Chaos and complexity through the lens of dynamics in Krylov space.

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What is this talk about?

- Dynamics in Krylov space emerged as a novel probe of chaos and complexity

  - universal operator growth hypothesis
  - operator growth (OTOC)
  - complexity

- What is behind all of this?

  - demystifying Krylov: it only knows about

$$C(t) = \langle A(t)A \rangle \text{ or } F(t) = \langle \psi(t)|\psi \rangle$$

  - relation between $C(t)$ and $b_n, K(t)$ is very intricate

Recursion method

- ingredients: Hamiltonian H, an operator A, two-point function $C(t) = \langle A(0)A(t) \rangle$
- iterative relation defines basis A_n in Krylov space

$$A_{n+1} = [H, A_n] - a_n A_n - b_{n-1}^2 A_{n-1}$$

  Lanczos coefficients a_n, b_n are fixed by requiring A_n are mutually orthogonal, $\quad b_n^2 = \langle A_{n+1} A_{n+1} \rangle / \langle A_n A_n \rangle$

- Liouvillian matrix $\mathcal{L}_{nm}$ is three-diagonal

$$[H, A_k] = \sum \mathcal{L}_{kl}\, A_l$$

  time evolution $e^{i\mathcal{L}t}$ is easy to evaluate numerically

Math of Krylov method

- Krylov space can be defined for any linear operator

$$v_0 = v, \qquad v_k = H^k v_0$$

  Krylov space includes all eigenvectors/eigenvalues of H, which have an overlap with v_0
- Lanczos method – choice of basis in Krylov space
- $\mathcal{L}$ is defined by $[H, A]$; choice of cor. function – choice of scalar product, affecting *representation* of $\mathcal{L}$

  isospectral deformation, integrable dynamics of a_n, b_n
  with Gorsky, PRB 102, 085137 (2020)
  temperature dependence – talk by Nick Angelinos
- Lanczos coefficients b_n – a way to rewrite $C(t)$

Chaos from two-point function

- power spectrum

$$f^2(\omega) = \frac{1}{2\pi} \int dt\, e^{-i\omega t}\, C(t)$$

- relation between f, C, and b_n

$$f^2(\omega) \sim e^{-\omega/\omega_0} \leftrightarrow C(i\tau) \sim (\tau - \omega_0^{-1})^{\Delta}$$

  consistent with $b_n \sim \pi\omega_0 n/2$

- signature of chaos?

  $f^2(\omega) \sim e^{-\omega/\omega_0}$ is a signature of (classical) chaos
  Elsayed, Hess, Fine, PRE 90, 022910 (2014)

  singularity of $|A(it)$ and $C(it)$ is expected in a generic quantum lattice model in $D \geq 2$
  with A. Avdoshkin, PRR 2, 043234 (2020)

Universal operator growth hypothesis

- in a generic quantum system Lanczos coefficients grow at maximal possible rate (consistent with locality)

$$b_n \sim \alpha\, n$$

smart reformulation of Elsayed et al.?

- Krylov complexity grows exponentially $K \sim e^{2\alpha t}$ and bounds OTOC (conjecture)

$$K(t) = \sum_k |c_k|^2 k, \qquad A(t) = \sum c_k \tilde{A}_k.$$

Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 9, 041017

partial proof Gu, Kitaev, Zhang JHEP 03 (2022) 133

Lanczos growth and chaos

- SYK model
  Parker et al.'2019
- universal bounds on $C(t)$ in lattice models
  with Avdoshkin'2020
- non-integrable 1D Ising model in magnetic field
  maximal growth of $b_n \sim n/\ln(n)$ (provided the behavior is smooth)
  Cao'2021
- faster then exponential decays of f^2 for XXZ model
  numerical evidence, Rigol et al.'2020

problems

- b_n grow linearly in free theories; also not smooth
- b_n probe scrambling, not chaos?
  Bhattacharjee et al.'2022

Thermal 2pt function in CFT

- "Wightman"-ordered thermal two point function

$$C(t) = \mathrm{Tr}(e^{-\beta H/2} A(t) e^{-\beta H/2} A)/Z(\beta).$$

- $C(t)$ always has a singularity at $t = i\beta/2$

  assuming b_n is smooth they must behave as

  $b_n \approx \pi(n + \Delta + 1/2)/\beta$, in which case $K \sim e^{\frac{2\pi}{\beta}t}$

