Mathematical reflections on locality

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joint work with Li Guo and Bin Zhang

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I. The concept of locality revisited

Locality principle

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- Propose a mathematical framework which encompasses the main features of the locality principle in QFT;
- use this framework to carry out renormalisation in accordance with the locality principle.

Causal separation

Light cone, past and future

In the Minkowski space $(\mathbb{R}^d, g)$, where $g(x,y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ is the Lorentzian scalar product,

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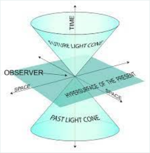

(picture downloaded from Wikipedia)

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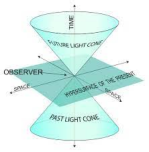

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Two sets S_1 and S_2 are causally separated $(S_1 || S_2)$ if and only if S_i does not lie in the future of S_j for $i \neq j$.

Locality in axiomatic QFT

The Wightman field $\varphi : \mathcal{S}(\mathbb{R}^d) \to \mathcal{O}(H)$ obeys the locality axiom

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The (relative) scattering matrix S_f satisfies the locality condition

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We introduce two binary relations

- on sets:

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Interpretation of (2)

$$f_1 \top f_2 \Longrightarrow S_f(f_1 + f_2) = S_f(f_1)\, S_f(f_2). \tag{6}$$

II. Locality as a symmetric binary relation

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set:

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For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

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Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$$L_1 \top L_2 :\iff L_1 \pitchfork L_2 \iff T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2.$$

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Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \iff m \wedge n = 1.$$

Locality category

Locality structures

- set X $\rightsquigarrow$ locality set $(X, \top)$; the polar set of U is $U^\top := \{x \in X, x \top u \quad \forall u \in U\}$
- semi-group (G, m_G) $\rightsquigarrow$ locality semi-group $(G, m_G, \top)$

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- algebra $(A, +, \cdot, m_A) \rightsquigarrow$ locality algebra $(A, +, \cdot, m_A, \top)$.

Locality morphisms: $f : (X, \top_X) \to (Y, \top_Y)$

- locality map:
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- locality semi-group morphism $f : (X, m_X, \top_X) \to (Y, m_Y, \top_Y)$:
  f is a locality map and $x_1 \top_X x_2 \Longrightarrow f(m_X(x_1, x_2)) = m_Y(f(x_1), f(x_2))$
  etc...

III. Locality relations are ubiquitious

Local functionals

are functionals F on test functions (fields) φ of the form $F(\varphi) = \int_M f\left(j_x^k(\varphi)\right)\, dx$ (here $j_x^k(\phi)$ is the k-th jet of ϕ at x): The localised version at φ:

$$F(\varphi + \psi) = F(\varphi) + \int_M f\left(j_x^k(\psi)\right)\, dx \quad \forall\psi. \tag{7}$$

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$$\varphi_1 \top\!\!\cap \varphi_2 \Longrightarrow F(\varphi_1 + \varphi + \varphi_2) = F(\varphi_1 + \varphi) - F(\varphi) + F(\varphi + \varphi_2) \quad \forall\, \varphi. \tag{8}$$

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Comparing the two [Brouder, Dang, Laurent-Gengoux, Rejzner (2018)]

Provided $D_\varphi F$ can be represented as a function $\nabla_\varphi F$ such that the map $\varphi \mapsto \nabla_\varphi F$ is smooth, then (8) $\Longleftrightarrow$ (7).

Locality and singularities

Separation of wavefront sets

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where we have set $\mathrm{WF}'(v) := \{(x, -\xi) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid (x, \xi) \in \mathrm{WF}(v)\}$.

Counterexample

Distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$. We have $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$

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Partial product of pseudodifferential operators of non-integer order

We equip $\Psi_{\mathrm{pgh}}^{\notin \mathbb{Z}}$ (the canonical trace TR is well defined) with the locality relation $A_1 \top^{\notin \mathbb{Z}} A_2 :\Leftrightarrow (\mathrm{ord}(A_1) + \mathrm{ord}(A_2) \notin \mathbb{Z}) \Rightarrow (\mathrm{TR}([A_1, A_2]) = 0)$.

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$(1/3, 1/3) \in (U^{\top} \times U^{\top}) \cap \top$ but $1/3 + 1/3 = 2/3 \notin U^{\top}$.

