Global well-posedness for the one-phase Muskat problem

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Abstract

We consider the free boundary problem for a 2D and 3D fluid filtered in porous media, which is known as the one-phase Muskat problem.

We show that if the initial free boundary is the graph of a periodic Lipschitz function, then there exists a unique global Lipschitz strong solution. The proof of the uniqueness relies on a new pointwise $C^{1,\alpha}$ estimate near the boundary for harmonic functions.

This is based on joint work with Francisco Gancedo (Universidad de Sevilla, Spain) and Huy Q. Nguyen (University of Maryland, USA).

Part I: Formulation of the problem and known results

Formulation of the problem

Consider a 2D (or 3D) incompressible fluid permeating a (homogeneous) porous medium, modeled by the classical Darcy law

$$\mu u(x,y,t) = -\nabla_{x,y} p(x,y,t) - \rho \cdot (0,1),$$

$$\nabla_{x,y} \cdot u(x,y,t) = 0, \quad (x,y) \in \Omega_t \subset \mathbb{R}^2,\ t \in \mathbb{R}_+.$$

Here u is the fluid velocity, p is the pressure (harmonic), and the positive constants μ and ρ are respectively the dynamic viscosity and fluid density (constants).

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- The free boundary $\Sigma_t = \partial\Omega_t$ moves with the fluid

$$\mathcal{V}(\Sigma_t) = u \cdot n,$$

where n is the outward pointing unit normal to Σ_t.

- We neglect the surface tension, so the pressure is continuous across the free boundary $p|_{\Sigma_t} = 0$.
- We are interested in the *geometry* and *regularity* of the free boundary Σ_t as time evolves.
- Two cases: graph and non-graph boundaries.
For both cases, the existence and uniqueness of local strong solution have been well established even for much more general settings, including multi-phase, with rigid boundaries, with surface tension, nonconstant permeability.

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Long-term dynamics

Global existence and uniqueness of solutions have been obtained when

- Σ_0 is the graph of a small function in various function spaces: Siegel–Caflisch–Howison (04), Córdoba–Gancedo (07), Escher–Matioc (11), Constantin–Gancedo–Shvydkoy–Vicol (17).
- "Medium data" in the Wiener algebra or the Lipschitz space: Constantin–Córdoba–Gancedo–Strain (13, 16), Gancedo–Garcia-Juarez–Patel–Strain (19), Cameron (19, 20).
- Small data in critical Sobolev spaces with large (or even infinite) Lipschitz norm (also critical): Córdoba–Lazar (18), Gancedo–Lazar (20), Alazard–Q.-H. Nguyen (20).
- Σ_0 is close to a circle: Xinfu Chen (93), Constantin–Pugh (93), Gancedo–Garcia-Juarez–Patel–Strain (19).

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Long-term dynamics

- For the two-phase problem, Deng-Lei-Lin (17) proved the existence (without uniqueness) of global weak solutions that are monotone in $\mathbb{R}$.
- There has not been any global well-posedness result for initial data of arbitrary size, either for weak or strong solutions.

Long-term dynamics

Finite-time singularity

- For the two-phase problem, initial graph interfaces with large slopes can turn over passing from a stable regime to an unstable regime (Castro–Córdoba–Fefferman (12)) and solutions loss regularity in finite time (Castro et. al. (12)).
- In contrast, starting from graph initial boundaries the free boundary of the one-phase problem cannot turn over.
- It was proved that for the one-phase problem, solutions can develop splash singularity (Castro–Córdoba–Fefferman–Gancedo (16)) from some non-graph initial boundary, while the two-phase problem cannot (Gancedo–Strain (14)).
- No splat singularity for both problems: particles on the free boundary cannot collide along a curve.

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Global solutions to the one-phase problem

Two fundamental questions for the one-phase problem:

1. Does there exist a unique global solution?
2. If yes, what is its long-term regularity?

In this work, we address the first problem: we proved

If Σ_0 is the graph of a periodic Lipschitz function, then there exists a global Lipschitz solution in the strong sense (and hence almost everywhere). Moreover, it is the unique viscosity solution.

