1} c_{ijkl} p_j p_k$$

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The polarization A is an eigenvector of the Christoffel matrix.

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In general, singularities of the elastic wave equation (mostly!) satisfy

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where c and ρ are allowed to depend on x.

The fastest singularities follow the geodesic flow of the Finsler metric $F^{qP} = [\lambda_1(\Gamma)^{1/2}]^*$.

Slowness polynomial and slowness surface

A reduced stiffness tensor $a_{ijkl} = \rho^{-1} c_{ijkl}$ defines

- a Christoffel matrix $\Gamma_a(p)$ and
- a slowness polynomial $P_a(p) = \det[\Gamma_a(p) - I]$.

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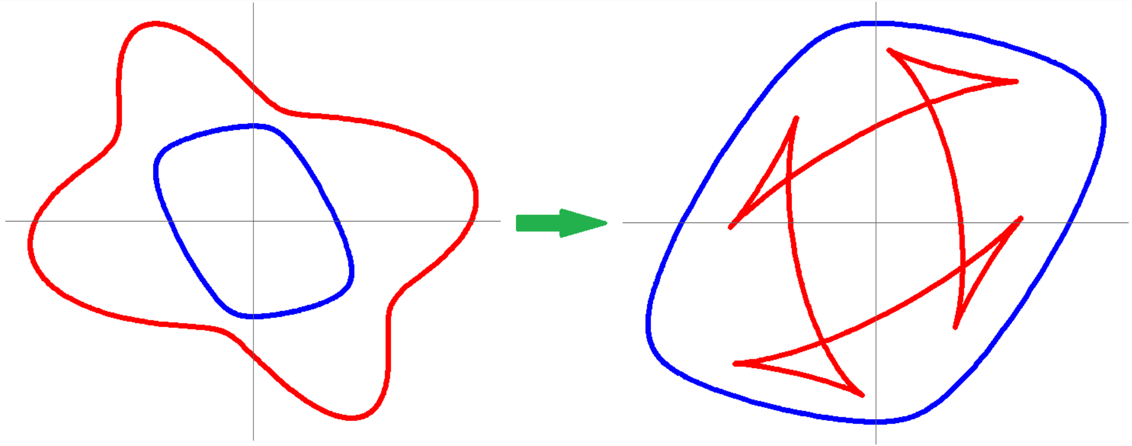
The set where singularities are possible is the slowness surface

$$\Sigma_a = \{p \in \mathbb{R}^n; P_a(p) = 0\}.$$

Knowing the slowness polynomial $\iff$ knowing the slowness surface.

Slowness polynomial and slowness surface



A slowness surface in 2D with its two branches, and the corresponding two Finsler norms.
The quasi pressure (qP) polarization behaves well.
Anisotropy $\iff$ dependence on direction $\iff$ not circles.

Geometrization of an analytic problem

Original inverse problem

Given information of the solutions to the elastic wave equation on $\partial\Omega$, find the parameters $c(x)$ and $\rho(x)$ for all $x \in \Omega$.

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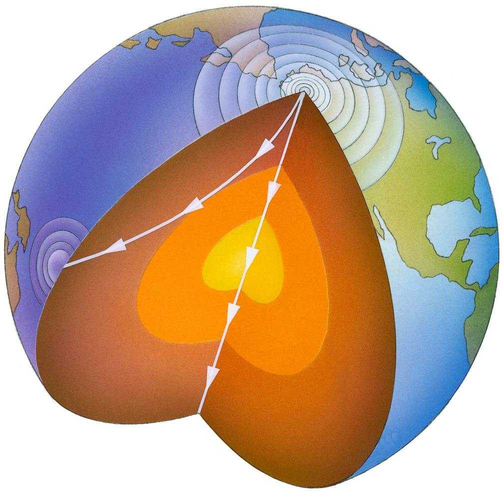
Remarks:

- Geometric inverse problems like this can be solved for qP geometries.
- Riemannian geometry is not enough; it can only handle a tiny subclass of physically valid and interesting stiffness tensors.
- Knowing the metric is the same as knowing the (co)sphere bundle: (M, g) or $(M, F) \iff (M, SM) \iff (M, S^*M)$.
- The cospheres of the Finsler geometry are the qP branches of the slowness surface.

Geometrization of an analytic problem



Rays follow geodesics and tell about the interior structure encoded as a geometry.

Outline

1. Inverse problems in elasticity
2. Geometry of slowness surfaces
   - Algebraic variety
   - Generic irreducibility
   - Generically unique reduced stiffness tensor
3. A two-layer model

Algebraic variety

Definition

A set $V \subset \mathbb{R}^n$ is an algebraic variety if it is the vanishing set of a collection of polynomials $\mathbb{R}^n \to \mathbb{R}$.

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The study of the geometry of varieties is a part of algebraic geometry.

Generic irreducibility

Definition

A variety $V \subset \mathbb{R}^n$ is reducible if it can be written as the union of two varieties in a non-trivial way.

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Let $n \in \{2,3\}$. There is an open and dense subset of stiffness tensors a so that the slowness polynomial P_a is irreducible.

