Are bosonic ghosts rational beings?

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Women in Mathematical Physics
Banff, Canada
August 16, 2023

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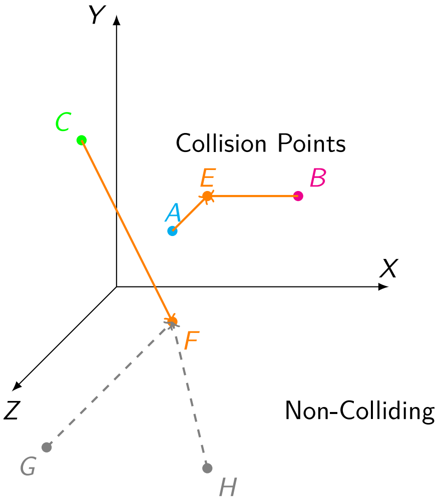
Introduction

- Representation theory goals: determine module categories
- Vertex (operator) algebra
    - Defined in Flor's talk.
    - Rational vertex algebras = semi-simple representation theory
    - Irrational vertex algebras = have indecomposable modules that are not simple
- Conformal flow = deformation of the conformal vector ω associated with a vertex operator algebra V to obtain a new conformal structure ω_μ on V, for $\mu \in \mathbb{C}$, a continuous parameter
    - example: Weyl vertex algebra

Rationality in pictures



* allowed collision combinations

* based on rationality

History

- Matsuo, Nagatomo: all possible conformal structures associated with the Heisenberg vertex algebra (free bosonic VA)
- Adamović, Milas: triplet algebras, an important example of C_2-cofinite but irrational vertex algebras
- Dong, Li, Mason: $\mathbb{Q}$–graded vertex algebras
- Dong-Mason, Laber-Mason: $\mathbb{C}$–graded vertex algebras
- Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna: a refinement of the various concepts of $\mathbb{C}$–grading for a vertex algebra.

Vacuum space

Vacuum space $\Omega(V)$ = vectors in V that are zero if they are acted on by any mode of V that lowers the real part of the weight.

- important: vacuum vector is not necessarily in the vacuum space. E.g. Weyl vertex algebra ${}_{\mu}M$ with $\mu \in \mathbb{R}$ and $\mu < 0$, e.g. $\mu = -\frac{1}{2}$, i.e. $c = 11$.

Graded vertex algebras

Definition 1

An Ω–generated $\mathbb{C}$–graded vertex algebra is a $\mathbb{C}$–graded vertex algebra $(V, Y, 1)$ such that every element $v \in V$ is a finite sum of elements of the form $v^k v^{k-1} \cdots v^2 u^0$, for $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{Z}$, $v^1, \ldots, v^k \in V$ and $u^0 \in \Omega(V)$.

Definition 2

An Ω–generated $\mathbb{C}_{Re>0}$–graded vertex algebra ($=$ nice vertex algebra) is an Ω–generated $\mathbb{C}$–graded vertex algebra such that the following notion of degree is well defined:

- degree of elements in $\Omega(V)$ is 0
- $deg(v^k_{n_k} \cdots v^1_{n_1} u^0) = \sum_{j=1}^k (|v^j| - n_j - 1)$, $v^1, \ldots, v^k \in V$, $n_1, \ldots, n_k \in \mathbb{Z}$, $u^0 \in \Omega(V)$.
- extend by linearity

Relationship between given graded VOA definitions

$$\Omega VOA(\mathbb{C}_{Re>0}(\mathcal{V})) \subset \Omega(\mathbb{C}(\mathcal{V})) \subset \mathbb{C}(\mathcal{V}) \subset \mathcal{V}$$

Here,

- $\mathcal{V}$ = set of vertex algebras,
- $\mathbb{C}(\mathcal{V})$ = set of $\mathbb{C}$–graded vertex algebras,
- $\Omega(\mathbb{C}(\mathcal{V}))$ = set of Ω–generated $\mathbb{C}$–graded vertex algebras,
- $\Omega VOA(\mathbb{C}_{Re>0}(\mathcal{V}))$ = set of Ω–generated $\mathbb{C}_{Re>0}$–graded vertex operator algebras

Also, if V is a nice vertex algebra, and we define $V(\lambda)$ to be the the space of all $v \in V$ with degree λ, then we have

$$V = V(0) \oplus \bigoplus V(\lambda)$$

$$= \Omega(V) \oplus \bigoplus V(\lambda)$$

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Introduction

- Free field theories:
  - Free bosons
  - Free fermions
  - $\beta\gamma$–ghost system (bosonic ghost system)
  - *bc*–ghost system (fermionic ghost system).
- gradings other than $\mathbb{Z}$–gradings, depend on the choice of conformal vector

Introduction

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Weyl vertex algebra

- conformal vector with $c = 2 \Longrightarrow \mathbb{Z}$–grading, but with infinite-dimensional weight spaces
  - no graded-traces
- general case $= \mathbb{C}$–grading, depending on the choice of conformal element
- there is a family of conformal vectors indexed by a complex parameter $\mu \in \mathbb{C}$
  - [BBOPY] the resulting family of conformal $\beta\gamma$–systems, denoted ${}_{\mu}M$, have very different resulting gradings, Zhu algebras, and representation theory.
- we defined and analyzed several different types of $\mathbb{C}$–graded VAs and proved general results for finitely Ω-generated $\mathbb{C}_{Re>0}$-graded VOAs:
  - these types of VAs are C_2–cofinite if the weights of the generating set are non integer.
  - if in addition, all simple modules of a V of this type are ordinary, then V is rational.