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Part II: Reformulation of the problem

Reformulation in terms of the D-N operator

Assume that

$$\Omega_t = \{(x, y) \in \mathbb{R}^2, \quad y < f(x, t)\}$$

for some function $f(x, t) : \mathbb{R} \times [0, T] \to \mathbb{R}$ that is 2π-periodic in x. Then f satisfies an equivalent (nonlocal) parabolic type equation

$$\partial_t f = -\kappa G(f)f, \quad \kappa = \rho/\mu.$$

For $f, g : \mathbb{T} \to \mathbb{R}$, the Dirichlet-Neumann operator $G(f)g$ is well defined with a quantitative bound, provided that $f \in W^{1,\infty}(\mathbb{T})$ and $g \in H^1(\mathbb{T})$:

$$\big(G(f)g\big)(x) = \partial_N \varphi(x, f(x)),$$

where $\varphi(x, y)$ solves the elliptic problem

$$\begin{cases} \Delta_{x,y}\varphi = 0 \quad \text{in } \Omega, \\ \varphi(x, f(x)) = g(x), \quad \nabla_{x,y}\varphi \in L^2(\Omega). \end{cases}$$

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Reformulation in terms of the D-N operator

Some examples of the D-N operator

- Half space, i.e, $f \equiv 0$:

$$\partial_n \varphi(x,0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(x+x')+g(x-x')-2g(x)}{|x'|^2}\,dx'$$

- In a disc $B_1(0)$,

$$\partial_n \varphi(e^{ix}) = -\frac{1}{8\pi}\int_{-\pi}^{\pi} \frac{g(x+x')+g(x-x')-2g(x)}{\sin^2(\frac{x'}{2})}dx'.$$

A simple property: $G(f+a)(g+b) = G(f)g$ for constants a, b.
Therefore, if f is a solution, $f + a$ and $f(x + x_0, t)$ are also solutions.

Main theorem in the 2D case

Theorem (D.-Gancedo-Nguyen, 2021, CPAM)

For any $f_0 \in W^{1,\infty}(\mathbb{T})$, there exists

$$f \in C(\mathbb{T} \times [0,\infty)) \cap L^\infty([0,\infty); W^{1,\infty}(\mathbb{T})), \quad \partial_t f \in L^\infty([0,\infty); L^2(\mathbb{T}))$$

such that $f|_{t=0} = f_0$, f satisfies the equation in $L^\infty_t L^2_x$, and

$$\|f(t)\|_{W^{1,\infty}(\mathbb{T})} \le \|f_0\|_{W^{1,\infty}(\mathbb{T})} \quad a.e.\ t > 0.$$

Moreover, f is the unique viscosity solution.

This appears to be the first global well-posedness result of the Muskat problem for initial data of arbitrary size.

A remark

Sufficiently smooth solutions obey the comparison principle:
if $f_0 \le \tilde{f}_0$, then $f(\cdot, t) \le \tilde{f}(\cdot, t)$ for any $t > 0$.

Consequently, the modulus of continuity of f_0 is preserved by $f(t)$ for all $t > 0$. Consequently, as long as the free boundary remains to be a graph, its slope is bounded by the initial slope.

Part III: Outline of the proof of the existence part

Viscosity solutions

A function $f : \mathbb{T} \times [0, T]$ is called a viscosity subsolution (resp. supersolution) on $(0, T)$ provided that
(i) f is upper semicontinuous (resp. lower semicontinuous) on $\mathbb{T} \times [0, T]$, and
(ii) for every $\psi : \mathbb{T} \times (0, T) \to \mathbb{R}$ with $\partial_t \psi \in C(\mathbb{T} \times (0, T))$ and $\psi \in C((0, T); C^{1,1}(\mathbb{T}))$, if $f - \psi$ attains a global maximum (resp. minimum) over $\mathbb{T} \times [t_0 - r, t_0]$ at $(x_0, t_0) \in \mathbb{T} \times (0, T)$ for some $r > 0$, then

$$\partial_t \psi(x_0, t_0) \le -\kappa\Big(G(\psi)\psi\Big)(x_0, t_0) \quad \text{(resp. } \ge\text{).}$$

A viscosity solution is both a viscosity subsolution and viscosity supersolution.