Open questions on locality [Clavier, Foissy, Lopez, S.P. 2022]

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IV. Evaluating meromorphic germs at poles in QFT

Functions of several variables in QFT

Speer's analytic renormalisation [JMP 1967] revisited

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Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$*, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...*

Brain teaser

(We assume the poles are at zero)
Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of Feynman functions $f : \mathbb{C}^k \to \mathbb{C}$,

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Evaluating a fraction with a linear pole at zero

$$f(z_1, z_2) = \frac{z_1 - z_2}{z_1 + z_2}|_{z_1=0, z_2=0} = \begin{cases} 1 \text{ or } -1\,? \\ 0\,? \\ 10000\,? \end{cases}$$

V. Locality on meromorphic germs comes to the rescue

Locality on multiparameter meromorphic germs

Multiparameter meromorphic germs with linear poles

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Locality: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k \in \mathbb{N}} \mathcal{M}(\mathbb{C}^k)$, $f_1 \, Q^\top \, f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

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Principle of locality revisited: locality evaluators

$f \perp^Q g \Longrightarrow \mathcal{E}(f \cdot g) = \mathcal{E}(f)\,\mathcal{E}(g)$ for two meromorphic germs f and g in an appropriate subalgebra $\mathcal{M}^\bullet$ of $\mathcal{M}$.

Speer's generalised evaluators

Reminder: Meromorphic germs in $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ have linear poles $L_i = \sum_{j_i \in J_i} j_i$.

Speer's evaluators consist of a family $\mathcal{E} = \{\mathcal{E}_k, \in \mathbb{N}\}$ of linear forms $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions

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Drawback: Speer's approach depends on the choice of coordinates

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Data

- $(\mathcal{M}^\bullet, \perp^Q)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen $\subset$ Speer $\subset$ Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}^\bullet$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- a group $\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)$ (for "Galois") of (locality) isomorphisms of $(\mathcal{M}^\bullet, \perp^Q)$ that leave holomorphic germs invariant;
- $\mathcal{M}_-^{\bullet Q}$ is generated by polar germs $f = \frac{h}{g}$ with $h \perp^Q g$.

Orthogonal projection

$\perp^Q$ induces a splitting [Berline and Vergne 2005, Guo, Zhang, S.P. 2015]

$$\mathcal{M}^\bullet = \mathcal{M}_+ \oplus^Q \mathcal{M}_-^{\bullet Q} \quad \text{and} \quad {\pi_+}^Q : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$$

VI. Classification of locality evaluators

Theorem [Guo, S.P., Zhang 2022]

Definition

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Theorem

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$$\mathcal{E} = \underbrace{\mathrm{ev}_0 \circ \pi_+{}^Q}_{\mathcal{E}^{\mathrm{MS}}} \quad \circ \quad \underbrace{T_{\mathcal{E}}}_{\mathrm{Gal}^Q(\mathcal{M}^\bullet/\mathcal{M}_+)} \quad .$$

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- locality Lyndon words with letters in X: locality Lyndon words form an algebraically independent generating set of the locality shuffle algebra generated by X.
- a locality isomorphism $u \mapsto x_u$ between the locality algebra generated by Chen-type poles $L_i = \sum_{j=1}^{i} \ell_{u_j} = \ell_{u_1} + \cdots \ell_{u_i}$ with $u \top v \Longrightarrow \ell_u \perp^Q \ell_v$ and a certain locality shuffle algebra.
- **Conclusion:** $\mathcal{M}^{\rm Chen}$ ($\mathcal{M}^{\rm Feyn}$) are locality polynomial algebras with locality "Lyndon fractions" as locality generators.

Final step of the proof

Since $\mathcal{M}^{\mathrm{Chen}}$, resp. $\mathcal{M}^{\mathrm{Feyn}}$ are $\perp$-local polynomial algebras, a generalised evaluator is uniquely determined by its values on the free generators.

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If $\mathcal{M}^\bullet$ is a free polynomial locality-algebra generated by $\mathcal{S}^\bullet$, then $T \in \mathrm{Gal}^{\perp}(\mathcal{M}^\bullet/\mathcal{M}_+)$ is uniquely determined by $\{T(S), S \in \mathcal{S}^\bullet\}$:
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$$\mathcal{E} = \underbrace{\mathrm{ev}_0 \circ \pi_+{}^{\perp}}_{\perp^Q-\text{minimal subtraction}} \circ \underbrace{T_\mathcal{E}}_{\text{Galois transformation}} .$$

THANK YOU FOR YOUR ATTENTION!

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