Rank 1 Weyl vertex algebra

Definition 3

Let $\mathcal{L}$ be the infinite-dimensional Lie algebra with generators $K, a(m)$, and $a^*(n)$ with $m, n \in \mathbb{Z}$ such that K is in the center and the bracket is given by

$$[a(m), a^*(n)] = \delta_{m+n,0} K.$$

We define the *rank one Weyl algebra* $\mathcal{A}_1$ to be the quotient

$$\mathcal{A}_1 = \frac{\mathcal{U}(\mathcal{L})}{\langle K-1 \rangle},$$

where $\mathcal{U}(\mathcal{L})$ = universal enveloping algebra of $\mathcal{L}$, $\langle K-1 \rangle$ = two sided ideal generated by $K-1$. Then $\mathcal{A}_1$ = associative algebra with generators $a(m)$, $a^*(n)$, for $m, n \in \mathbb{Z}$, and relations

$$[a(m), a^*(n)] = \delta_{m+n,0} \quad (1)$$

$$[a(m), a(n)] = [a^*(m), a^*(n)] = 0 \quad (2)$$

Weyl vertex algebra

- There is a unique vertex algebra structure on M, given by $(M, Y, \mathbf{1})$ with vertex operator map $Y : M \to End(M)[[z, z^{-1}]]$ such that

$$Y(a(-1)\mathbf{1}, z) = a(z), \qquad Y(a^*(0)\mathbf{1}, z) = a^*(z),$$
$$a(z) = \textstyle\sum_{n\in\mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n) z^{-n}. \tag{3}$$

- The fields $a(z)$ and $a^*(z)$ are usually denoted by $\beta(z)$ and $\gamma(z)$ in the physics literature (up to a choice of sign) where the vertex algebra M is referred to as the $\beta\gamma$ vertex algebra or $\beta\gamma$ system.

Conformal elements

- The vertex algebra M admits a family of Virasoro vectors

$$\omega_\mu = (1-\mu)a(-1)a^*(-1)\mathbf{1} - \mu a(-2)a^*(0)\mathbf{1}, \quad \text{for } \mu \in \mathbb{C},$$

  of central charge

$$c_\mu = 2(6\mu(\mu-1)+1).$$

- The corresponding Virasoro field is

$$L^\mu(z) = (1-\mu) : a(z)\partial a^*(z) : -\mu : \partial a(z)a^*(z) :$$

- This gives a $\mathbb{C}$–grading on M, denote by

$$({}_\mu M, Y, \mathbf{1}, \omega_\mu),$$

  or just ${}_\mu M$.

Weyl vertex algebra gradings

We have a $\mathbb{C}$–grading

$$_{\mu}M = \amalg_{\lambda \in \mathbb{C}} \ {}_{\mu}M_{\lambda}, \quad |v| = \lambda, \ v \in_{\mu} M],$$

and similarly there are $_{\mu}M$–modules

$$W = \amalg_{\lambda \in \mathbb{C}} W_{\lambda} m \quad L_{\mu}(0)w = \lambda w.$$

Nice positive energy grading is the resulting $\mathbb{C}$–grading of $_{\mu}M$ and its modules if

- truncated from below
- $|Im(\lambda)| > Re(\lambda)$, for finitely many λ
- $dim W_{\lambda} < \infty$

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Various regions which give different structures on ${}_{\mu}M$

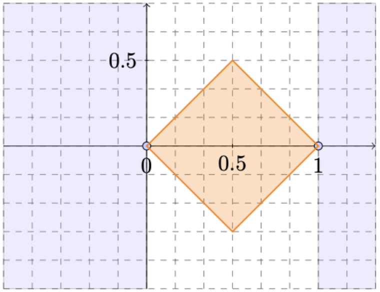

Figure 1: Values of $\mu \in \mathbb{C}$ for which ${}_{\mu}M$ has different $\mathbb{C}$-graded VA structures.

Theorem (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna):

- In the orange diamond shaped region, ${}_{\mu}M$ has a "nice positive energy" grading. Here, ${}_{\mu}M$ is rational with only 1 simple admissible module. No graded pseudo-traces.

Various regions which give different structures on ${}_\mu M$

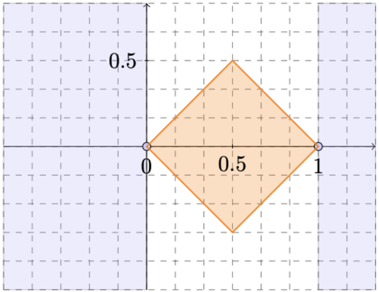

Figure 2: Values of $\mu \in \mathbb{C}$ for which ${}_\mu M$ has different $\mathbb{C}$-graded VA structures.