Viscosity solutions

We construct solutions by the viscosity regularization approach: for small $\varepsilon > 0$, consider the approximate equation

$$\partial_t f^\varepsilon = -\kappa G(f^\varepsilon) f^\varepsilon + \varepsilon \partial_x^2 f^\varepsilon.$$

To solve for f^ε, we use the layer potential representation of $G(f)g$.

Layer potential representation

Newtonian kernel for $\mathbb{T} \times \mathbb{R}$:

$$\mathcal{N}(z) = (4\pi)^{-1} \ln\Big(\cosh y - \cos x\Big), \quad z = (x, y) \in \mathbb{T} \times \mathbb{R}.$$

Double layer potential for a function $h : \mathbb{T} \to \mathbb{R}$:

$$\begin{aligned}\mathcal{K}[f]h(z) &:= -\int_{\Sigma} (\partial_{n(x')}\mathcal{N})(z - z')\widetilde{h}(z')dz' \\ &= \frac{1}{4\pi}\int_{\mathbb{T}} \frac{\sin(x - x')\partial_x f(x') - \sinh(y - f(x'))}{\cosh(y - f(x')) - \cos(x - x')} h(x')dx'.\end{aligned}$$

Single layer potential:

$$S[f]h(x, y) = \frac{1}{4\pi}\int_{\mathbb{T}} \ln\Big(\cosh(y - f(x')) - \cos(x - x')\Big)h(x')dx'.$$

Layer potential representation

The unique solution φ of the Dirichlet problem is then given by

$$\varphi = \mathcal{K}(\frac{1}{2}I + K)^{-1}g.$$

For $f \in \mathrm{Lip}(\mathbb{T})$ and $g \in H^1(\mathbb{T})$, we have for a.e. $x \in \mathbb{T}$ that

$$\begin{aligned}(G(f)g)(x) &= (1, \partial_x f(x)) \cdot \nabla S[f]\theta(x, f(x)) \\ &= \frac{1}{4\pi} p.v. \int_{\mathbb{T}} \frac{\sin(x - x') + \sinh(f(x) - f(x'))\partial_x f(x)}{\cosh(f(x) - f(x')) - \cos(x - x')} \theta(x') dx' \\ &= \frac{1}{4\pi} p.v. \int_{\mathbb{T}} \partial_x \ln\Big(\cosh(f(x) - f(x')) - \cos(x - x')\Big)\theta(x') dx',\end{aligned}$$

where

$$\theta = \partial_x(\frac{1}{2}I + K)^{-1}g = (\frac{1}{2}I - K^*)^{-1}(\partial_x g).$$

Quantitative bounds

Verchota (84) proved that $\frac{1}{2}I - K^* : L^2_0(\mathbb{T}) \to L^2_0(\mathbb{T})$ is invertible provided that the boundary $f \in \mathrm{Lip}$.

We obtained the following quantitative estimates, which are needed for the solvability of the equation.

There exists a universal constant $C > 0$ such that

$$\|(\frac{1}{2}I \pm K^*)^{-1}\|_{L^2_0(\mathbb{T}) \to L^2_0(\mathbb{T})} \leq C(1 + \|f\|_{\mathrm{Lip}(\mathbb{T})})^{5/2}.$$

Moreover, for any $g \in \dot{H}^1(\mathbb{T})$,

$$\|G(f)g\|_{L^2(\mathbb{T})} \leq C(1 + \|f\|_{\mathrm{Lip}(\mathbb{T})})^2 \|\partial_x g\|_{L^2(\mathbb{T})}$$

With these estimates, the existence of solutions is proved by using the contraction mapping method and (quite involved) energy method: L^2, $\dot{H}^1$, $\dot{H}^2$, and finally $\dot{H}^s$ estimates for $s > 2$ (depending on ε).