- conjectural bound on OTOC in terms of Krylov complexity becomes MSS bound

$$\lambda_{OTOC} \leq \frac{2\pi}{\beta}$$

Universality of b_n in QFT

- singularity at $C(i\beta/2)$ or $f(\omega) \propto e^{-\beta\omega/2}$ is dictated by locality
- relation between b_n and $C^{(n)}$ – sum over Dyck paths

$$M_{2k} = C^{(2k)}/C = \sum_{h_1 \dots h_{2k}} \prod b_{(h_i+h_{i+1})/2}$$

- integral over Dyck paths

$$C^{(2k)}/C = \int \mathcal{D}f e^S, \quad S = 2k \int_0^1 dt S_2((f'(t)+1)/2)+\ln b(2kf(t))$$

with A. Avdoshkin'2019

$$b_n \approx \pi(n+\Delta+1/2)/\beta$$

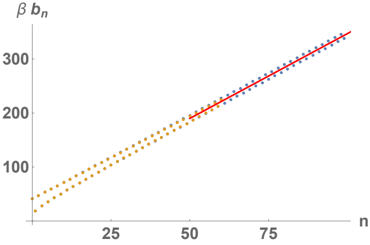
with Smolkin, PRD 104, L081702 (2021)

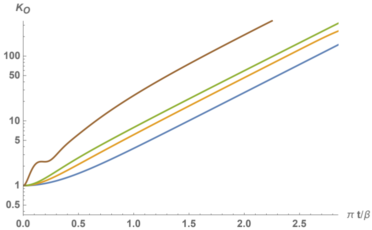
Free fields in various dimensions

- thermal 2pt function

$$C(-i\tau) \sim \zeta(2\Delta, 1/2 + \tau/\beta) + \zeta(2\Delta, 1/2 - \tau/\beta)$$

- for scalar $d = 4$ and $d = 6$ b_n are known; for other d and fermions theories – numerical results
- b_n exhibit "staggering" but are sufficiently smooth; $K(t)$ grows exponentially at the "MSS" rate



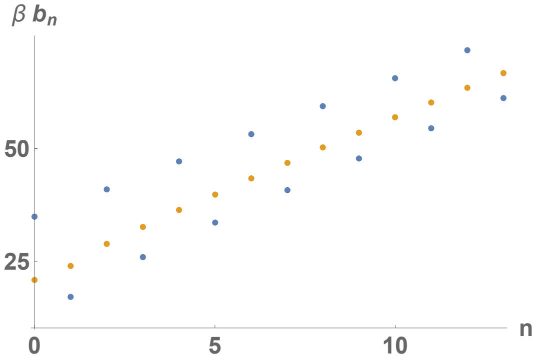


with M. Smolkin, 2021

Holographic thermal 2pt function

- thermal 2pt function evaluated numerically

  the behavior of b_n asymptote to $\pi(n + \Delta + 1/2)/\beta$



exponential growth of Krylov complexity with $\lambda_k = 2\pi/\beta$

Effects of β and UV cutoff

- $f^2(\omega) \propto e^{-\beta\omega/2}$ is fixed kinematically, because 2pt function is singular when two operators collide
- another look: $f^2(\omega) \propto e^{-\beta\omega/2}$ is because $\langle E+\omega/2|\phi|E-\omega/2\rangle$ is unsuppressed for large ω – due to Heisenberg uncertainty principle!
- conclusion: for any *local* ϕ behavior $f^2(\omega) \propto e^{-\beta\omega/2}$ and hence linear growth of b_n follow automatically; asymptotic behavior of b_n is not fixed when UV cutoff is introduced
- prediction: at low T coefficients b_n should exhibit linear growth, even if the model is integrable!

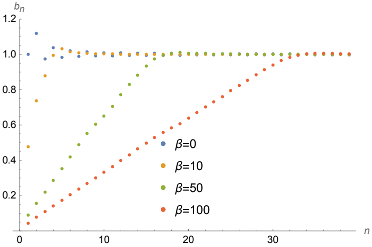
  more on temperature dependence: talk by Nick Angelinos

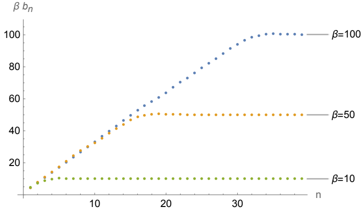
Spin-chain at different temperatures

- XY-model

$$H = \sum_i X_i X_{i+1} + Y_i Y_{i+1}$$

- linear growth of b_n at small T, saturation at UV-cutoff





small T behavior is universal (field theory), real asymptote of b_n is controlled by UV physics

Universal growth hypothesis: upshot

- maximal *asymptotic* growth of b_n for generic non-integrable systems

  finite Λ_{UV}, $b_n \sim \Lambda_{UV}$, and thermodynamic limit, which reduces to divergence of $|A(it)|$

- coarse grained universality of b_n from $\rho(E)$
  as well OTOC, which also probes scrambling

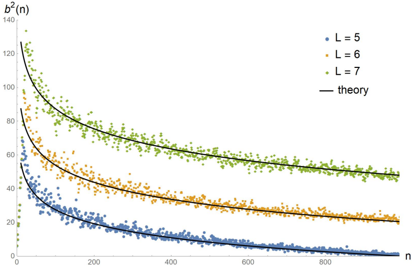
  Bhattacharjee, Cao, Nandy, Pathak '2022

- possibility of exotic behavior (several branches of b_n)?