Remarks:

- This is not true for all a — this fails at least when one of the geometries is Riemannian.
- Typically for a family of polynomials the set of irreducible ones is Zariski-open.
  We thus only need an example.

Generic irreducibility

Corollary (de Hoop–I.–Lassas–Várilly-Alvarado)

When the slowness surface Σ_a is irreducible, any (Euclidean) relatively open subset determines the whole slowness surface.
If $n \in \{2, 3\}$, this is generically true.

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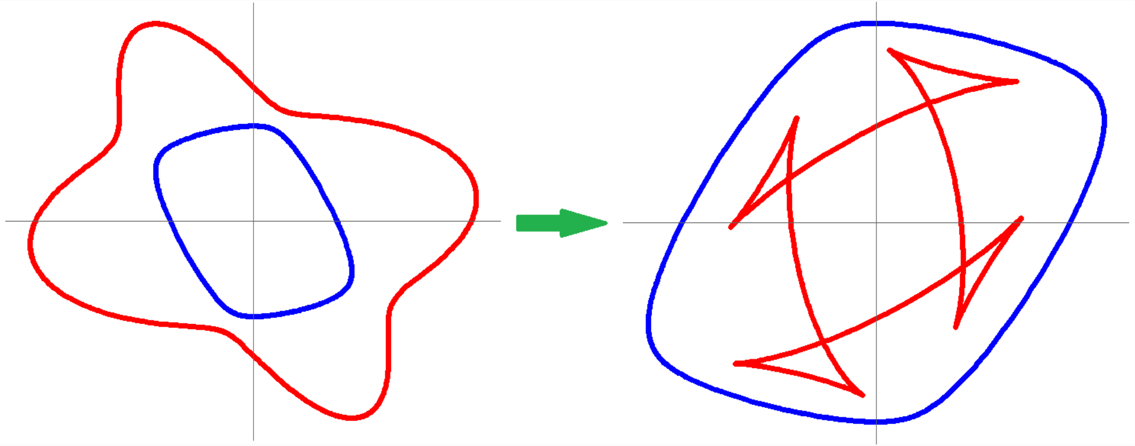
It suffices to measure the well-behaved qP branch!

Generic irreducibility



Any small part of the well-behaved quasi pressure branch determines the whole thing via Zariski closure.

Generically unique reduced stiffness tensor

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Corollary (de Hoop–I.–Lassas–Várilly-Alvarado)

Let $n \in \{2,3\}$. Generically any small subset of the slowness surface Σ_a determines a.

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1. Denote by $A \approx \mathbb{R}^N$ the space of stiffness tensors and $C \approx \mathbb{R}^M$ the space of degree $2n$ even polynomials on $\mathbb{R}^n$.

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6. The fiber $f^{-1}(c)$ generically contains only finitely many points. The number of these points is upper semicontinuous.
7. We find an explicit $c \in f(A)$ for which the number is one, so it is generically one.
8. The generic preimage on the image is thus a singleton, so the map f is generically injective. □

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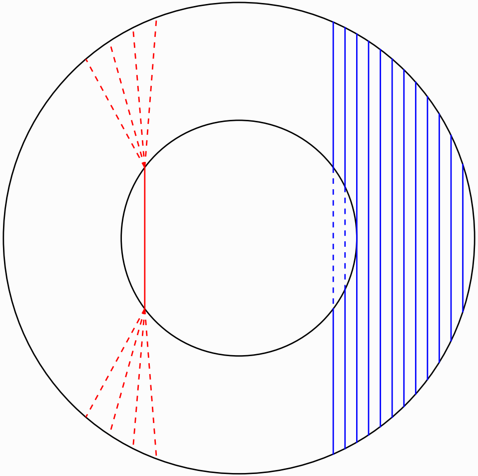
Measurement: Travel times and directions of waves between all surface points, for all polarizations.

Result: The measurement generically determines the model completely!

The proof



First find outer stiffness and boundary, then inner stiffness.

Today's highlights

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But orthorhombic stiffness tensors are not unique!

Theorem (de Hoop–I.–Lassas–Várilly-Alvarado, 2023)

Suppose the planet is piecewise homogeneous (but anisotropic) with two layers. Measurements of travel times of qP (or all) rays generically determine the whole model:

- *stiffness tensor in the mantle,*
- *stiffness tensor in the core,*
- *the core–mantle boundary.*

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Zariski topology

Given any set F of functions $f\colon \mathbb{R}^n \to \mathbb{R}$, we can define a closure for all $A \subset \mathbb{R}^n$:

$$\mathrm{cl}_F(A) = \{x \in \mathbb{R}^n; \forall f \in F : f|_A = 0 \implies f(x) = 0\}.$$

(This satisfies the Kuratowski axioms if F is a unital ring.)

Examples:

- $F = C(\mathbb{R}^n) \rightsquigarrow$ standard Euclidean topology.
- $F = C^\infty(\mathbb{R}^n) \rightsquigarrow$ standard Euclidean topology.
- $F = \{\text{polynomial functions}\} \rightsquigarrow$ Zariski topology.

A variety is the same as a Zariski-closed set.