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Part IV: Proof of the uniqueness

Comparison principle for viscosity solutions

The uniqueness of viscosity solutions follows from the comparison principle below by using the inf/sup convolutions.

Theorem

Assume that $f, g : \mathbb{T} \times [0, T] \to \mathbb{R}$ *are respectively a bounded viscosity subsolution and supersolution on* $(0, T)$. *If* $f(x, 0) \le g(x, 0)$ *for all* $x \in \mathbb{T}$, *then* $f(x, t) \le g(x, t)$ *for all* $(x, t) \in \mathbb{T} \times [0, T]$.

The theorem above is a consequence of the consistency result:

If a viscosity solution is $C^{1,1}$ at a point (x_0, t_0) then it satisfies the equation classically at the same point.

A key step in the proof of the consistency result is a pointwise $C^{1,\alpha}$ estimate, which allows us to pass to the limit in the integral representation of the D-N mapping.

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A pointwise boundary $C^{1,\alpha}$ estimate

Suppose that Ω is a Lipschitz domain in $\mathbb{R}^2$. For $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$, we denote

$$\Omega_r(x_0, y_0) = B_r(x_0, y_0) \cap \Omega \quad \text{and} \quad \Omega_r = \Omega_r(0).$$

We also define the half ball as

$$B_r^+(x_0, y_0) = \{(x, y) \in B_r(x_0, y_0) : y > y_0\}.$$

We assume that $0 \in \partial\Omega$. Suppose that there exists some $r_0 > 0$ such that in a coordinate system, $\partial\Omega \cap B_{2r_0}$ can be represented by a Lipschitz graph with Lipschitz constant $L > 0$.

Let u be a harmonic function in Ω, which vanishes on $\partial\Omega$.

A pointwise boundary $C^{1,\alpha}$ estimate

Theorem

Suppose that there exist constants $M_0, r_0 > 0$ and function ψ in $(-r_0, r_0)$ such that in a coordinate system

$$\psi(0) = \psi'(0) = 0, \quad \Omega_{r_0} = \{(x, y) \in B_{r_0} : \, y > \psi(x)\},$$

and ψ is $C^{1,1}$ at the origin.

Then u is $C^{1,\alpha}$ at 0, i.e., for any $(x, y) \in \Omega$ such that $\sqrt{x^2 + y^2} < r_0$,

$$|u(x, y) - (x, y) \cdot \nabla_{x,y} u(0)| \le C|x^2 + y^2|^{\frac{1+\alpha}{2}} r_0^{-2-\alpha} \|u\|_{L^2(\Omega_{2r_0})},$$

where $C > 0$ is a constant depending only on $M_0 r_0$ and L, and $\alpha \in (0, 1)$ is a small constant depending only on L.

Some remarks

- The conditions can be relaxed to $\psi \in C^{1,\beta}$ at 0 for some $\beta \in (0, 1)$.
- By using simple barrier argument, we know that in any dimension, u is Lipschitz at 0.
- With a bit more work, one can show that u is C^1 in any non-tangential direction (see Caffarelli-Salsa (05)), again in any dimension.
- Unfortunately, the C^1 regularity is insufficient for our purpose: we need $C^{1,\alpha}$ regularity or at least $C^{1,\text{Dini}}$.

Proof of $C^{1,\alpha}$ estimate

We first recall a global $C^{1/2+\varepsilon_0}$ estimate when the domain is Lipschitz.

Lemma

Under the Lipschitz conditions, there exist $\varepsilon_0 = \varepsilon_0(L) > 0$ *and* $M_1 = M_1(L) > 0$ *such that* $u \in C^{\frac{1}{2}+\varepsilon_0}(\Omega_{r_0})$ *and*

$$\|u\|_{C^{\frac{1}{2}+\varepsilon_0}(\Omega_{r_0})} \le M_1 r_0^{-\frac{3}{2}-\varepsilon_0} \|u\|_{L^2(\Omega_{2r_0})}.$$

For the proof, we compare u with $\mathrm{Re}(z^\beta)$, where $\beta \in (1/2, 1)$.