Reformulation in terms of Krylov space opened new connections with level statistics, OTOC, complexity, etc.

Dynamics in Krylov space and OTOC

- does $C(t)$ and b_n know about chaos (level statistics)?
  Yes! spectrum of $\mathcal{L}$ is $E_n - E_m$
- coarse grained universality of b_n from $\rho(E)$



imprint of $\langle \rho(E)\rho(E') \rangle$ is work in progress

talk by Javier Magan, Balasubramanian, Magan, Wu'22
Erdmenger, Jian, Xian'23
Hashimoto, Murata, Tanahashi, Watanabe'23

Dynamics in Krylov space and OTOC

- b_n grow at maximal rate compatible with locality, $b_n \sim \alpha\, n$
- Krylov complexity grows exponentially $K \sim e^{\lambda_K t}$, bounds OTOC

$$\lambda_{OTOC} \leq \lambda_K (= 2\alpha) \leq \frac{2\pi}{\beta}$$

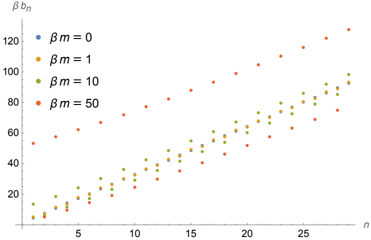
left inequality, conjectured by Parker et al., proved for $f^2(\omega) \sim e^{-\pi\omega(2\alpha)}$ by Gu, Kitaev, Zhang'2021

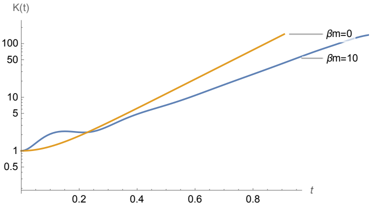
right inequality, generalization when $\lambda_K \neq 2\alpha$, with Avdoshkin'2020

Free massive scalar

- mass introduces "persistent staggering"

$$\beta b_n = \alpha_0 \, n + \alpha_1 + (-1)^n \alpha_2$$





Krylov complexity grows exponentially with $\lambda_K < 2\pi/\beta$

- for massive fermion qualitatively the same

  with Avdoshkin and Smolkin, 2212.14429

  also talk by Keun-Young Kim,

  Camargo, Jahnke, Kima, Nishidac'22

Krylov complexity

Dynamics in Krylov space and complexity

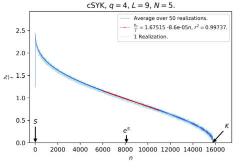
- growth of $K(t)$ should be compared with holographic/computational complexity

  Barbon, Rabinovici, Shir, Sinha'2019

- initial (exponential) growth, after that $K(t)$ grows approximately linearly for an exponential time until saturates at $K(t) \sim e^{O(S)}$

  numerical evidence from SYK model, spin chains, etc., bulk interpretation for SYK

  Rabinovici, Sanchez-Garrido, Shir, Sonner'2021, 2022,2023



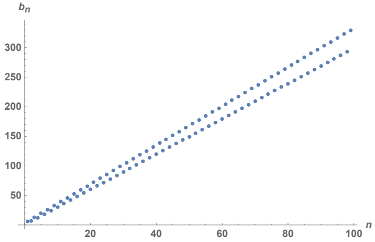
- connection between $K(t)$ and complexity, Caputa, . . .

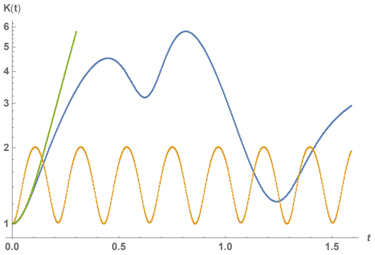
Free scalar on $\mathbb{S}^3 \times \mathbb{S}^1$

- what if the space is compact

$$C(-i\tau) \sim \sum_{\ell} \frac{\pi^2}{\cosh^2(\pi(\ell R + i\tau)/\beta)} - \frac{2\pi\beta}{R}$$

- for finite R asymptotic behavior changes!





