Anomalous dissipation in fluid dynamics

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European Research Council
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Universität Basel

FNSNF
FONDS NATIONAL SUISSE
SCHWEIZERISCHER NATIONALFONDS
FONDO NAZIONALE SVIZZERO
SWISS NATIONAL SCIENCE FOUNDATION

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- Experimentally and numerically validated to a large extent.

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Extreme nonuniqueness. Can we restore uniqueness by selection?
(This is the case for scalar conservation laws (Burgers).)

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Yaglom's relation: $u \in C^\alpha$ and $\vartheta \in C^\beta$ with

$$\alpha + 2\beta \quad \begin{array}{ll} > 1 & \text{subcritical} \\ = 1 & \text{critical} \\ < 1 & \text{supercritical} \end{array}$$

Obukhov-Corrsin predictions (1949-1951)

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- Supercritical: which question exactly?
  - One velocity field and one initial datum.
  - One velocity field and all initial data.
  - Statistical statement.

Result and approach [MC-GC-MS]

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W is a Brownian motion, informally (in this context)

- a (probabilistically) parametrized family of trajectories
- with Gaussian increments
- and isotropically distributed.

Chessboard mixing à la Depauw-Bressan

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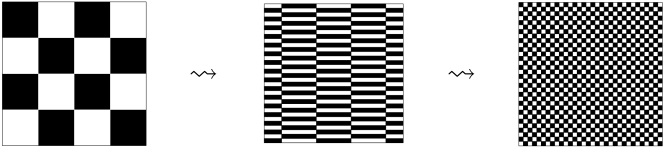

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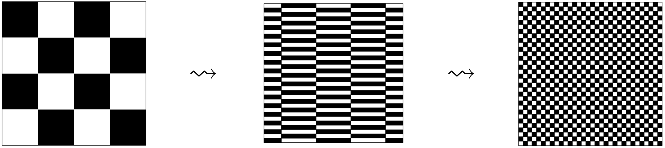

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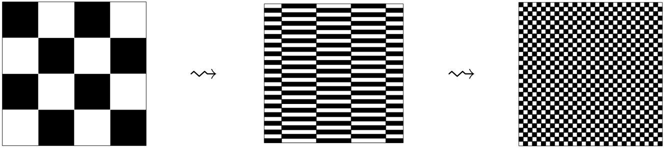

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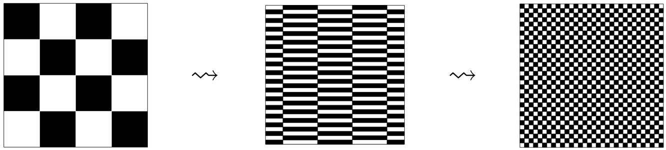

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  - "separate scales", and
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- Perfect mixing at $t = 1$. Reconstruct chessboard for $t \in [1, 2]$.

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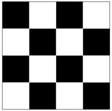
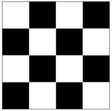
 $\rightsquigarrow$ 

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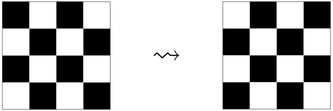


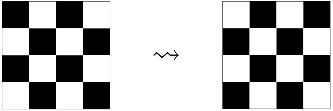
- Depending on the "parity of ε", the solution ϑ^ε will converge for $\varepsilon \downarrow 0$ to an even or odd chessboard for $t \sim 2$.

Filtering (convolution)

Convolution $u \rightsquigarrow u^\varepsilon = u * \varphi_\varepsilon$ in pure advection

$$\partial_t \vartheta^\varepsilon + u^\varepsilon \cdot \nabla \vartheta^\varepsilon = 0\,.$$

- Convolution "filters" shear flows with scale below ε.
- ϑ^ε does not get fully mixed at $t = 1$.
- ϑ^ε stays fixed-scale $\sim \varepsilon$, then large scales are reconstructed.
- For $t > 1$ introduce a "swap" move at each step:



- Depending on the "parity of ε", the solution ϑ^ε will converge for $\varepsilon \downarrow 0$ to an even or odd chessboard for $t \sim 2$.
- Lack of selection by convolution (cp. [C-C-S] and [DL-G]).

Filtering vs. diffusing

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Surprisingly, for our effects...

infinite propagation speed	finite propagation speed (with high probability)

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- Enhanced diffusion. Baby example: both

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solve the heat equation $\partial_t f - \partial_{xx} f = 0$.

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- We get: dissipation of a fixed amount of the solution.

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- Recently [A-V]: Anomalous diffusion by fractal homogeniz.
  [H-T]: Energy can increase
  [E-L]: Universality

Anomalous dissipation for forced Navier-Stokes [B-C-C-DL-S]

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Take u and ϑ from before and notice that

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Hence anomalous dissipation and lack of selection for

$$\partial_t v^\nu + v^\nu \cdot \nabla v^\nu = -\nabla q^\nu + \nu \Delta v^\nu + F_\nu \qquad F_\nu = (f_\nu, 0)\,.$$

for v^ν uniformly Onsager-supercritical.

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- Open: two-dimensional case
  (very special due to vorticity transport! can have selection?)

Thank you for your attention!