Step 1

By scaling, we may assume that $r_0 = 1$ and $\|u\|_{L^2(\Omega_2)} = 1$.

Using the Lipschitz estimate and the reverse Hölder's inequality, there exists $p_0 = p_0(L) > 2$ such that

$$\|\nabla_{x,y} u\|_{L^{p_0}(\Omega_r)} \leq C r^{\frac{2}{p_0}}.$$

Step 2

Take a smooth domain E such that $B^+_{2/3} \subset E \subset B^+_{3/4}$. For any $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$, denote

$$E_r(x_0, y_0) = \Big\{(x, y) \in \mathbb{R}^2 : r^{-1}(x - x_0, y - y_0) \in E\Big\},$$
$$\Gamma_r(x_0, y_0) = \Big\{(x, y) \in \partial E_r(x_0, y_0) : y = y_0\Big\}.$$

For r sufficiently small, we have $E_r(0, M_0 r^2) \subset \Omega_r$.

Take a smooth function $\eta = \eta(s)$ on $\mathbb{R}$ such that $\eta(s) = 0$ in $(-\infty, 1)$ and $\eta(s) = 1$ in $(2, \infty)$. Denote $\eta_r(s) = \eta(s/(M_0 r^2))$. A simple calculation reveals that $u(x, y)\eta_r(y)$ satisfies

$$\Delta_{x,y}(u(x, y)\eta_r(y)) = \partial_y(u\eta_r') + \partial_y u\eta_r' \quad \text{in } E_r(0, M_0 r^2)$$

and $u\eta_r = 0$ on $\Gamma_r(0, M_0 r^2)$. Note that the right-hand side is supported in a narrow strip $\{(x, y) \in \Omega_r : M_0 r^2 < y < 2M_0 r^2\}$.

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Step 3

We decompose $u\eta_r = w + v$ in $E_r(0, M_0 r^2)$, where $w = w_r$ be a weak solution to

$$\Delta_{x,y} w = \partial_y(u\eta_r') + \partial_y u\eta_r' \quad \text{in } E_r(0, M_0 r^2)$$

with the zero Dirichlet boundary condition on $\partial E_r(0, M_0 r^2)$.
Then $v = u\eta_r - w$ is harmonic in $E_r(0, M_0 r^2)$ and $v = 0$ on $\Gamma_r(0, M_0 r^2)$.

By using the $W^{1,p}$ estimate, Hardy's inequality, and a duality argument,

$$\|\nabla_{x,y} w\|_{L^p(E_r(0,M_0 r^2))} \le C\|\nabla_{x,y} u\|_{L^p(\Omega_r \cap \{y < 2M_0 r^2\})}.$$

Fix $p = \frac{(2+p_0)}{2}$ and let $q > 1$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_0}$.
Using Hölder's inequality,

$$\|\nabla_{x,y} w\|_{L^p(E_r(0,M_0 r^2))} \le C\|\nabla_{x,y} u\|_{L^{p_0}(\Omega_r \cap \{y < 2M_0 r^2\})} r^{\frac{3}{q}} \le C r^{\frac{2}{p_0} + \frac{3}{q}}.$$

By the Morrey embedding,

$$\|w\|_{L^\infty(E_r(0,M_0 r^2))} \le C r^{1+\frac{1}{q}}.$$

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By using the $W^{1,p}$ estimate, Hardy's inequality, and a duality argument,

$$\|\nabla_{x,y} w\|_{L^p(E_r(0,M_0 r^2))} \le C\|\nabla_{x,y} u\|_{L^p(\Omega_r \cap \{y < 2M_0 r^2\})}.$$

Fix $p = \frac{(2+p_0)}{2}$ and let $q > 1$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_0}$.
Using Hölder's inequality,

$$\|\nabla_{x,y} w\|_{L^p(E_r(0,M_0 r^2))} \le C\|\nabla_{x,y} u\|_{L^{p_0}(\Omega_r \cap \{y < 2M_0 r^2\})} r^{\frac{3}{q}} \le C r^{\frac{2}{p_0} + \frac{3}{q}}.$$