Krylov complexity is bounded from above, $K(t) \leq K_{\max}(R/\beta)$

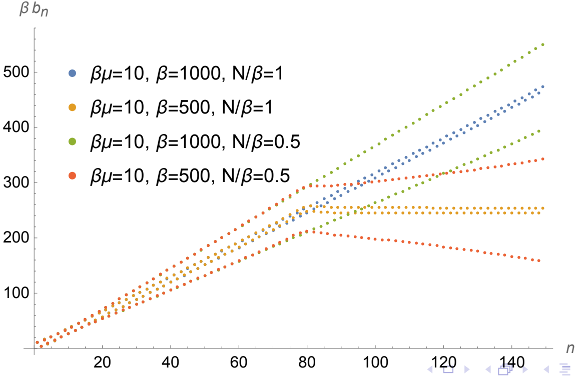
- same qualitative picture for holographic model in TAdS

1D bosons on the lattice

- a model with mass, compact space, UV-cutoff

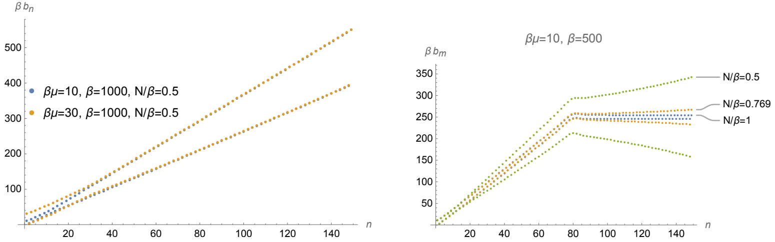
$$H = \sum_{i=1}^{N} \dot{\phi}_i^2 + (\phi_{i+1} - \phi_i)^2 + \mu^2 \phi_i^2$$

- slopes, controlled by N/β and UV cutoff



1D bosons on the lattice

- ‘staggering” controlled by $\beta\mu$, slopes controlled by N/β



by tuning N/β two slopes will emerge well before the UV-cutoff

- if R/β is sufficiently small, $K(t)$ is bounded from above by some UV-independent value!

Conclusions

- Krylov space is a new avenue to *unify* approaches to chaos
- extension of MSS bound

$$\lambda_{OTOC} \leq \lambda_K \leq \frac{2\pi}{\beta}$$

  non-trivial checks

- behavior of b_n as a probe of chaos, need for UV cutoff

  imprint of level statistics?

- Krylov complexity vs holographic/computational complexity

  suggestive but non-universal results

Auxiliary slides

Upper bound on infinity-norm

- Euclidean time evolution

$$A(t) \equiv e^{tH} A\, e^{-tH} = \sum_k \underbrace{[H, \ldots [H, A]]}_{k \text{ times}} \frac{t^k}{k!}$$

- locality of interaction

$$H = \sum_I h_I, \qquad [H, \ldots [H, A]] = \sum_{I_1, \ldots, I_k} [h_{I_k}, \ldots [h_{I_1}, A]]$$

- the bound

$$|A(t)| \le |A| f(t), \quad f(t) = \sum_{\text{clusters}} \sum_k n(k) \frac{(2J|t|)^k}{k!}$$

$n(k)$ – number of sets $I_1, \ldots, I_k$, which satisfy adjacency condition, associated with a given cluster (lattice animal)

Counting the sets $I_1, \dots, I_k$

- Each set $I_1, \dots, I_k$ defines lattice animal *history*

$$\{I\} \equiv I_1, \dots, I_k \to \{J\} \equiv J_1, \dots, J_j, \qquad j \le k$$

- the map $\{I\} \to \{J\}$ defines a partition of k objects into j groups, and vice versa

$$n(k) = S(k, j)\phi(j)$$

$n(k)$ – number of sets $\{I\}$ associated with a given cluster
$\phi(j)$ – number this cluster's histories $\{J\}$

$$N(k) = \sum_{\text{clusters}} n(k) = \sum_j S(k, j)\phi(j)$$

Summing over histories

- Stirling transform

$$N(k) = \sum_j S(k,j)\phi(j), \quad \phi(j) = \sum_k s(k,j)N(k)$$

- Stirling transform

$$f(t) \equiv \sum_k N(k)\frac{t^k}{k!} = \sum_j \phi(j)\frac{q^j}{j!}, \quad q \equiv e^t - 1.$$

- summing over histories – new expansion parameter

$$|A(t)| \leq |A| f(t), \qquad f = \sum_j \phi(j)\frac{q^j}{j!}, \quad q \equiv e^{2J|t|} - 1.$$