Theorem (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna):

- In the blue regions we lose truncation: ${}_\mu M$ has ${}_\mu M_\lambda \neq 0$ for an infinite arbitrary $Re(\lambda) > 0$ and $Re(\lambda) < 0$.

Various regions which give different structures on ${}_{\mu}M$

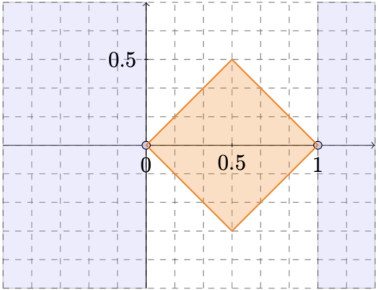

Figure 3: Values of $\mu \in \mathbb{C}$ for which ${}_{\mu}M$ has different $\mathbb{C}$-graded VA structures.

Theorem (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna):

- In the white region, ${}_{\mu}M$ has $|Im(\lambda)| > Re(\lambda)$, for an infinite number of λ

Various regions which give different structures on ${}_{\mu}M$

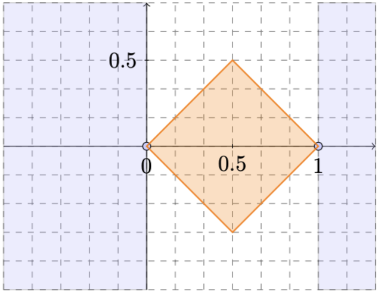

Figure 4: Values of $\mu \in \mathbb{C}$ for which ${}_{\mu}M$ has different $\mathbb{C}$-graded VA structures.

Theorem (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna):

• At $\mu = 0, 1$ the only thing that fails is that we lose $dim_{\mu} M_{\lambda} < \infty$. However, $\mathbb{N}$–graded and irrational $\implies$ have meaningful graded pseudo-traces if add another grading operator

Various regions which give different structures on ${}_\mu M$

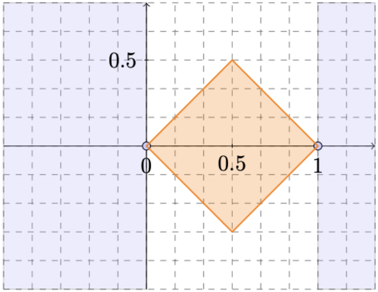

Figure 5: Values of $\mu \in \mathbb{C}$ for which ${}_\mu M$ has different $\mathbb{C}$-graded VA structures.

Theorem (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna):
//: On the lines corresponding to $Re(\lambda) = 0$ or 1, but $Im(\lambda) \neq 0$, we have $|Im(\lambda)| > Re(\lambda)$, for an infinite number of λ.

Main Theorems: Rationality for certain $\mathbb{C}$-graded VOAs and applications to Weyl VAs

Theorem 4 (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna (JMP 2022))

Let V be an Ω-generated $\mathbb{C}_{Re>0}$-graded VOA that is finitely generated by $v^1, \ldots, v^k$, and in addition satisfies the following:

- *For each $j \in 1, \ldots, k$, $|v^j|$ is not an integer;*
- $V^0 = \bigoplus_{n=0}^{\infty} V_n$;
- *Every simple $\mathbb{C}_{Re>0}$-graded V–module is ordinary.*

Then V is rational and has only one simple $\mathbb{C}_{Re>0}$-graded V–module.

Main Theorems: Rationality for certain $\mathbb{C}$-graded VOAs and applications to Weyl VAs

Theorem 5 (Barron-Batistelli-Orosz Hunziker-VPT-Yamskulna (JMP 2022))

Let $\mu \in \mathbb{C}$ such that one of the following holds:

(i) $0 < \mathrm{Re}(\mu) \leq 1/2$ *and* $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(\mu)$,

(ii) $0 < \mathrm{Re}(1-\mu) < 1/2$ *and* $|\mathrm{Im}(\mu)| \leq \mathrm{Re}(1-\mu)$,

i.e., $\mu \in$ orange region.
Then $({}_{\mu}M, \omega_{\mu})$ is a rational Ω-generated $\mathbb{C}_{Re>0}$-graded VOA and has only one simple $\mathbb{C}Re > 0$-graded module, which is, in fact, a simple ordinary ${}_{\mu}M$-module, namely ${}_{\mu}M$ itself.

Tools for proof

The proofs involve:

- The Zhu algebra $A(V)$, appropriately defined in this setting,
- A filtration of $A(V)$ to relate it to an abelian poisson algebra $VC(V)$.
- Another Theorem involving the structure of certain Zhu algebras and implications for the rationality of certain VAs.

Thank You!

References I

[Bar+22] Katrina Barron et al. “On rationality of C-graded vertex algebras and applications to Weyl vertex algebras under conformal flow”. In: *Journal of Mathematical Physics* 63.9 (Sept. 2022), p. 091706. ISSN: 0022-2488. DOI: 10.1063/5.0117895. eprint: https://pubs.aip.org/aip/jmp/article-pdf/doi/10.1063/5.0117895/16563713/091706_1_online.pdf. URL: https://doi.org/10.1063/5.0117895.