By the Morrey embedding,

$$\|w\|_{L^\infty(E_r(0,M_0 r^2))} \le C r^{1 + \frac{1}{q}}.$$

Step 3, Cont'ed

By the boundary estimate for harmonic functions,

$$\|\nabla_{x,y} v\|_{L^\infty(B^+_{r/4}(0,M_0 r^2))} \le C r^{-1} \|v\|_{L^\infty(B^+_{r/2}(0,M_0 r^2))},$$

which together with the Lipschitz regularity of u at 0 implies that

$$\|\nabla_{x,y} v\|_{L^\infty(B^+_{r/4}(0,M_0 r^2))} \le C.$$

Moreover, for any linear function ℓ of y,

$$\|\nabla^2_{x,y} v\|_{L^\infty(B^+_{r/4}(0,M_0 r^2))} \le C r^{-2} \|v - \ell\|_{L^\infty(B^+_{r/2}(0,M_0 r^2))}.$$

Thus by the mean value theorem and $v(0, M_0 r^2) = \partial_x v(0, M_0 r^2) = 0$, for any $\kappa \in (0, 1/4)$,

$$\begin{aligned} &\|v - (y - M_0 r^2)\partial_y v(0, M_0 r^2)\|_{L^\infty(B^+_{\kappa r}(0,M_0 r^2))} \\ &\le C\kappa^2 \|v - \ell\|_{L^\infty(B^+_{r/2}(0,M_0 r^2))}. \end{aligned}$$

Step 4 (last step)

Recalling $u\eta_r = w + v$ in $E_r(0, M_0 r^2)$, we have

$$\|u\eta_r - (y - M_0 r^2)\partial_y v(0, M_0 r^2)\|_{L^\infty(B^+_{\kappa r}(0,M_0 r^2))}$$
$$\le C\kappa^2 \inf_{a,b\in\mathbb{R}} \|u\eta_r - (a + by)\|_{L^\infty(B^+_{r/2}(0,M_0 r^2))} + Cr^{1+\frac{1}{q}}.$$

By the $C^{1/2+\varepsilon_0}$ estimate,

$$\|u(1-\eta_r)\|_{L^\infty(\Omega_r)} \le \sup_{\Omega_r\cap\{y<2M_0 r^2\}} |u(x,y)| = \sup_{\Omega_r\cap\{y<2M_0 r^2\}} |u(x,y) - u(x,\psi(x))| \le Cr^{1+2\varepsilon_0}.$$

Thus,

$$\inf_{a,b\in\mathbb{R}} \|u - (a+by)\|_{L^\infty(\Omega_{\kappa r})} \le C\kappa^2 \inf_{a,b\in\mathbb{R}} \|u - (a+by)\|_{L^\infty(\Omega_r)} + Cr^{1+\alpha},$$

where $\alpha = \min\{2\varepsilon_0, 1/q\}$.
By a standard iteration argument,

$$\inf_{a,b\in\mathbb{R}} \|u - (a+by)\|_{L^\infty(\Omega_r)} \le Cr^{1+\alpha}.$$

Global wellposedness in 3D

Recently, we established the global wellposedness in the 3D case.

- Compared to the 2D case, in 3D the fundamental solution is implicit.
- While the H^1 regularity result due to Verchota suffices in the 2D case, in 3D this regularity turns out to be critical and thus inadequate.
- Instead, our proof relies on the $W^{1,2+\varepsilon}$ layer potential estimates in Lipschitz domains by Dahlberg-Kenig (1987) and Mitrea-Taylor (1999).
- For the proof of the pointwise $C^{1,\alpha}$ regularity, we used the $W^{1,3+\varepsilon}$ estimate for harmonic functions in Lipschitz domains due to Jerison-Kenig (1995).

Further questions

- Smoothness of strong solutions.
  For example, does the solution become C^1 and smooth in finite time? Note that there is no instantaneous smoothing of solutions (S. Wu et. al. (2022)).
- Equations in the whole space.
- Equations with surface tension.

Thank you for your attention!