Bound for Bethe lattices

- Bethe lattice of coordination number $z \geq 2$
- exact number of lattice animal histories

$$\phi(j) = (z-2)^j \frac{\Gamma(j + z/(z-2))}{\Gamma(z/(z-2))}$$

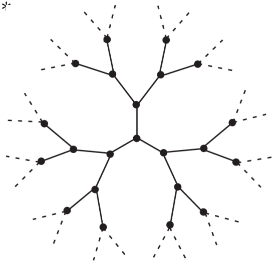
- the bound

  $f = (1-(z-2)q)^{-z/(z-2)}$

  for $z > 2$ there is a pole at some $t = \beta^*$

  for $z = 2$, i.e. 1D lattices, $f = e^{2q}$

- for arbitrary lattices Bethe lattices provide an upper bound



Euclidean operator growth and chaos

- generic non-integrable quantum lattice models
  $D \geq 2$, singularity at finite $t = \beta^*$

$$|A(t)| \lesssim \frac{|A|}{(1 - q/q_0)}$$

  $D = 1$, double-exponential growth

$$|A(t)| \lesssim |A|e^{2q}$$

- Euclidean Lieb-Robinson
  $D \geq 2$, operators spread to spatial infinity at finite $t = \beta^*$
  $D = 1$, operators spread exponentially, $t \sim \ln(\ell)$

$$|[A(t), B]| \leq 2|A||B|e^{q}\frac{q^{\ell}}{\ell!}$$

Euclidean operator growth and OTOC

- location of the singularity of $C(\beta^* = \pi/(2\alpha))$ – slope of Lanczos coefficients growth $b_n \propto \alpha\, n$ bounds $\lambda_{\rm OTOC}$

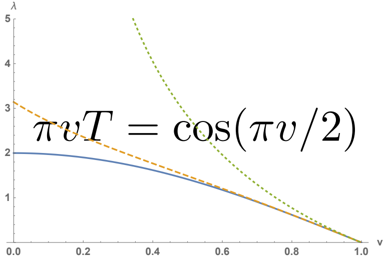
$$\lambda_{\rm OTOC} \le 2\alpha$$

Parker, Cao, Avdoshkin, Scaffidi, Altman'18
Murthy, Srednicki'19

- improved bound on chaos for large T

$$\lambda_{\rm OTOC} \le \frac{2\pi T}{1+2\beta^* T}$$

  - exact SYK
  - improved bound
  - MSS bound



$$\pi v T = \cos(\pi v/2)$$

Singularity of $C(t)$

- scalar product in the space of operators

$$\langle B|A\rangle := \frac{1}{N}\mathrm{Tr}(AB^\dagger), \qquad C(t) = \langle A|A(t)\rangle$$

- adjoint action $[H,\ \ ]$ is self-adjoint with $\langle\ |\ \rangle$

$$C(t_1+t_2) = \langle A(t_1)|A(t_2)\rangle = \langle A(t_1/2)|e^{t_2[H,\ ]}|A(t_1/2)\rangle$$

- assuming $A(t/2)$ is typical

$$C(t+\beta) = ||A(t)||^2\frac{Z(\beta)Z(-\beta)}{Z(0)^2} = ||A(t)||^2 e^{2F(0)-F(\beta)-F(-\beta)}$$

qualitatively, singularity of $C(t)$ is associated with $A(t)$ spreading within Krylov space and becoming more typical

Chaos vs localization in Krylov space

- when the system is chaotic and $C(t)$ has a singularity at $t = t^*$, $A(t)$ delocalizes in Krylov space at $t = t^*/2$
- when the system is integrable and $C(t)$ is analytic, IPR is finite and the operator is Localized

$$C(t) \propto e^{at^2/2}, \qquad I \propto t,$$
$$C(t) \propto e^{ae^{mt}}, \qquad I \propto e^{mt}$$

qualitatively similar to: localization/ergodicity in physical space = localization / delocalization in Fock space

Altshuler, Gefen, Kamenev, Levitov'97

Basko, Aleiner, Altshuler'06

Main results

- universal bounds on the operator norm growth in lattice models, Euclidean Lieb-Robinson bound
- Toda chain interpretation of the recursion method, time-correlation function
- chaos in the underlying quantum many-body system as delocalization in Krylov space

Outlook

- Connection between Euclidean and Minkowski dynamics
- Can Toda help connect different manifestations of chaos?

  - connection with OTOC

  - connection with spectral properties
- Chaos as delocalization? Connection to BH physics?